

Manifestly covariant Jüttner distribution and equipartition theorem

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The relativistic equilibrium velocity distribution plays a key role in describing several high-energy and astrophysical effects. Recently, computer simulations favored Jüttner's as the relativistic generalization of Maxwell's distribution for $d=1,2,3$ spatial dimensions and pointed to an invariant temperature. In this work, we argue an invariant temperature naturally follows from manifest covariance. We present a derivation of the manifestly covariant Jüttner's distribution and equipartition theorem. The standard procedure to get the equilibrium distribution as a solution of the relativistic Boltzmann's equation, which holds for dilute gases, is here adopted. However, contrary to previous analysis, we use Cartesian coordinates in $d+1$ momentum space, with d spatial components. The use of the multiplication theorem of Bessel functions turns crucial to regain the known invariant form of Jüttner's distribution. Since equilibrium kinetic-theory results should agree with thermodynamics in the comoving frame to the gas the covariant pseudonorm of a vector entering the distribution can be identified with the reciprocal of temperature in such comoving frame. Then by combining the covariant statistical moments of Jüttner's distribution a form of the equipartition theorem is advanced which also accommodates the invariant comoving temperature and it contains, as a particular case, a previous not manifestly covariant form.

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I. INTRODUCTION

Incorporating the relativity principles in kinetic theory is crucial not only to understand the theoretical grounds in the description of relativistic many-particles systems [1,2] but to interpret relativistic high-energy experiments like those involving heavy-ion collisions [3] as well as phenomena in the astrophysical [4] and cosmological realms [5]. These include, for instance, the use of the relativistic Boltzmann equation to understand the thermal history of the universe [6,7] and the structure of the cosmic microwave background radiation spectrum associated to its interaction with hot electrons in galaxy clusters [8].

In the case of equilibrium the history of the relativistic analog of Maxwell's velocity distribution goes back to F. Jüttner, who in 1911 turned to relativity to consistently get rid of the contribution of particles with speeds exceeding that of light in vacuum, denoted by c , and which are contained in Maxwell's distribution. This was achieved upon replacement of the nonrelativistic energy of the free particles by its relativistic form and a maximum entropy principle thus yielding the so called one-particle Jüttner's distribution [9]

$$f = \frac{n}{4\pi kTm^2 c K_2(mc^2/kT)} e^{-\sqrt{\mathbf{p}^2 c^2 + m^2 c^4}/kT}, \quad (1)$$

which is written here in momentum rather than velocity space. m is the rest mass of any of the particles forming the

gas, k is Boltzmann's constant, and K_2 is the modified Bessel function of order two [10]. The remaining quantities: particle number density n , temperature T , and the relativistic spatial momentum of the particle \mathbf{p} , should be here understood as determined by an observer comoving with the gas as a whole. The mass shell condition, $(E/c)^2 - \mathbf{p}^2 = m^2 c^2$, is assumed along the sequel. In the nonrelativistic regime $|\mathbf{p}| \ll mc$, $kT \ll mc^2$, we have $f \approx f_{\text{Maxwell}}$, where f_{Maxwell} stands for Maxwell's velocity distribution. Thus particles's speeds beyond c are just an artifact of the nonrelativistic approximation.

Jüttner's distribution in form (1) can be disadvantageous in that it is not manifestly covariant, namely, it is not expressed in terms of Lorentz tensors which in turn explicitly show its behavior under Lorentz transformations to investigate its description for frames in relative motion. To make it manifestly covariant two key information pieces are needed: the transformation under change in frame of f and that of T . Both of them have been considered in the literature at large [11,12] and [1,13–15,17], for temperature and distribution function, respectively.

Upon multiplying (1) by the characteristic function for the box confining the gas so that $\chi_{\text{Box}}(\mathbf{x})=1$ for \mathbf{x} within the box and zero otherwise, the resulting distribution $\bar{f}(\mathbf{x}, \mathbf{p}) := \chi_{\text{Box}}(\mathbf{x})f(\mathbf{p})$ is defined on the single-particle phase space. An observer would thus determine $\bar{f}d^3x d^3p$ particles of the gas in a volume d^3x located at \mathbf{x} and having momentum \mathbf{p} within range d^3p . Moreover \bar{f} must be Lorentz invariant [13,16,17]. This is readily seen in the case of equilibrium [18] (for the nonequilibrium case see [13]): for a simple gas the number of particles \mathcal{N} is invariant and so $\mathcal{N} = \int d\mu \bar{f}$, with $d\mu = d^3x d^3p$, must be. Since $d\mu$ is a Lorentz invariant mea-

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sure due to a compensation between the transformations of the (mass shell) spatial momentum and space measure, \bar{f} must be so too. Now since χ_{Box} is invariant [48] then f is. Hence a manifestly covariant form of f amounts to manifest invariance and, in particular, it should involve the behavior of temperature under Lorentz transformations. The manifestly invariant Jüttner distribution was given long ago [2,14,15,19]. It was determined by introducing spherical coordinates on the pseudosphere associated to the mass shell in relativistic four-momentum space. This provided a rather elegant and fruitful treatment which however does not seem to have been fully appreciated in particular regarding temperature.

Alternatives to Eq. (1) have been proposed recently for which Lorentz covariance is incorporated in a different manner. In [20] a quantum mechanical approach including an invariant “historical time” was considered to derive a manifestly covariant Boltzmann equation with stationary solution implying a different ultrarelativistic mean energy-temperature relation whereas in [21–23] a maximum relative entropy principle was combined with an invariant group theoretical measure approach to obtain an equilibrium distribution. Both alternatives have been recently criticized in [24]. Moreover recent molecular dynamics simulations of a two component one dimensional simple relativistic gas showed agreement with Eq. (1) and temperature defined through an equipartition theorem was shown to be invariant under change in frame [25]. The study of the two-dimensional case along these lines has been reported in [26,27]. For three spatial dimensions Monte Carlo simulations have been considered favoring also Jüttner’s distribution [28]. Amusingly, as for temperature, such kind of analysis take us back to the long standing discussion of whether a moving object appears cooler [11,12,25,26].

