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Dynamics after a sweep through a quantum critical point

Frank Pollmann,[1] Subroto Mukerjee,[1,2,3] Andrew G. Green,[4] and Joel E. Moore[1,2]
[1]*Department of Physics, University of California, Berkeley, California 94720, USA*
[2]*Materials Science Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*
[3]*Department of Physics, Indian Institute of Science, Bangalore 560012, India*
[4]*School of Physics and Astronomy, University of St. Andrews, North Haugh, St. Andrews, KY16 9SS Scotland, United Kingdom*
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The coherent quantum evolution of a one-dimensional many-particle system after slowly sweeping the Hamiltonian through a critical point is studied using a generalized quantum Ising model containing both integrable and nonintegrable regimes. It is known from previous work that universal power laws of the sweep rate appear in such quantities as the mean number of excitations created by the sweep. Several other phenomena are found that are not reflected by such averages: there are two different scaling behaviors of the entanglement entropy and a relaxation that is power law in time rather than exponential. The final state of evolution after the quench is not characterized by any effective temperature, and the Loschmidt echo converges algebraically for long times, with cusplike singularities in the integrable case that are dynamically broadened by nonintegrable perturbations.

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A many-particle quantum system evolving at zero temperature can demonstrate various forms of approach to equilibrium even with no loss of phase coherence. This phenomenon has been studied in most detail experimentally [1] and theoretically [2–8] for systems prepared by a quantum quench across a phase transition. A system is prepared in the ground state for certain parameter values, which are then rapidly changed to values for which the ground state (GS) is in a different phase. Ultracold atomic systems are especially valuable for experiments in this area because they can be treated as closed quantum systems on rather long time scales compared to the basic dynamical time scales of the system. Two basic notions in the literature are that many properties equilibrate after the quench exponentially in time and that the system thermalizes (i.e., the final state can be described by an effective temperature). The notion of thermalization is generally not a well-defined one for integrable systems and only nonintegrable systems in the thermodynamic limit can be described by an effective temperature after a quench [9]. Nonintegrability alone may not be a sufficient condition for thermalization in finite systems [10].

This Rapid Communication concerns the coherent zero-temperature dynamics of states prepared through a different process that has also attracted recent attention [11–15]: rather than quenching instantly across a quantum phase transition, the Hamiltonian is changed smoothly across the transition at a constant rate Γ. For a second-order transition (critical point), various averaged physical quantities such as the excitation density and energy show power laws in rate Γ as $\Gamma \to 0$, with exponents determined by the quantum critical point's universal physics. The system evolves after such a sweep to a steady state for some quantities, such as the Loschmidt echo defined below, but its energy distribution remains nonthermal. The main result of this Rapid Communication, using a generalized quantum Ising model as an example, is that many features of the resulting evolution including the approach to equilibrium differ from the case of an instant quench and are not captured by the simple averaged quantities studied previously. Integrable and nonintegrable systems evolve differently and our results suggest a sharp dynamical probe of these differences.

We consider a system with a tunable parameter g that becomes critical at a value g_c and study the dynamics as the Hamiltonian is swept from one side of the critical point to the other. A particular example of such a system is the generalized quantum Ising model in a field described by the Hamiltonian

$$H = J\sum_i [\sigma_i^x - \sigma_i^z\sigma_{i+1}^z + g(\cos\phi\sigma_i^x + \sin\phi\sigma_i^z)]. \qquad (1)$$

The system is critical at $g=g_c=0$ for any value of ϕ. For $\phi=0$, the model Hamiltonian is the Ising model in a purely transverse field [the "transverse Ising (TI) model"] which is integrable. For any $\phi \neq 0$, the lattice model is no longer integrable (except at the critical point where $g=0$). The correlation length goes as $\xi(g) \sim |g-g_c|^{-\nu}$ and the gap as $\Delta(g) \sim \xi^{-z} \sim |g-g_c|^{z\nu}$, close to the critical point, where z is the dynamical exponent and ν the correlation length exponent (there are two values of ν depending on ϕ, $\nu=1$ for $\phi=0,\pi$ and $\nu=8/15$ otherwise). At each g, the system has a GS $|\psi_0(g)\rangle$ with energy $\epsilon_0(g)$ and excited states $|\psi_n(g)\rangle$; the lowest excited state $|\psi_1(g)\rangle$ has energy $\Delta(g)$ (Fig. 1).

