Sampling from scale-free networks and the matchmaking paradox

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Consider a large finite scale-free network consisting of $M \gg 1$ nodes and $N \gg 1$ links, in which the degree distribution of links per bond is governed by a power-law $P(n) \simeq n^{-1-\alpha}$ with exponent $0 < \alpha < 1$. A subset of $m \ll M$ nodes is sampled arbitrarily, yielding the empirical sample mean η: the average number of links per node, within the sampled subset. We explore the statistics of the sample mean η and show that its fluctuations around the network mean $\nu = N/M$ are extremely broad and strongly skewed—yielding typical values, which are systematically and significantly smaller than the network mean ν. Applying these results to the case of bipartite scale-free networks, we show that the sample means of the two parts of these networks generally differ—a fact we refer to as the "matchmaking paradox."

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I. INTRODUCTION

In this work we address the problem of sampling finite scale-free networks. Consider a network consisting of $M \gg 1$ nodes connected by $N \gg 1$ links, and assume that the distribution of the number of links per node follows a power-law

$$P(n) \simeq n^{-1-\alpha}, \quad (1)$$

with exponent $0 < \alpha < 1$. Note that in this exponent range the mathematical expectation diverges: $\sum_{n=1}^{\infty} nP(n) = \infty$. Examples of such scale-free networks include the network of words co-occurrence [1,2] (with $\alpha = 0.8$), and the network of homosexual contacts [3] (with $\alpha = 0.6$).

Most "real-world" scale-free networks discussed in the literature are vast, yet finite: biological populations, texts and movies, the Internet and the World Wide Web, etc. In all such networks both the overall number of nodes M and the number of links N—though being huge—is finite. Consequently, the average number of links per node $\nu = N/M$ is well defined and finite.

Consider now a random sample of size $m < M$ nodes, drawn from the overall population of M nodes. We inquire the distribution of the sample mean $\eta = \frac{1}{m}\sum_{i=1}^{m} n_i$, where n_i denotes the number of links connected to the i-th node of the sample. In particular, it is of interest to know whether the sample mean η is typically larger or smaller than the network mean ν, and how do its statistics change as the sample size m is increased.

In what follows we consider large, yet finite, scale-free networks governed by the power-law degree distribution of Eq. (1), and possessing a finite network mean ν. We note that the finite-size effect due to a finite network mean ν has to be distinguished from the finite-size effects in growing networks, where the growth process poses additional constraints on the degree distribution—see [4] and references therein.

The aforementioned sampling problem is closely related to the "matchmaking problem." In the mid eighties several research groups were conducting investigations on the distribution of the number of sexual partners in human populations—these investigations promoted by the necessity to point out the risk groups in the AIDS epidemic. A "Nature" editorial by Maddox stated that [5], "the figures so far show that the average number of heterosexual partners of men in the course of a lifetime is 11.0 and of women 2.9." In response to Maddox's editorial, Gurman published a note explaining the nonsense of having different means in the two populations connected by well-defined one-to-one links [6], "a heterosexual union is analogous to a heteronuclear chemical bond, and the total number must be the same if viewed from the male or female end."

To model the matchmaking problem consider two sets consisting of $M \gg 1$ nodes each—say red and blue nodes (or boys and girls). The nodes of the two sets are connected by $N \gg 1$ links having a red node on one side and a blue one on the other side. Although the overall number of the links N is the same when viewed from the red and from the blue side, the distributions of the number of links attached to red and to blue nodes may differ. In the aforementioned scale-free setting the red and the blue degree distributions will admit the power-laws

$$P_{\mathrm{red}}(n) \simeq n^{-1-\alpha_{\mathrm{red}}}, \quad P_{\mathrm{blue}}(n) \simeq n^{-1-\alpha_{\mathrm{blue}}} \quad (2)$$

(with, in general, exponents $\alpha_{\mathrm{red}} \neq \alpha_{\mathrm{blue}}$). In such a situation—when sampling from the red and blue populations—how different can the red and the blue sample means be?