In this work we shall obtain the manifestly invariant Jüttner distribution by adopting “Cartesian” rather than spherical coordinates in relativistic $(d+1)$ -momentum space [14,19]. Our approach can be considered as an alternative to such previous derivations of the Jüttner’s distribution and to others [1,29] in the sense that we avoid adopting any frame along the sequel which otherwise would raise the question on the Lorentz transformation of temperature. We only allude to the frame comoving with the gas at the end of the analysis to relate the kinetic description with thermodynamics in particular to identify temperature with the Lorentz invariant norm of a thermal vector. We once more obtain in this way the thermal four vector introduced long ago [2,14,15,19] which is formed by the product of the inverse comoving temperature with the four velocity of the gas as a whole. Also the lower dimensional cases recently studied [25–28] are contained in our results. Hence comoving temperature is seen to play a role analog to rest mass of a particle. The compatibility of such an interpretation is further investigated in relation with a manifestly covariant form of the equipartition theorem.

The paper is organized as follows. For the sake of completeness Sec. II briefly reviews the derivation of the Jüttner distribution as an equilibrium solution of the relativistic Boltzmann equation. This includes a brief description of the

two types of approaches: the manifestly covariant one adopting spherical coordinates on the mass shell pseudosphere in momentum space and the one adopting the comoving frame and Cartesian components. Section III provides the details of our analysis in which we use “Cartesian” coordinates in momentum space to get the manifestly invariant distribution. In particular the thermal Lorentz vector is here characterized. Its role in the relativistic covariant equipartition theorem is the subject of Sec. IV. Finally Sec. V summarizes our results. We also describe the behavior of a Planckian distribution when use is made of invariant temperature. The difficulties to define a noncomoving temperature for a gas of massive particles is also touched upon to further stress that invariant comoving temperature seems to be the only consistent possibility to define temperature according to the standard relativistic kinetic-theory framework.

Our conventions are the following. Space-time components of tensors are denoted by greek indices: $\mu, \nu, \dots = 0, 1, 2, 3$, the zero component referring to time whereas spatial components are denoted by latin indices $i, j, \dots = 1, 2, 3$. Einstein sum convention is assumed for both types of indices throughout. The Minkowski metric has components $\eta_{\mu\nu} = \text{diag}\{+, -, -, -\}$.

II. JÜTTNER DISTRIBUTION AND THE RELATIVISTIC BOLTZMANN EQUATION

Consider a simple dilute relativistic gas composed by identical particles of mass m . To each particle correspond space time and four-momentum coordinates, respectively, given by x^μ and p^μ . The evolution of the one-particle distribution function of the gas is governed by the relativistic Boltzmann equation[1,29,30].

$$p^\mu \frac{\partial \bar{f}}{\partial x^\mu} + m \frac{\partial \bar{f} \mathcal{K}^\mu}{\partial p^\mu} = \mathcal{C}[\bar{f}^*, \bar{f}]$$

$$\mathcal{C}[\bar{f}^*, \bar{f}] := \int (\bar{f}_1^* \bar{f}_1 - \bar{f}_1 \bar{f}_1^*) F, \sigma, d\Omega \frac{d^3 p_1}{p_1^0}. \quad (2)$$

Here $\mathcal{C}[\bar{f}^*, \bar{f}]$ is the so called collision term and $\bar{f}_1^* \equiv \bar{f}(\mathbf{x}, \mathbf{p}_1^*, t)$, * implying a quantity is evaluated after the collision. σ is the differential cross section of the scattering process whereas Ω is the solid angle. \mathcal{K}^μ denotes an external four-force while F is the so called invariant flux, $F = \sqrt{(p_1^\mu p_{1\mu})^2 - m^4 c^4}$, and $\frac{d^3 p_1}{p_1^0}$ the invariant measure on mass shell. We shall be interested in the case $\mathcal{K}^\mu = 0$.

The equilibrium state can be defined so that the entropy production vanishes [1,14,15,29]. In covariant form this means $\frac{\partial S^\mu}{\partial x^\mu} = 0$, with the entropy flux given by

$$S^\mu = -kc \int \frac{d^3 p}{p^0} p^\mu \bar{f} \ln(h^3 \bar{f}), \quad (3)$$

and h is a constant with dimensions to make the argument of the logarithm dimensionless. Zero entropy production requires $\bar{f}_1^* \bar{f}_1 = \bar{f}_1 \bar{f}_1^*$ and so the collision term $\mathcal{C}[\bar{f}^*, \bar{f}]$ becomes zero too. Such condition can be written as

$$\ln f^* + \ln f_1^* = \ln f + \ln f_1, \quad (4)$$

where χ_{Box} canceled out. Equation (4) is fulfilled by the *collisional invariants* [29] which for the present case become the four-momentum up to a constant (independent of p^μ), so that an f solving (4) takes on the form

$$\ln f = -[\Lambda(x) + \tilde{\Theta}_\mu(x)p^\mu] \Leftrightarrow f = \alpha(x)\exp[-\tilde{\Theta}_\mu(x)p^\mu]. \quad (5)$$

Here the negative sign has been introduced for later convenience, p^μ is the four-momentum of a particle of the gas and $\alpha = e^{-\Lambda}$. The equilibrium distribution function is fully obtained by requiring consistency between Eq. (5) and the left hand side of Eq. (2) equated to zero. Such substitution yields Λ independent of x^μ and $\partial_\nu \tilde{\Theta}_\mu(x) + \partial_\mu \tilde{\Theta}_\nu(x) = 0$, whose solution is $\tilde{\Theta}_\mu(x) = \omega_{\mu\nu}x^\nu + \Theta_\mu$ and $\omega_{\mu\nu} = -\omega_{\nu\mu}$. These correspond to the ten Killing vectors (six for $\omega_{\mu\nu}x^\nu$ and four for Θ_μ) under which Minkowski space time metric is invariant. They are associated with Lorentz transformations and space time translations, respectively [31]. As we will see below Θ_μ can be identified with the four velocity of the fluid as a whole and thus it inherits space time symmetries in the form of rigid motions: world lines of neighbor fluid elements would keep their separation whenever they lie along Killing vectors [14]. We shall restrict such motions to translations in the present work and so only Θ_μ is considered.