We study the dynamics of energy, entanglement entropy

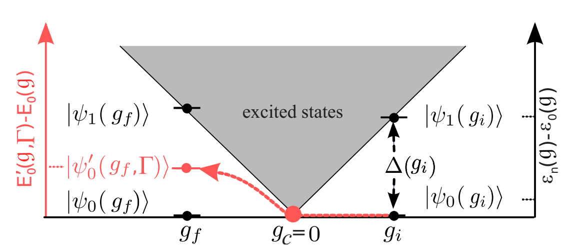

FIG. 1. (Color online) The panel shows schematically the energy levels $\epsilon_n(g)$ and gaps of eigenstates $|\psi_n(g)\rangle$. The energy density $E_0'(g,\Gamma)$ of a wave function $|\psi_0'(g,\Gamma)\rangle$ resulting from a linear sweep at a constant rate Γ and the GS energy density $E_0(g)$ are shown in red (light gray).

(EE), and wave-function overlap during and after an adiabatic sweep across a quantum critical point. We derive several analytic expressions and compare to numerical results for a sweep through the critical point of Hamiltonian (1). We assume that the parameter g depends on time t as $g_t = g_i - \Gamma t/J$, where g_i is some initial value of g and $\hbar=1$, until some final value g_f is reached, after which the Hamiltonian is constant for some "wait period." We are interested in the adiabatic limit $\Gamma \to 0$. For the numerical calculations, we use the recently introduced *infinite time-evolving block decimation* (iTEBD) algorithm [16]. This method uses variational wave functions based on matrix product states (MPSs) and exploits translational invariance for efficient simulation of infinite one-dimensional (1D) systems. The method is formally working with infinite systems and the errors result from finite dimensional matrices which are used to represent the state [17,18]. Away from a critical point, the EE of the *exact* wave function is finite, and the iTEBD algorithm becomes very accurate. As a critical point is neared, the EE of the exact wave function increases and the dimension of the matrices has to be chosen increasingly large to represent the state accurately. We estimate the error resulting from the finite matrices by two different methods. We calculate the truncation error, i.e., the truncated weight of the wave function at a time step, which gives an upper bound for the truncation effects on local expectation values ($<10^{-8}$ for all simulations presented in this Rapid Communication). In addition we checked the dependence of the measured observables on the matrix dimension and the time steps used in the simulations.

The expected energy density $E_0'(g_t,\Gamma)$ of $|\psi_0'(g_t,\Gamma)\rangle$ at a time t when the system has crossed the critical point is a natural quantity to consider. We use a scaling relation for slow sweeps in this model derived in Refs. [13,14]. Let the system be in the state $|\psi_0(g_i)\rangle$ at $t=0$ and change g_t linearly with rate $\Gamma \to 0$. While g_t is on the same side of the critical point as g_i, the system will be in state $|\psi_0(g_t)\rangle$ with exponential accuracy as guaranteed by the adiabatic theorem. However, adiabaticity breaks down at the critical point since the gap vanishes, and once g_t is on the other side, the system is in a state $|\psi_0'(g_t,\Gamma)\rangle$. Thus, excitations (well defined in the $\phi=0$ model) are created in the system upon crossing the critical point. The number of excitations is $n_{\mathrm{ex}} = C\Gamma^{d\nu/(z\nu+1)}$ [14], where d is the spatial dimension. Note that the scaling parameter in the above equation is the rate Γ and not $|g - g_c|$ since n_{ex} is nonzero only as a consequence of sweeping across the critical point at a finite rate. Since the excitations in the TI model can be interpreted as free domain-wall excitations, the energy density of the final state is expected to be proportional to n_{ex}, so

$$\Delta E = E_0'(g_t,\Gamma) - E_0(g_t) \sim n_{\mathrm{ex}}\Delta(g_t) \sim \Gamma^{d\nu/(z\nu+1)}\Delta(g_t). \quad (2)$$

The gap $\Delta(g_t)$ depends only on the instantaneous value of g_t and does not scale with Γ. Equation (2) follows from noting that as $\Gamma \to 0$, there are a small number of excitations that exist only in a small band of states of vanishing width above $|\psi_1(g)\rangle$. Numerical results are shown in Fig. 2. The energy difference ΔE between the actual GS and the state after the sweep for the TI model ($\phi=0$) is approximately proportional