Empirical studies showed that the distributions of the number of sexual partners are extremely broad [7]. If at least one of the exponents (α_{red} or α_{blue}) is in the range $0 < \alpha < 1$ then—as we shall establish in this paper—the corresponding sample mean depends systematically on the sample size m. Consequently, the male and the female sample means *have to differ* for small samples in order to match each other for the population as a whole. This key point is what we refer to as the "*matchmaking paradox*." For exponents in the range $\alpha > 1$ this is no more the case, and the sample means have to match. Up to our best knowledge, this exponent-dependency aspect of the matchmaking problem was never considered in detail (probably due to the lack of an adequate mathematical toolbox). Both the problems of sampling and matchmaking are of considerable interest—especially taking into account the overall importance of the sampling procedures in networks [8]. Moreover, the problem has much in common with the phenomena of *weak ergodicity breaking*—in which sample means that should "normally" be the same exhibit, in effect, universal fluctuations [9–11].

The main issues explored in this research are the following: What is a distribution of the sample mean η? How does this distribution depend on the sample size m? And how does the sample mean η relate to the network mean ν? These issues are intimately connected to the statistics of *Lévy random probabilities*, studied in [12]—but have several unique aspects which are worth a separate and detailed investigation.

II. MODEL

We follow Gurman's setup with a static and finite bipartite population. Such bipartite structures appear quite naturally in many situations [1]. To begin with, we have to establish a model yielding, in a natural way, matched power-law degree distributions.

Reference [1] proposes the following algorithm for the generation of bipartite networks with prescribed degree-distributions: create two sets of nodes (red and blue ones) with stubs, with the numbers of stubs per node distributed according to Eq. (2). Then connect the stubs of the nodes of the two types at random. The problem which arises when using this algorithm is that even if the two distributions are theoretically consistent, the sums of the degrees of the red and the blue nodes might be different due to statistical fluctuations. To resolve this problem additional nodes need be added to compensate for the difference. If the mathematical expectations of the degree-distributions are finite, then the theoretical matching condition is trivial, these expectations have to be the same (this is exactly the situation referred to in [1]). However, in the case of infinite mathematical expectations—at least on one side—such a condition is not known *a priori*. Therefore, we propose a different model of creating a bipartite network with automatically imposed matchmaking conditions, and then discuss the properties of the corresponding sample means.

Let us first explain the model in the case of a simple (rather than bipartite) network with a given degree distribution. Instead of concentrating on the nodes, we focus on the links and consider a network with exactly M nodes and N links—with M and N being consistent with the corresponding network average via $N=\nu M$. We assign each node j of the network an *attractiveness level* A_j, which has to be taken proportional to its desired degree [for example by taking it at random from the degree-distribution $P(n)$]. We further attach, at random, the ends of each link to two different nodes—the random attaching taking place with probability proportional to the nodes' attractiveness levels. The attractiveness model, thus, generates a realization from a *microcanonical* ensemble of networks, with given M and $N=\nu M$, while the model presented in [1] generates a realization from the corresponding *canonical* ensemble of similar networks (M is fixed, but N fluctuates).

The microcanonical nature of the attractiveness model is of great value when modeling situations, in which the global matching conditions have to be met—especially when modeling bipartite networks. Consider now a large bipartite population consisting of $2M \gg 1$ nodes—M "red" and M "blue"—and $N \gg 1$ links connecting the red and blue nodes. Each red node i has an attractiveness level f_i, and each blue node j has an attractiveness level g_j—the random attractiveness levels governed by one-sided Lévy distribution with respective exponents α_{red} and α_{blue}. Each link connects—on each red/blue side—to a single node, the probability of connecting being proportional to the attractiveness levels. Hence, the probabilities ϕ_i and γ_j that the ends of a given link are connected to the red node i and to the blue node j are given by

$$\phi_i = \frac{f_i}{\sum_{k=1}^{M} f_k}, \quad \gamma_j = \frac{g_j}{\sum_{k=1}^{M} g_k}. \tag{3}$$