To determine α and Θ^μ one assumes that typical physical quantities are related to the statistical moments of the distribution. For instance the particle number density flux and the energy-momentum tensor

$$N^\mu = c \int \frac{d^3p}{p^0} p^\mu \bar{f}, \quad (6)$$

$$T^{\mu\nu} = c \int p^\mu p^\nu \bar{f} \frac{d^d p}{p^0}, \quad (7)$$

can be, respectively, written as

$$N^\mu = -\alpha \chi_{\text{Box}} c \frac{\partial \mathcal{I}}{\partial \Theta_\mu}, \quad (8)$$

$$T^{\mu\nu} = \alpha \chi_{\text{Box}} c \frac{\partial^2 \mathcal{I}}{\partial \Theta_\mu \partial \Theta_\nu}, \quad (9)$$

$$\mathcal{I} := \int \frac{d^d p}{p^0} e^{-\Theta_\alpha p^\alpha}, \quad (10)$$

from which the denomination of \mathcal{I} as generating functional suggests itself [14]. Having explicitly defined \mathcal{I} will allow to determine α . On the other hand to obtain Θ^μ one relates the kinetic theory form for thermodynamical quantities with equilibrium thermodynamic equations assumed to hold in a frame comoving with the gas as a whole. This will relate Θ_μ with a comoving temperature.

To evaluate (10) one can express it in spherical coordinates on the mass shell in momentum space [14,15,19], namely, since $p^0 = \sqrt{\mathbf{p}^2 + m^2 c^2}$, one has for the components of the particle's momentum

$$p^\mu = (mc \cosh \chi, mc \sinh \chi \sin \theta \cos \phi, mc \sinh \chi \sin \theta \sin \phi, mc \sinh \chi \cos \theta),$$

$$0 \leq \chi \leq \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad (11)$$

on the pseudosphere $p^\mu p_\mu = m^2 c^2$. Hence Eq. (10) becomes

$$\mathcal{I} = \int d\Omega^{(3)} d\chi \sinh^2 \chi e^{-mc\Theta \cosh \chi} = 4\pi \frac{K_1(mc\Theta)}{mc\Theta}, \quad (12)$$

$$\Theta^\mu \Theta_\mu = \Theta^2,$$

where $d\Omega^{(3)}$ is the element of solid angle in three spatial dimensions, K_1 is the modified Bessel function of order one [10], and Θ^μ has been chosen to lie along the $\chi=0$ axis. Upon use of the identity between Bessel functions $\frac{d}{du}[u^{-n}K_n(u)] = -u^{-n}K_{n+1}(u)$ to evaluate the components of the statistical moments (8) and (9) allows to relate their components in the comoving frame to get the equation of state for the gas $\mathcal{P} = \rho c^2 \Theta$, with \mathcal{P} the pressure and ρ the density of energy, both in the comoving frame. This led Israel [14] to identify $\Theta = \frac{c}{kT}$, with T the invariant comoving temperature.

A different way to deal with Eq. (10) is to consider that, being invariant, it can be calculated in a convenient frame, say one in which $\Theta^\mu = (\Theta^0, \mathbf{\Theta} = \mathbf{0})$. Here cartesian coordinates have been used in momentum space [1,29]. Noticing that (i) Θ_μ in Eq. (6) should be timelike to make the integral to converge and (ii) the only available timelike vector for the gas as a whole is its velocity \mathcal{U}^μ , one is led to propose $\Theta^\mu = \frac{\mathcal{U}^\mu}{kT}$, so that T is identified with a quantity in a frame in which \mathcal{U} , the spatial part of \mathcal{U}^μ , is zero. Such frame is the comoving frame to the gas. This approach in which one picks the comoving frame from scratch begs the question about which is the Lorentz transformation of the temperature.

It would be desirable to be able to combine the power of the spherical components of the manifestly covariant approach mentioned afore with the more intuitive and easier to handle Cartesian ones and still be able to investigate the behavior of temperature under Lorentz transformations. This is indeed possible and will be discussed in the next section.

III. MANIFESTLY COVARIANT GENERATING FUNCTIONAL WITH CARTESIAN COORDINATES

Since our analysis remains the same independently of the number of spatial dimensions we consider such general case at once. Hence unless otherwise stated, tensor indices run as follows: $\mu, \nu \dots = 0, 1, 2, \dots, d$, for d spatial dimensions.

A. Determination of α

Let us consider a frame noncomoving with the gas so that the spatial components of Θ^μ , namely $\mathbf{\Theta}$, are nonzero. In d -dimensional space the integral \mathcal{I} , Eq. (10), just requires to

change the measure from d^3p to $d^d p$. Now we adopt spherical coordinates only for the spatial part of the momentum and assume that $\Theta \cdot \mathbf{p} = |\Theta| |\mathbf{p}| \cos \vartheta_1$. We then have $d^d p = |\mathbf{p}|^{d-1} d|\mathbf{p}| d\Omega^{(d)}$, where

$$d\Omega^{(d)} = (\sin \vartheta_1)^{d-2} (\sin \vartheta_2)^{d-3} \cdots (\sin \vartheta_{d-2})^1 \\ \times d\vartheta_1 d\vartheta_2 \cdots d\vartheta_{d-2} d\varphi = \prod_{i=1}^{d-2} (\sin \vartheta_i)^{d-i-1} d\vartheta_i d\varphi,$$

where $0 < \varphi < 2\pi$ and $0 < \vartheta_i < \pi$ [32]. Thus \mathcal{I} can be written as

$$\mathcal{I} = S_{d-1} \int \frac{d|\mathbf{p}| |\mathbf{p}|^{d-1}}{p^0} (\sin \vartheta_1)^{d-2} d\vartheta_1 e^{-\Theta_0 p^0} e^{|\Theta| |\mathbf{p}| \cos \vartheta_1}. \quad (13)$$