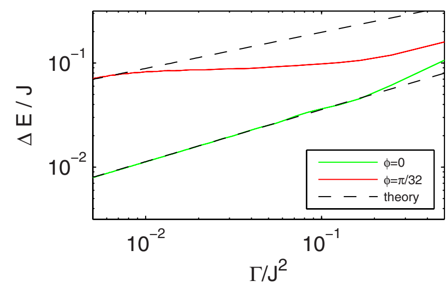

FIG. 2. (Color online) The difference ΔE between the energy density $E_0'(g_f,\Gamma)$ of the final state after sweeping with a rate Γ from $g_i=0.5$ to $g_f=-0.5$ through the critical point and the GS energy density $E_0(g_f)$. The dashed lines show the expected asymptotic behaviors $\Delta E \sim \sqrt{\Gamma}$ ($\phi=0$) and $\Delta E \sim \Gamma^{8/23}$ ($\phi>0$) resulting from different correlation length exponents in the "thermal" and "magnetic" directions.

to $\sqrt{\Gamma}$. The critical exponents for the TI model are $z=1$ and $\nu=1$ and $n_{\mathrm{ex}} \propto \sqrt{\Gamma}$ (see Refs. [12,13]). Thus the numerical results are consistent with the scaling in Eq. (2).

We now consider the EE between the left and right halves of the infinite system: the Hilbert space is partitioned so that all sites to the left of some bond are in one subsystem and all sites to the right are in the other. We find one scaling law immediately after the sweep and another in the time dependence of the swept state under the final Hamiltonian, in addition to oscillatory behavior. For critical points with conformal invariance ($z=1$) in one dimension ($d=1$), the EE of a GS with large finite correlation length diverges as $S = \frac{c}{6}\log \xi + \cdots$, where c is the central charge of the critical point [19]. To obtain the EE, we have to find the correlation length of the state $|\psi_0'(g_t,\Gamma)\rangle$. The process of sweeping generates a gap in the system $\Delta'(g_t,\Gamma)$ different from $\Delta(g_t)$ just as $|\psi_0'(g_t,\Gamma)\rangle$ is different from $|\psi_0(g_t)\rangle$. Polkovnikov has calculated the scaling form of the "typical gap" $\Delta'(g_t,\Gamma)$, which is $\Delta' \sim \Gamma^{z\nu/(z\nu+1)}$. Now, combining this with the correlation length $\xi' \sim \Delta'^{(-z)}$ and using $z=1$, we obtain

$$S = -\frac{c\nu}{6(\nu+1)}\log \Gamma + \mathrm{const.} \quad (3)$$

For the TI model with central charge $c=1/2$, we find that $S = -1/24 \ln(\Gamma) + \mathrm{const}$, which is consistent with the overall slope of the numerically calculated EE for the TI model ($\phi=0$) after the sweep in Fig. 3(a). A similar expression for the TI model can also be obtained for the EE of a finite block embedded in an infinite chain [20]. If the swept state continues to evolve in the (now constant) final Hamiltonian, the EE oscillates around a linearly increasing mean [Fig. 3(b)]; this linear increase is as predicted by Calabrese and Cardy for a "global quench" [21], but in the swept case the slope depends on the number of excitations created during the sweep.

In our numerical study of the weakly nonintegrable model with $\phi=\pi/32$, entanglement grows faster than in the integrable case, which is related to the interactions between "excitations" of the integrable model. The fast growth of entanglement makes the numerical simulations more difficult because much larger matrices have to be used in the iTEBD algorithm [16]. Because the resulting state $|\psi_0'(g_t,\Gamma)\rangle$ in-

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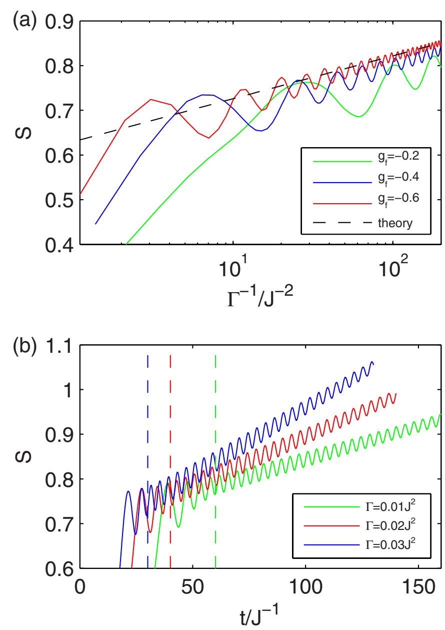