III. SAMPLE MEAN DISTRIBUTION

Let us concentrate henceforth on the red side of the bipartite network. As a statistical sample we chose at random a set of $m < M$ of the red nodes. The probability that a given link is connected to one of the sample nodes is given by

$$p = \frac{\sum_{i=1}^{m} f_i}{\sum_{j=1}^{M} f_j} = \frac{\sum_{i=1}^{m} f_i}{\sum_{j=1}^{m} f_j + \sum_{j=m+1}^{M} f_j} = \frac{1}{1 + Y/X}, \tag{4}$$

where $X = \sum_{i=1}^{m} f_i$ and $Y = \sum_{j=m+1}^{M} f_j$. Note that X and Y are the sums of independent, identically distributed random variables governed by a one-sided Lévy distribution. Hence, X and Y are the independent one-sided Lévy random variables with exponent $\alpha = \alpha_{\mathrm{red}}$, and with respective scaling parameters $m^{1/\alpha}$ and $(M-m)^{1/\alpha}$. The value of p—the *Lévy random probability* [12]—is thus a random variable, which coincides in law with the random variable

$$z = \frac{1}{1 + (M/m - 1)^{1/\alpha} R}, \tag{5}$$

where R is quotient of two independent one-sided Lévy variables with exponent $\alpha = \alpha_{\mathrm{red}}$, and with a unit scaling parameter. Henceforth, we set the shorthand notation $x = (M/m - 1)^{1/\alpha}$. Note that the random variable z admits values in the unit interval (0,1). Moreover, we emphasize that even if the distributions of the attractiveness levels f_i deviate from the one-sided Lévy—but yet possess power-law asymptotics with exponent α—then the distribution of z for $m, M \gg 1$ is universal (in the sense of the corresponding central limit theorem [13]). Hence, our analysis does not depend on the precise form of distributions of the attractiveness levels f_i.

The probability density function (pdf) p_R of the quotient R is known [12]: Its Laplace transform is given by

$$\mathcal{L}(p_R(R)) = E_\alpha(-u^\alpha), \tag{6}$$

where $E_\alpha(\cdot)$ denotes the Mittag–Leffler function [13], and u denotes the Laplace variable. The derivation of Eq. (6) is based on the analytic methods of Chap. XIV in [13]. For the sake of completeness a variant of the full derivation is given in the Appendix. The asymptotic behavior of p_R for R large and small is obtained—via Tauberian theorems—from the asymptotics of the Mittag-Leffler function. For R large we have

$$p_R(R) \simeq \frac{1}{\Gamma^2(\alpha)} R^{-1-\alpha}, \tag{7}$$

($R \gg 1$), where $\Gamma(\cdot)$ denotes the Gamma function. Note that since R is a quotient of two identically distributed non-negative random variables, the distributions of R and $1/R$ are the same (i.e., the pdf of $\ln(R)$ is an even function), which fact will be repeatedly used in what follows.

Let $h=\Sigma_{i=1}^m n_i$ denote the number of "hits" in the sample—i.e., the overall number of links connected to the sampled nodes. Given the value of Lévy random probability z, the probability that h of N links hit the sample is given by the conditional binomial distribution

$$p(h|z) = \frac{N!}{h!(N-h)!} z^n (1-z)^{N-h}. \tag{8}$$

Hence, the unconditional probability distribution of h is given by

$$p_h(h) = \int_0^\infty \frac{N!}{h!(N-h)!} z^n (1-z)^{N-h} p_z(z) dz. \tag{9}$$

For $N \gg 1$ the binomial distribution is actually extremely narrow: Its standard deviation is much smaller than its mean, so that $[N!/h!(N-h)!]z^h(1-z)^{N-h} \approx \delta(h-Nz)$. Thus, we can take $h=Nz$; the distribution of h follows from that of the Lévy random probability z by change of variables. The distribution of the sample mean $\eta=h/m=Nz/m$, in turn, is given by

$$p_\eta(\eta) \approx \frac{m}{N} p_z\left(\eta \frac{m}{N}\right). \tag{10}$$

This fact can be proved by an explicit calculation of the generating function of the probability distribution p_h—evaluating it in the range $1 \ll h \ll N$ using Tauberian theorems.