Here $S_{d-1} = \frac{2\pi^{d/2}}{\Gamma(\frac{d-1}{2})}$ is the hypersurface of the $d-1$ -dimensional sphere [32] resulting from integrating over φ and $d\Omega^{(d-1)}$, which excludes ϑ_1 . To integrate over ϑ_1 it is better to use the series form of the spatial exponential $e^{|\Theta| |\mathbf{p}| \cos \vartheta_1} = \sum_{k=0}^{\infty} \frac{(|\Theta| |\mathbf{p}| \cos \vartheta_1)^k}{k!}$. In this way (13) becomes

$$\mathcal{I} = S_{d-1} \sum_{k=0}^{\infty} \frac{|\Theta|^k}{k!} \int \frac{|\mathbf{p}|^{d-1+k} d|\mathbf{p}|}{p^0} e^{-\Theta_0 p^0} \\ \times \int_0^\pi \sin \vartheta_1^{d-2} \cos \vartheta_1^k d\vartheta_1. \quad (14)$$

The remaining angular integral is nonzero only for $k = 2n$, $n=0, 1, 2, \dots$ and, moreover, it can be related to the beta functions $B\left(\frac{2n+1}{2}, \frac{d-1}{2}\right)$ [10] so that

$$\mathcal{I} = S_{d-1} \sum_{n=0}^{\infty} \frac{|\Theta|^{2n}}{(2n)!} B\left(\frac{2n+1}{2}, \frac{d-1}{2}\right) \\ \times \int e^{-\Theta_0 p^0} [p^{02} - m^2 c^2]^{(2n+d-2)/2} dp^0. \quad (15)$$

Where we have exchanged the independent variable $|\mathbf{p}|$ by p^0 by using the mass shell condition. The further change $y^0 = p^0/mc$ and $z_0 = mc\Theta_0$ together with the integral form $\int_1^\infty dx e^{-ax} (x^2-1)^{j-1/2} = \left(\frac{2}{a}\right)^j \frac{\Gamma(j+1/2)}{\Gamma(1/2)} K_j(a)$ for the modified Bessel function [10] leads to

$$\mathcal{I} = S_{d-1} \sum_{n=0}^{\infty} \frac{|\Theta|^{2n}}{(2n)!} B\left(\frac{2n+1}{2}, \frac{d-1}{2}\right) (mc)^{2n+d-1} \\ \times \left(\frac{2}{z_0}\right)^{(2n+d-1)/2} \frac{\Gamma\left(n + \frac{d}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} K_{(2n+d-1)/2}(z_0). \quad (16)$$

At this point we make the following considerations: (i) we use the known expression of the beta function in terms of the gamma functions [10] and then (ii) introduce $z^\mu = mc\Theta^\mu$ together with $\beta := |\mathbf{z}|/z^0$, $0 \leq \beta < 1$, the latter following from the fact that Θ^μ is timelike. Thereby Eq. (16) may be rewritten as

$$\mathcal{I} = 2^{(d+1)/2} (mc)^{d-1} \left(\frac{\pi}{z_0}\right)^{(d-1)/2} \sum_{n=0}^{\infty} \beta^{2n} z_0^n \frac{K_{n+[(d-1)/2]}(z_0)}{2^n n!}. \quad (17)$$

Here we arrive at a critical point from the technical perspective. The sum in Eq. (17) can be further reduced to a modified Bessel function upon using the multiplication theorem of the Bessel functions [33]

$$K_\nu(\lambda x) = \lambda^\nu \sum_{l=0}^{\infty} \frac{(-1)^l (\lambda^2 - 1)^l \left(\frac{1}{2}x\right)^l}{l!} K_{\nu+l}(x), \quad (18)$$

with $|\lambda^2 - 1| < 1$. This finally produces

$$\mathcal{I} = 2(mc)^{d-1} \left(2\frac{\pi}{z}\right)^{(d-1)/2} K_{(d-1)/2}(z). \quad (19)$$

where $z \equiv \sqrt{z^\mu z_\mu} = mc\Theta$. Formula (18) is essential to trade the Bessel series in Eq. (17) by a single modified Bessel function in Eq. (19). This becomes Eq. (12) when $d=3$. We see that \mathcal{I} is only function of the invariant z . Now we can obtain the relativistic particle number density flux from Eq. (8) producing

$$N^\mu = 2m^d c^{d+1} \alpha \chi_{\text{Box}} \left(2\frac{\pi}{z}\right)^{(d-1)/2} K_{(d+1)/2}(z) \frac{z^\mu}{z}. \quad (20)$$

Such equation can be solved for α giving

$$\alpha \chi_{\text{Box}} := \frac{N}{2c(mc)^d K_{(d+1)/2}(mc\Theta)} \left(\frac{mc\Theta}{2\pi}\right)^{(d-1)/2}. \quad (21)$$

It is clear from Eq. (20) that z^μ and N^μ point in the same direction. Let us notice that $N^\mu = N \mathcal{U}^\mu / c$, where $N = \sqrt{N^\mu N_\mu}$. In the comoving frame $\mathcal{U}^\mu \rightarrow (c, \mathbf{0})$, and then $N = nc$, where n is the number density in the comoving frame [see Eq. (1)]. Then we have

$$\Theta^\mu = \frac{\Theta}{c} \mathcal{U}^\mu. \quad (22)$$

The physical significance of Θ within our approach is discussed in the next subsection.