FIG. 3. (Color online) (a) The entanglement entropy as a function of the inverse rate for the parameter $\phi=0$ and $g_i=0.4$. The dashed line in the upper panel shows the expected asymptotic behavior $S=-1/24\,\ln(\Gamma)+\text{const.}$ (b) Entanglement entropy as a function of time for a sweep at $\phi=0$ from $g_i=0.4$ to $g_f=-0.4$. The dashed lines indicate the time at which the final value of $g_f=-0.4$ has been reached, and the Hamiltonian remains unchanged thereafter.

cludes excited states, both S and the expectation values of operators that do not commute with the Hamiltonian will oscillate as a function of the sweep rate and time. While the system is on the same side of the critical point as g_i, the state will evolve adiabatically. Once the critical point is reached (at time t_c), the expectation value of any observable that mixes the GS with the band of excited states oscillates with frequency $\frac{d\theta(t)}{dt}\approx\Delta(g_t)$. Using the scaling of the gap $\Delta(g_t)\sim|g_t-g_c|^{z\nu}=\Gamma^{z\nu}(t-t_c)^{z\nu}$ and integrating with respect to time, we obtain

$$\theta(t)\sim\Gamma^{z\nu}(t-t_c)^{z\nu+1}\sim\frac{\Delta(g_t)^{(z\nu+1)/z\nu}}{\Gamma}. \quad (4)$$

These findings are consistent with the numerical simulations of the sweep in the TI model in Fig. 3(a) as they show oscillations with a frequency that corresponds to the magnitude of $\Delta(g_t)$. The EE shows oscillations in real time [Fig. 3(b)] which have a constant frequency equal to the gap $\Delta(g)$. The oscillations in the rate dependence in Fig. 3(a) are consistent with Eq. (4).

In order to study these oscillations and their damping without reference to a particular observable, we use a version of the Loschmidt echo (see Ref. [22]). We square the inner product between the wave function immediately after the sweep and the wave function after an additional time t:

$$|\langle\psi_0'(g_f,\Gamma)|\psi_0'(g_f,\Gamma,t)\rangle|^2=|\langle e^{-iH_ft}\rangle_{\psi_0'(g_f,\Gamma)}|^2=e^{-\alpha(t)L}. \quad (5)$$

Here we have defined $\alpha(t)$ because this overlap typically goes to zero exponentially for translation-invariant states in the thermodynamic limit (i.e., when the length of the chain $L\to\infty$) unless $|\psi_0'(g_f,\Gamma)\rangle$ and $|\psi_0'(g_f,\Gamma,t)\rangle$ are identical. This overlap gives a direct probe of the magnitude and decay of quantum oscillations in the many-body state and it is suitable for computation using iTEBD [16]. It has been shown to determine the statistics of work done in a quantum quench [23]. Our main results are that the overlap oscillations have an unusual cusp structure, arising from the integrability of the TI model, and decay even in the exact zero-temperature evolution of an infinite system (because there is a continuum of excitation energies) algebraically rather than exponentially.

An analytical approach can be developed from independent Landau-Zener tunneling [24,25] at each k [13]. In terms of the Bogoliubov excitations (see Ref. [13] for details), the wave function $|\psi_0'(g_f,\Gamma)\rangle$ for the TI model ($\phi=0$) is a product $|\psi_0'(g_f,\Gamma)\rangle=\Pi_k(u_k|0\rangle+v_k|k,-k\rangle)$, where $|0\rangle$ is the vacuum and the state $|k,-k\rangle$ contains a pair of quasiparticles with pseudomomenta $(k,-k)$. The Landau-Zener formula gives for the small k's that dominate for slow rate,

$$|v_k|^2=P_k=1-|u_k|^2=\exp\left(-\frac{2\pi J^2k^2}{\Gamma}\right). \quad (6)$$

During the wait period, each fixed-k wave function $|k,-k\rangle$ has an energy $\Delta_f(k)=2\sqrt{\Delta^2+4J^2gk^2}$ (this is an approximation for low energies), with lattice spacing $a=1$. So up to an overall phase

$$|\psi_0'(g_f,\Gamma,t)\rangle=\prod_k\left(u_k|0\rangle+e^{-i\Delta_f(k)t}v_k|k,-k\rangle\right). \quad (7)$$