Note that for $m \ll M$ the prefactor in front of R in the denominator of Eq. (5) is large, so that one can assume $z \simeq (M/m)^{-1/\alpha} R^{-1}$ for all z not too close to unity. Noting that the variables R and R^{-1} have the same distribution, we get that the distribution p_z of the Lévy random probability z practically follows the distribution of $(M/m)^{-1/\alpha}R$, and is a power law. Taking $m=1$ we arrive at the (continuous approximation for the) distribution of the number of links per node. The power-law spreads over the domain of $1 \ll h \ll N$ and is truncated for $h > N$—as it is evident from the fact that $p_z(z)$ vanishes for $z > 1$ [14]. The sample mean η is therefore a random variable, and the properties of its distribution are discussed below.

The mathematical expectation $\langle \eta \rangle$ of the sample mean η is *equal* to the network mean ν. Indeed,

$$\langle \eta \rangle = \left\langle \frac{h}{m} \right\rangle = \frac{N}{m} \langle z \rangle, \tag{11}$$

and

$$\langle z \rangle = \int_0^\infty z(R) p_R(R) dR = \int_0^\infty \frac{1}{1+xR} p_R(R) dR. \tag{12}$$

Noting that $1/(1+xR)=x^{-1}(1/x+R)^{-1}$ and substituting the integral representation

$$\frac{1}{1/x+R} = \int_0^\infty e^{-u/x} e^{-uR} du \tag{13}$$

into Eq. (12), while interchanging the order of integration, yields:

$$\langle z \rangle = \frac{1}{x} \int_0^\infty du e^{-u/x} \int_0^\infty dR e^{-uR} p_R(R) = \frac{1}{x} \int_0^\infty du e^{-u/x} E_\alpha(-u^\alpha). \tag{14}$$

The right-hand-side of Eq. (14) is the Laplace transform of the Mittag-Leffler function. This Laplace transform is known to be given by $\mathcal{L}\{E_\alpha(-u^\alpha)\}=s^{\alpha-1}/(s^\alpha+1)$ [13], and hence setting $s=1/x$ we arrive at

$$\langle z \rangle = \frac{1}{x^\alpha + 1}. \tag{15}$$

Finally, recalling that $x=(M/m-1)^{1/\alpha}$ we obtain that $\langle z \rangle = m/M$ and

$$\langle \eta \rangle = \frac{N}{m} \frac{m}{M} = \frac{N}{M} = \nu. \tag{16}$$

Thus, we have asserted that, on average, the sample mean η coincides with the network mean ν. In other words, the "mean sample mean" $\langle \eta \rangle$ equals the network mean ν.

The distribution of the sample mean η, however, is extremely broad—as seen from its variance. To calculate the variance we note that $\langle \eta^2 \rangle = (N^2/m^2)\langle z^2 \rangle$ and

$$\langle z^2 \rangle = \int_0^\infty z^2(R) p_R(R) dR = \int_0^\infty \frac{1}{(1+xR)^2} p_R(R) dR. \tag{17}$$

Using the fact that $(1+xR)^2 = \frac{d}{dx}(1/x+R)^{-1}$, and the integral representation given by Eq. (13), we obtain

$$\langle z^2 \rangle = \frac{d}{dx} \frac{x}{1+x^\alpha} = \frac{m^2}{M^2}\left[1 + (1-\alpha)\left(\frac{M}{m} - 1\right)\right]. \tag{18}$$

From this we conclude that the variance of η is given by

$$\sigma^2 = \langle \eta^2 \rangle - \langle \eta \rangle^2 = (1-\alpha)\frac{N^2}{M^2}\left(\frac{M}{m} - 1\right) \simeq (1-\alpha)\nu^2 \frac{M}{m}. \tag{19}$$

Hence, the standard deviation σ of η is of the order of magnitude $\sqrt{M/m} \gg 1$—i.e., far larger than its mean $\langle \eta \rangle$. Therefore, it is highly improbable to obtain an accurate estimate of the network mean ν from a sample with size m much smaller than the population size M.