B. Θ^μ and the invariant temperature

Now we seek to identify Θ^μ with a thermodynamical quantity. Let us take as our starting point the Gibbs form of the second law of thermodynamics for a closed system [15,34], which we shall assume to hold in the comoving frame

$$\delta U = T \delta S - P \delta V. \quad (23)$$

Clearly we must relate the relativistic kinetic expressions like the energy-momentum tensor and entropy flux with internal energy, U , entropy, S , pressure \mathcal{P} and volume V appearing in Eq. (23). To begin with let us introduce the distribution function (5) and Eq. (21), in the expression for the energy-momentum tensor (9). This leads to

$$T^{\mu\nu} = -\frac{N}{\Theta} \left(\eta^{\mu\nu} - \frac{K^{\frac{d+3}{2}}(z)}{K^{\frac{d+1}{2}}(z)} \frac{\Theta^\mu \Theta^\nu}{\Theta} mc \right). \quad (24)$$

The comoving pressure can be obtained from the energy-momentum tensor in d -dimensions as

$$\mathcal{P} \equiv -\frac{1}{d} \Delta_{\mu\nu} T^{\mu\nu} = \frac{N}{\Theta}, \quad (25)$$

where the projector is $\Delta^{\mu\nu} \equiv \eta^{\mu\nu} - \mathcal{U}^\mu \mathcal{U}^\nu c^{-2}$. The corresponding d dimensional entropy flow (3) can be conveniently re-expressed as

$$S^\mu = -k[\ln(\alpha h^d) N^\mu - T^{\mu\nu} \Theta_\nu]. \quad (26)$$

It is worth stressing that the quantities N^μ , $T^{\mu\nu}$, and S^μ are densities, and therefore do not depend on the size of the system. It is however rewarding to include the fact that the gas is inside a box and to do so we make use of the characteristic function χ_{Box} defined in the introduction. In particular integrating over d spatial dimensions

$$\mathcal{N} = c^{-1} \int N^\mu d\sigma_\mu, \quad (27)$$

$$\mathcal{S} = c^{-1} \int S^\mu d\sigma_\mu, \quad (28)$$

$$\mathcal{G}^\mu = c^{-1} \int T^{\mu\nu} d\sigma_\nu. \quad (29)$$

In the previous expressions (27)–(29), $d\sigma_\mu = n_\mu d^d x$, with n_μ a unit normal to the spacelike hypersurface, and therefore a timelike vector. Recall that the only dependence on x^μ is through the characteristic function χ_{Box} , which is implicit in the definitions of S^μ , N^μ , and $T^{\mu\nu}$, Eqs. (3), (8), and (9), respectively. To calculate the above spatial integrals we take a hyper plane comoving with the gas so that its unit normal is given by $n^\mu = \frac{\mathcal{U}^\mu}{c} = \frac{\Theta^\mu}{\Theta}$. Since \mathcal{U}^μ is an intrinsic property of the system this choice provides a truly covariant form of such integrals [35]. Notice that for a different choice such as $n^\mu = \delta_0^\mu$ that corresponds to the case of a fixed time hyperplane, the integral of the momentum Eq. (29) would be only an apparent rather than true Lorentz vector [49]. Therefore, the number of particles \mathcal{N} and the entropy \mathcal{S} are obviously Lorentz scalars while \mathcal{G}^μ , the relativistic momentum of the gas, is a Lorentz vector.

Integrating (26) with use of Eqs. (27)–(29) yields

$$\mathcal{S} = -k[\mathcal{N} \ln(\alpha h^d) - \Theta_\mu \mathcal{G}^\mu], \quad (30)$$

Now we can make contact with Eq. (23) by going to the comoving frame. Consider Eq. (30) for which such process means, in light of Eq. (22), $\Theta_\mu \mathcal{G}^\mu \rightarrow \Theta_0 \mathcal{G}^0 = \Theta \mathcal{G}^0$. The differential of the entropy, Eq. (30), in thermodynamic space (Θ constant) becomes hereby

$$\delta \mathcal{S} = k \Theta \left[\delta \mathcal{G}^0 - \frac{\mathcal{N}}{c} \frac{\delta N}{N^2} \right],$$

$$= \frac{k \Theta}{c} [\delta(c \mathcal{G}^0) + \mathcal{P} \delta V], \quad (31)$$

where use has been made of Eqs. (21) and (25) and the relation $\delta V = -\frac{\mathcal{N} \delta N}{N^2}$. Comparison of Eq. (23) with Eq. (31) gives rise to the identification

$$\Theta = \frac{c}{kT}. \quad (32)$$

with T the comoving temperature by interpreting $c \mathcal{G}^0$ as the relativistic generalization of internal energy U ; this clearly holds in the low speed regime in which $c \mathcal{G}^0 \approx \mathcal{N} m c^2 + U_{\text{non-rel}}$, with $U_{\text{non-rel}}$ the usual nonrelativistic internal energy of the gas. Thus Eqs. (21), (22), and (32) complete our analysis of the manifestly covariant distribution function which is expressed in terms of invariant quantities, $N, \Theta, m, c, \Theta^\mu p_\mu$, in the fashion

$$f(p) = \frac{N}{2c(m c)^d K^{\frac{d+1}{2}}(m c \Theta)} \left(\frac{m c \Theta}{2\pi} \right)^{(d-1)/2} \exp(-\Theta_\mu p^\mu). \quad (33)$$

Equation (33) for $d=1$ and 2 reduces to those in [25–27], respectively. Equation (1) is obtained from Eq. (33) by considering $d=3$ and using the comoving frame to the gas. Moreover, Eq. (33) coincide with the obtained in [23] for arbitrary d .

It should be stressed however that in deriving Eq. (33) no assumption was made in regard to the Lorentz transformation of temperature. The latter came from the requirement to reproduce equilibrium thermodynamics in the comoving frame.