Not only is the distribution of η extremely broad—it is also extremely skewed. As we now proceed to show, the median of η lays far below its mathematical expectation $\langle\eta\rangle=\nu$. And, finding values of η which are larger than its mathematical expectation $\langle\eta\rangle=\nu$ is highly improbable. Hence, a typical result of a statistical measurement of η will be much smaller than the network mean ν.

Since z and therefore η are monotonous functions of R, their medians follow from the median of R. The random variable R is a quotient of two identically distributed random variables—hence, the distribution of R is the same as the distribution of $1/R$. The random variable $\ln(R)$ is therefore symmetric and, consequently, its median is zero—implying, in turn, that the median of R is unity: $R_{1/2}=1$. Substituting the median $R_{1/2}=1$ into the expressions for z and $\eta=(N/m)z$ we obtain that the median $\eta_{1/2}$ of the sample mean η is given by:

$$\eta_{1/2}=\frac{N}{m}\frac{1}{1+x}\simeq\nu\left(\frac{m}{M}\right)^{1/\alpha-1} \tag{20}$$

[Eq. (20) holding for all $m\ll M$]. Clearly, the median $\eta_{1/2}$ is much smaller than the network mean ν.

Let us turn now to calculate the probability P_+ that the sample mean η is larger than the network mean ν—i.e., the probability of the random event $\{\eta>\nu\}$. Using the fact that for $m\ll M$ the distribution p_z of the Lévy random probability z practically follows the distribution of $(M/m)^{-1/\alpha}R$, we get

$$P_+=\int_{m/M}^{\infty}p_z(z)dz=\int_{R_0}^{\infty}p_R(R)dR \tag{21}$$

with $R_0=(M/m-1)^{1-1/\alpha}$. Further using Eq. (7), we obtain that

$$P_+\simeq\frac{1}{\alpha\Gamma^2(\alpha)}\left(\frac{M}{m}-1\right)^{\alpha-1} \tag{22}$$

[Eq. (22) holding for all $m\ll M$]. Clearly, the probability P_+ is very small.

Thus, in a the matchmaking problem, the sample means in different subpopulations not only fluctuate strongly, but also display a systematic skew. Moreover, for the same sample size, a population with smaller α—i.e., the one with a broader distribution—will typically show a smaller sample mean.

IV. AN EXAMPLE

It is instructive to consider an analytically solvable example: The Lévy-Smirnov case—corresponding to the exponent value $\alpha=0.5$. In this special case the corresponding distributions can be obtained exactly, without use of asymptotic realtions applied in the previous section, which may serve as the additional proof of the quality of approximations. Moreover, this example is of special interest due to the fact that the exponent $\alpha=0.5$ is not too far from the exponent $\alpha\approx0.6$ observed in the network of homosexual contacts [3].

The Lévy-Smirnov pdf of the attractiveness levels is given by

$$p(f)=\frac{1}{2\sqrt{\pi}f^{3/2}}\exp\left(-\frac{1}{4f}\right), \tag{23}$$

for which the probability density $p_R(R)$ is given by the inverse Laplace transform of $E_{1/2}(-u^{1/2})=e^u\,\mathrm{erfc}(\sqrt{u})$, with $\mathrm{erfc}(x)$ being the complementary error function. The inverse Laplace transform of this function is known to be

$$p_R(R)=\frac{1}{\pi}\frac{1}{\sqrt{R}(1+R)}, \tag{24}$$

i.e., $p_R(R)\simeq\pi^{-1}R^{-3/2}$ for R large. Now, performing the change of variables as prescribed by Eq. (5) one obtains

$$p_z(z)=\frac{1}{\pi}\sqrt{\frac{x}{z(1-z)}}\frac{1}{1+(x-1)z}. \tag{25}$$

This $p_z(z)$ behaves as $p_z(z)\simeq(1/\pi\sqrt{x})z^{-3/2}$ for $x\gg1$ and z not too close to unity.

The quantiles of the corresponding distributions can be calculated explicitly:

$$z_q=\frac{\tan^2(q\pi/2)}{x+\tan^2(q\pi/2)}, \tag{26}$$

implying, in particular, that for $x\gg1$ the sample mean η lays with probability 0.5 within the interval

$$(\sqrt{2}-1)^2\nu\frac{m}{M}<\eta<(\sqrt{2}+1)^2\nu\frac{m}{M}. \tag{27}$$

Namely, for $m\ll M$ the sample mean η is typically considerably smaller than the network mean ν. Only as $m\to M$ (and $x\to0$) does the sample mean converge to the network mean ν. On the other hand, the distribution over samples is very skewed, and the probability that the sample mean η be greater than the network mean ν is given by $P_+\simeq(2/\pi)\sqrt{m/M}$. Namely, P_+ is very small for sample sizes m which are considerably smaller than the population size M.

A simple numerical example elucidates the situation: For $\alpha=1/2$, $M=10^6$ and $m=1000$ the variance is approximately 45 times larger than the sample mean η, and the median is approximately 32 times smaller than the sample mean η. The discussion above also gives a possibility to roughly estimate the unknown network mean ν from the typically much smaller sample mean η. Such an extrapolation is given by Eq. (20) [or by Eq. (27)—in the special Lévy-Smirnov case]. Indeed, since for $\alpha=1/2$ the value of η lays with the 50% probability within the interval between $0.172\cdot10^{-3}\nu$ and $5.828\cdot10^{-3}\nu$, the estimate $\nu\sim10^3\eta$ will give a correct order of magnitude of ν. This estimate is not too bad, especially when taking into account that the corresponding confidence interval (being an order of magnitude for $p=50\%$) can be explicitly evaluated.

This "anomalous behavior" is typical in the cases of power-law distributions with divergent mathematical expectation: $P(n)\simeq n^{-1-\alpha}$ with $0<\alpha<1$. For exponents in the range $\alpha>1$ the sample mean shows no systematic fluctuations around the network mean. This can be anticipated by considering the limiting behavior of $\langle \eta \rangle = \nu$ and $\eta_{1/2} \to \nu$ for $\alpha \to 1$. The results for $\alpha > 1$ are: In the range $1 < \alpha < 2$ the fluctuations are Lévy distributed, and of the order $O(m^{1/\alpha - 1})$. And, in the range $\alpha > 2$ these fluctuations are Normally distributed, and of the order $O(1/\sqrt{m})$. The proofs of these results involve a rather different methodology and analysis than the ones employed here [15].

Note that the type of behavior following from the discussion of the matchmaking paradox—i.e., that a subpopulation with broader distribution (smaller α) typically yields a smaller sample mean—is essentially *opposite* to what is observed (the male population, the one with smaller α, gives rise to a larger sample mean). Therefore the reason for the sample-mean deviations in the reported number of heterosexual partners is not of purely statistical nature and should be looked for elsewhere [16]. Moreover, recent estimates for the exponent α tend to values slightly above unity [3,17], so that the simpler situation with equal means applies.

V. SUMMARY

In this paper we considered the problem of sampling from large scale-free networks, and the "matchmaking problem" of bipartite large scale-free networks—in the case of power-law degree-distributions with exponents α in the range $0 < \alpha < 1$. We have shown that the sample means—in case of sample sizes which are considerably smaller than the population size—fluctuate strongly and display systematic deviations from the network mean. These fluctuations and deviations were explicitly quantified, and an order-of-magnitude estimation of the network mean—if unknown—was obtained from the sample means.