IV. COVARIANT EQUIPARTITION THEOREM

Equipartition theorems are usually considered to connect temperature with averages of say kinetic energy in the non-relativistic case and certain functions of momenta in the relativistic case [12,38]. Now, as we have shown in the previous section a manifestly covariant approach to relativistic equilibrium distribution function unveils the convenience of using an invariant temperature which is the one associated with the comoving frame of the gas. Thus it is of interest to investigate the role an invariant temperature plays within a manifestly covariant equipartition theorem.

The nonmanifestly covariant but relativistic equipartition theorem seems to have been first considered by Tolman [38] and later by others [39]. Its use as a criterion to determine the Lorentz transformation of temperature was criticized by Landsberg [12] who stressed that it can accommodate both an invariant as well as a transforming temperature under a change in frame. Recently Cubero *et al.* performed numerical simulations indicating the existence of an invariant temperature [25] on the basis of the relativistic equipartition theorem of [12] thus hinting at the invariant temperature included in Landsberg's analysis. In this section, we shall provide a manifestly covariant form of the equipartition theorem which not only contains an invariant temperature but in-

cludes Landsberg's version. Moreover our form of the theorem is neatly expressed in terms of the total momentum of the gas.

Let us start by reexpressing the generating functional \mathcal{I} (10) in $(d+1)$ -dimensional momentum space [1]

$$\mathcal{I} = 2 \int e^{-\Theta_\mu p^\mu} \delta(p_\sigma p^\sigma - m^2 c^2) H(p^0) d^{d+1} p, \quad (34)$$

where $H(p^0)$ is the Heaviside function that restricts (34) to positive energies. The mass shell condition is accounted for by Dirac's delta.

By covariantly extending the argument in [38], we next integrate by parts $d+1$ times one for each p^μ and discarding the boundary terms at infinity we obtain

$$\mathcal{I} = -\frac{2}{d+1} \int p^\nu \frac{\partial}{\partial p^\nu} [e^{-\Theta_\mu p^\mu} \delta(p_\sigma p^\sigma - m^2 c^2)] H(p^0) d^{d+1} p. \quad (35)$$

To integrate over p^0 we apply the properties of the Dirac delta which finally yields

$$\mathcal{I} = \frac{1}{d} \int e^{-\Theta_\mu p^\mu} \left[\left(\frac{|\mathbf{p}|}{p^0} \right)^2 + \Theta_\mu \frac{\partial p^\mu}{\partial p^i} p^i \right] \frac{d^d p}{p^0}. \quad (36)$$

Remarkably, notwithstanding the second term in Eq. (36) containing the spatial components p_i it is Lorentz invariant overall.

To connect with the physical framework Eq. (36) is plugged back into Eqs. (8) and (9) to yield

$$N^\alpha = \frac{c}{d} \int \frac{d^d p}{p^0} f \left\{ p^\alpha \left[\left(\frac{|\mathbf{p}|}{p^0} \right)^2 + p^i \frac{\partial p^\mu}{\partial p^i} \Theta_\mu \right] - p^i \frac{\partial p^\alpha}{\partial p^i} \right\}, \quad (37)$$

$$T^{\alpha\beta} = \frac{c}{d} \int \frac{d^d p}{p^0} f \left\{ p^\alpha p^\beta \left[\left(\frac{|\mathbf{p}|}{p^0} \right)^2 + p^i \frac{\partial p^\mu}{\partial p^i} \Theta_\mu \right] - p^i \frac{\partial p^\alpha p^\beta}{\partial p^i} \right\}. \quad (38)$$

A further spatial integration of Eqs. (37) and (38) according to Eqs. (27) and (29) gives rise to

$$d = \Theta_\mu \left\langle \left\langle p^i \frac{\partial p^\mu}{\partial p^i} \right\rangle \right\rangle, \quad (39)$$

$$\mathcal{G}^\alpha = \frac{\mathcal{N}}{d} \left[\Theta_\mu \left\langle \left\langle p^i \frac{\partial p^\mu}{\partial p^i} p^\alpha \right\rangle \right\rangle - \left\langle \left\langle p^i \frac{\partial p^\alpha}{\partial p^i} \right\rangle \right\rangle \right]. \quad (40)$$

where $\langle \langle \cdot \rangle \rangle := \frac{1}{\mathcal{N}} \int \cdot f d^d \mu$ has been used. It is worth stressing that Eq. (39) is just the manifestly covariant form of the equipartition theorem corresponding to that advanced by Tolman and Landsberg [12,38], respectively, expressed using the invariant comoving temperature $T = c/k\Theta$, Eq. (32). A comment regarding covariance of Eq. (39) is here in order. Although it contains the spatial sum $p^i \frac{\partial}{\partial p^i}$, we arrive to it from the manifestly covariant Eq. (34). This is analogous to recognizing the invariance of $\frac{d^d p}{p^0}$. Clearly the argument holds also for both terms in the right-hand side of Eq. (40).

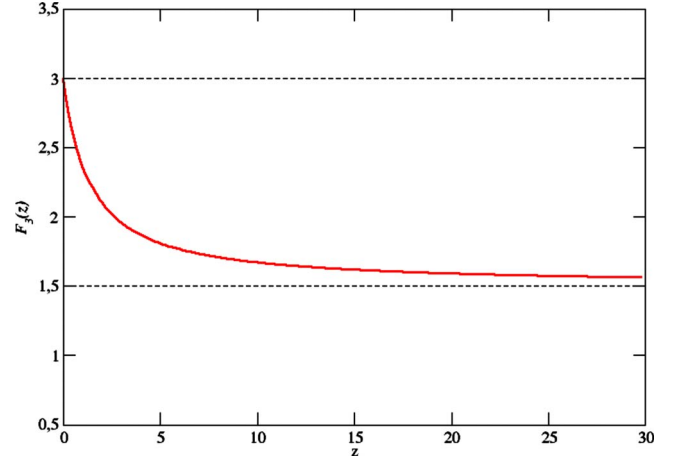


FIG. 1. (Color online) The graph shows $\mathcal{F}_d(z)$ vs $z = mc^2/kT$ for $d=3$. $\mathcal{F}_3(z)$ corresponds to the quotient of the average of the relativistic kinetic energy per particle and the comoving temperature. We see that for $z \gg 1$ and $z \ll 1$ we get the appropriate limits, while for intermediate values of z , the ratio is a fraction between $3 < \mathcal{F}_3(z) < 3/2$.

Observing that Eqs. (39) and (40) contain a common term suggests that the relativistic momentum can be made to enter the covariant equipartition theorem. To proceed further we need to determine the first term in Eq. (40). First we combine Eqs. (24) and (29) to obtain the explicit form for the momentum

$$\mathcal{G}^\alpha = \mathcal{N} \frac{\Theta^\alpha}{\Theta^2} \left[mc \Theta \frac{K^{d+3}(mc\Theta)}{K^{d+1}(mc\Theta)} - 1 \right]. \quad (41)$$

By equating the projections $\Theta_\alpha \mathcal{G}^\alpha$ of Eqs. (40) and (41) we identify

$$\frac{\Theta_\alpha \Theta_\mu}{d} \left\langle \left\langle p^i \frac{\partial p^\mu}{\partial p^i} p^\alpha \right\rangle \right\rangle = \frac{K^{d+3}(mc\Theta)}{K^{d+1}(mc\Theta)} mc \Theta. \quad (42)$$

From here we finally arrive at

$$\frac{\Theta_\mu \mathcal{G}^\mu}{\mathcal{N}} - z = \mathcal{F}_d(z), \quad (43)$$

$$\mathcal{F}_d(z) := z \frac{K^{d+3}(z)}{K^{d+1}(z)} - 1 - z. \quad (44)$$

Amusingly just in the same way as the energy of a particle having momentum p_{particle}^μ is determined by an observer having velocity U_{obs}^ν is given by $E_{\text{obs}} = p_{\text{particle}}^\nu U_{\text{obs}}^\nu$ we interpret $\frac{1}{\Theta} \Theta_\mu \mathcal{G}^\mu$ as the energy of the gas determined by the comoving observer.

$\mathcal{F}_d(z)$ reduces to the usual values of the equipartition theorem in the limiting cases (see Fig. 1).

$$\mathcal{F}_d(z) = \begin{cases} \frac{d}{2}, & z \gg 1 \\ d, & z \ll 1. \end{cases} \quad (45)$$

Notice that although a relativistic equipartition theorem had been considered before by other authors [19,40] their approach was not manifestly covariant. Their work and ours coincide in terms of the function \mathcal{F}_d (44), in particular the behavior in the nonrelativistic as well as in the ultrarelativistic limits. Remarkably, recent numerical calculations adopting Monte Carlo methods [28] confirm \mathcal{F}_d as giving the relativistic kinetic energy divided by kT .

V. DISCUSSION

The interest in incorporating the relativity principles into kinetic theory goes beyond the theoretical foundations: Actual observations and experiments like for instance in high-energy physics [3], astrophysics [4], and cosmology [5] require a description of relativistic many-particles systems in terms of, say, Boltzmann's equation and the corresponding equilibrium distribution [6–8]. Recently, Cubero *et al.* [25] (see also references there) have developed numerical simulations based on molecular dynamics pointing to Jüttner's as the equilibrium distribution in agreement with their numerical analysis, as opposed to other proposals in the literature, for one and possibly other number of spatial dimensions. This was further confirmed for two [26,27] and three [28] spatial dimensions, the latter adopting Monte Carlo simulations instead. In [25], the old problem regarding the relativistic transformation of temperature [11] was also considered in connection with a previous relativistic version of the equipartition theorem proposed by Landsberg [12]. Based upon such theorem Cubero *et al.* determined, within their approach, that temperature should possess an invariant character. It is worth mentioning that such analysis were not formulated in a manifestly covariant form and indeed some further insight was needed to interpret the contributions entering such theorem as well as temperature.

In this work we have obtained a derivation of the manifestly invariant Jüttner's relativistic distribution function (33). This is based on Cartesian coordinates in $(d+1)$ -momentum space (d spatial dimensions) in contrast with the known results developed using spherical coordinates. This was made possible by the use of the multiplication theorem for Bessel's functions (18) which simplified the treatment of a series involving Bessel functions [33]. In this approach no assumption is made *a priori* of any specific relativistic character for temperature. The latter appears through the invariant norm, Θ , of a four vector, Θ^μ (22), and it is invariant just for the same reason a point particle's rest mass is. Indeed because of the assumption that relativistic equilibrium kinetic theory should agree with standard thermodynamics for an observer comoving with the system under study the pseudonorm becomes $\Theta = c/kT$, Eq. (32), T being the comoving temperature of the gas. Finally we have developed a manifestly covariant equipartition theorem, Eq. (43), in which the average of the energy-momentum of the gas as determined by the comoving observer is given by a

function \mathcal{F}_d , Eq. (44) of the invariant temperature. Indeed this is analogous to the case of a point particle for which the energy is obtained by projecting momentum along the four velocity of the observer. Here we have the thermal vector $\Theta^\mu = \frac{c}{kT} \mathcal{U}^\mu$, with \mathcal{U}^μ the four velocity of the gas as whole which defines a comoving observer reading the invariant temperature T . A further comment is here in order regarding the difference between our approach and previous ones. While previous versions of the equipartition theorem [12,38] relate temperature with a peculiar combination of relativistic quantities [see for example Eq. (39)], here we interpret that combination as included in Eq. (43) which relates the invariant temperature with the averaged relativistic momentum, G^μ .