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APPENDIX

We present a derivation of Eq. (6) based on a straightforward calculation. Let $R = x/y$ be the quotient of the two identically distributed, one-sided Lévy random variables x and y, whose probability density $L_\alpha(x)$ has a Laplace-transform

$$\mathcal{L}(L_\alpha(x)) = \int_0^\infty L_\alpha(x) e^{-ux} dx = \exp(-u^\alpha). \tag{28}$$

The Laplace transform of the probability density function of R, $p(u) = \int_0^\infty p(R) e^{-uR} dR$ can be represented as the mean value $p(u) = \langle e^{-uR} \rangle$ and therefore has an alternative representation,

$$\begin{aligned} p(u) &= \langle e^{-uR} \rangle_R \\ &= \left\langle \exp\left(-u\frac{x}{y}\right) \right\rangle_{x,y} \\ &= \int_0^\infty \int_0^\infty \exp\left(-u\frac{x}{y}\right) L_\alpha(x) L_\alpha(y) dx dy \\ &= \int_0^\infty \exp\left[-\left(\frac{u}{y}\right)^\alpha\right] L_\alpha(y) dy, \end{aligned} \tag{29}$$

where we make explicit use of Eq. (28) when integrating over x. Changing the variable of integration to $z = (u/y)^\alpha$ we can rewrite the last integral as

$$p(u) = \int_0^\infty \frac{1}{\alpha} \frac{u}{z^{1/\alpha + 1}} L_\alpha\left(\frac{u}{z^{1/\alpha}}\right) e^{-z} dz. \tag{30}$$

Let $F_\alpha(x) = \int_0^x L_\alpha(x') dx'$ be the cumulative distribution function of the one-sided Lévy distribution. Let us further introduce the auxiliary function

$$G_\alpha(u,z) = 1 - F_\alpha(u/z^{1/\alpha}). \tag{31}$$

Since

$$\frac{d}{dz} G_\alpha(u,z) = \frac{1}{\alpha} \frac{u}{z^{1/\alpha + 1}} L_\alpha\left(\frac{u}{z^{1/\alpha}}\right), \tag{32}$$

$p(u)$ can be represented as

$$p(u) = \int_0^\infty \frac{d}{dz} G_\alpha(u,z) e^{-z} dz = \int_0^\infty G_\alpha(u,z) e^{-z} dz, \tag{33}$$

where the second expression follows by integration by parts, and noting that $G(u,z) \to 0$ for $z \to 0$.

To evaluate this integral we first consider the Laplace transform of $p(u)$ in its u variable, $p(s) = \int_0^\infty p(u) e^{-su} du$, represent it as a double integral, $p(s) = \int_0^\infty \int_0^\infty G_\alpha(u,z) e^{-z} e^{-su} dz du$, interchange the sequence of integrations in u and z,

$$p(s) = \int_0^\infty dz e^{-z} \int_0^\infty G_\alpha(u,z) e^{-su} du, \tag{34}$$

and then perform the inverse Laplace transform in s.

The internal integral in Eq. (34) is easy to take by using the definition of G, Eq. (31), the standard formula for the Laplace-transform of the integral, and by applying Eq. (28), so that

$$\int_0^\infty G_\alpha(u,z) e^{-su} du = \frac{1}{s}(1 - e^{zs^\alpha}). \tag{35}$$

The second integral follows elementary and reads

$$p(s) = \int_0^\infty dz e^{-tz} \frac{1}{s}(1 - e^{zs^\alpha}) = \frac{s^{\alpha-1}}{s^\alpha + 1}. \tag{36}$$

The inverse Laplace transform of this expression is known to be a Mittag-Leffler function, [13],

$$\mathcal{L}^{-1}[p(s)] = p(u) = E_\alpha(-u^\alpha). \tag{37}$$

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[14] Although it might seem that for $1 \ll m \ll M$ the corresponding distribution of η (whose value is proportional to the one of the sum of the numbers of links of sampled nodes, which follow a power-law) might be approximated by a Lévy one, it is actually not the case. The reason is the truncation of the corresponding distributions due to finite size effects. As an example consider the explicit form corresponding to $\alpha=1/2$, Eq. (25), which differs from the corresponding Lévy distribution, Eq. (23) not only asymptotically, but also in its body.
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