In a nutshell the manifestly covariant form of Jüttner's distribution leads naturally to consider the invariant comoving temperature to characterize the equilibrium regime. As an illustrative example of how this result applies to known cases let us consider the case of black body radiation. Let us recall the standard analysis of this problem [5,7,18]: the Lorentz invariance of $\frac{I_\nu}{\nu^3}$, containing the specific intensity I_ν and the frequency ν of the photons, implies the invariance of the Planckian distribution

$$\frac{I_\nu}{2h\nu^3} = \frac{1}{e^{h\nu/kT} - 1}. \quad (46)$$

Indeed for two frames in relative motion one should have

$$\frac{h\nu}{kT} = \frac{h\nu'}{kT'}, \quad (47)$$

where primed quantities refers to an observer that moves with respect to radiation and, in particular, T' is suggested as the noncomoving temperature. Since the photons suffer a Doppler shift

$$\frac{\nu}{\nu'} = \gamma(1 - \beta \cos \delta), \quad (48)$$

where δ is the angle between the momentum of the photon and the gas velocity whereas β is the ratio between the gas speed and c . The quantity T' takes the anisotropic form

$$T' = \frac{T}{\gamma(1 - \beta \cos \delta)}. \quad (49)$$

Notice that we could alternatively write a manifestly invariant form for Eq. (46) as $(e^{\Theta_\mu p^\mu} - 1)^{-1}$. In this way one can focus in the invariant product $\Theta_\mu p^\mu$. By evaluating it in the two frames mentioned above and using that for photons $p'^0 = |\mathbf{p}'|$

$$\frac{h\nu}{kT} = \Theta'^0 p'^0 (1 - \beta \cos \delta). \quad (50)$$

Considering the invariant comoving temperature T the components Θ'^μ become $\frac{\gamma}{kT}(c, \mathcal{U})$ and thus Eq. (50) reduces to Eq. (48). This shows the consistency of adopting such invariant comoving temperature, through Θ^μ , in the description of the black body radiation without resorting to T' , Eq. (49). The latter can be regarded only as an auxiliary quantity for

the following reasons. First, actual observations of the cosmic microwave background radiation (CMBR) [41] reveal there is a frame in which it presents the black body structure. Measurements however involve brightness depending on frequency rather than a noncomoving temperature (49). Second, in the case of massive particles one finds an obstacle to handle (49) (see for instance [42]). For massive particles we would have

$$T' = \frac{T}{\gamma(1 - \beta_p \beta \cos \delta)},$$

$$\beta_p = \frac{|\mathbf{p}|}{p^0}, \quad (51)$$

which makes no sense due to the dependence on the particles' momentum. However adopting an invariant comoving temperature is viable just for the same reason that the case for photons described above works.

Our work then adds to recent claims elaborated on the basis of an Unruh-DeWitt detector [43] pointing to the impossibility to have a relativistic transformation of temperature [44]. In any case all these results reinforce the idea that temperature makes undisputable sense in the comoving frame.

There are several possible extensions of the present work which could be of interest. One possibility is to extend the analysis in the present work to the case of relativistic Fokker-

Planck equation, say along the lines of [45]. In other direction, we have seen that in order to solve Boltzmann's equation for the equilibrium distribution the concept of rigid motions appeared. They can be seen related to the symmetries of Minkowski space time that we have here considered and specifically to the existence of Killing vectors. Although symmetric curved space times have been studied previously for instance in [2,6] there seems to be no attempt to incorporate the analog of Noether's theorem in regard to the kinetic theory in particular noticing that the latter can be linked to relativistic quantum field theory [1,24]. This would be particularly useful in the manifestly covariant approach. Generally covariant approaches to statistical mechanics have also been studied to investigate many particle systems in general relativistic theories [46]. It is a challenge to incorporate similar ideas to kinetic theory. Furthermore the very notion of space time has been considered in these terms [47].

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- [48] Consider two frames S, S' with the gas container lying on S . For relative speed v , $\gamma = 1/\sqrt{1-v^2/c^2}$, Lorentz volume contraction $V' = \gamma^{-1}V$ yields $\int d^3x' \chi'_{\text{Box}}(x') = \int d^3x \chi_{\text{Box}}(x)$, $d^3x' = \gamma^{-1}d^3x$ thus proving invariance of χ_{Box} .
- [49] In a classical paper by van Kampen [36], it is stated that the energy and momentum of the gas do not form a true Lorentz vector because the gas is not a closed system. Only the total momentum, P_{Total}^μ , of the composite system formed by the gas and the box containing it, is a true Lorentz vector. To really grasp the meaning of such statement it is rewarding to follow Rohrlich's analysis [37]: he shows that P_{Total}^μ can be expressed as the sum of either apparent vectors or true vectors. The notion of apparent or true is linked to the choice of the hypersurface normal in the integral (29). By choosing $n^\mu = \delta_0^\mu$ one gets $P_{\text{Total}}^\mu = \mathcal{G}^{(\mu)} + \Pi_{\text{Box}}^{(\mu)}$, where $\mathcal{G}^{(\mu)}$ and $\Pi_{\text{Box}}^{(\mu)}$ are the contributions of the gas and the box, respectively, and the parenthesis in (μ) indicate such objects are not true Lorentz vectors although the sum is. Explicitly we have

$$\mathcal{G}^{(\mu)} = \frac{\mathcal{N}}{\Theta} \left[mc \frac{K_{(d+3)/2}(mc\Theta)}{K_{(d+1)/2}(mc\Theta)} \frac{\Theta^0}{\Theta} \Theta^\mu - \eta^{\mu 0} \right].$$

Hence in order to get the true vector P_{Total}^μ , starting from $\mathcal{G}^{(\mu)}$, the contribution of the box is essential. This is the contents of van Kampen's statement in [36]. Moreover, although the box is necessary to explain the equilibrium of the system, it only plays a role in the vector character of the relativistic momentum when using apparent vectors. For instance, the choice $n^\mu = \frac{U^\mu}{c}$ makes \mathcal{G}^μ a true vector without reference to the box. Of course it is possible to express $P_{\text{Total}}^\mu = \mathcal{G}^\mu + \Pi_{\text{Box}}^\mu$, where both terms on the right hand side are true Lorentz vectors.