Now, using Eq. (6), the squared overlap in Eq. (5) is rewritten as

$$\prod_k\left[1+4\sin^2\left(\frac{\Delta_f(k)t}{2}\right)P_k(P_k-1)\right]. \quad (8)$$

In the continuum limit, the logarithm of this product becomes an integral for $\alpha(t)$. Taking the momentum cutoff to ∞,

$$\alpha(t)=\frac{1}{2\pi}\int_0^\infty dk\,\log\left[1+4\sin^2\left(\frac{\Delta_f(k)t}{2}\right)P_k(P_k-1)\right]. \quad (9)$$

This is compared to the numerical results in Fig. 4(a). The cusplike minima arise from the momentum value k^* where the tunneling probability $P_{k^*}=1/2$. These give a singularity in $\alpha(t)$ at times $\Delta_f(k^*)t=(2n+1)\pi$. This singular behavior is smeared out in the nonintegrable model at $\phi=\pi/32$, which does not have sharp excitations at finite energy [Fig. 4(b)]. This gives a probe of integrable versus nonintegrable behavior that could be useful for experiments [26]. Asymptotics of Eq. (9) lead to the following results: if we have $\Gamma N\ll\Delta^2$ for some number of periods $N\gtrsim1$, then the overlap has peaks shifted from the points $\Delta t=\pi N$, with maxima $\alpha_N=-\frac{\sqrt{2}}{\pi}\frac{g^2\Gamma^{5/2}}{J\Delta^4}\left[\frac{7}{32}-\frac{\sqrt{2}}{16}\right]N^2$. At long times, α approaches a constant independent of the final gap: $\alpha(\infty)\approx-0.0564\sqrt{\Gamma}/J$.

In the integrable case, the amplitude of oscillations in the Loschmidt echo falls off as $1/t$ for large times (not shown), which is different from the case of a quasithermal initial state

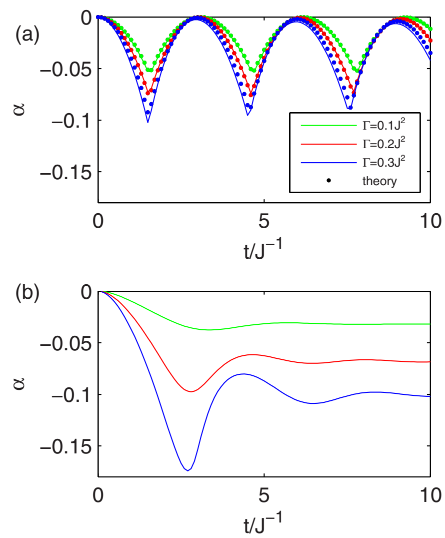

FIG. 4. (Color online) Exponent α in the overlap $|\langle\psi_0'(g_f,\Gamma)|\psi_0'(g_f,\Gamma,t)\rangle|^2=\exp(\alpha L)$ after a sweep from $g_i=0.5$ to $g_f=-0.5$. Panel (a) shows results for $\phi=0$ and panel (b) for $\phi=\pi/32$. The overlap is taken of the wave function $|\psi_0'(g_f,\Gamma)\rangle$ (immediately after the sweep with the rate parameter Γ) and the wave function $|\psi_0'(g_f,\Gamma,t)\rangle$ (after an additional evolution for the time t at fixed $g=g_f$). The dots in the upper ($\phi=0$) panel show the analytical results using Eq. (9).

whose excitation probabilities are given by Boltzmann weights. The latter has no cusps and damping as $1/\sqrt{t}$. This power-law decay of the oscillations rather than an exponential decay, is an important difference between the evolution found here and the decay of some observables after a sudden quench [6]. We note that the (pure) state as $t\to\infty$ does not resemble a (mixed) thermal state at any effective temperature of the final Hamiltonian since the excited state occupation probabilities remain determined by their tunneling probability P_k, while their energy is $\Delta_f(k)$; the final state has more occupation of low-k excited states than a thermal state. Sweeping can also induce *spatial* oscillations in correlation functions of local operators [27].

These results are modified in several ways for nonintegrable systems. In any system, the Loschmidt echo depends on both the initial state and the Fourier transform of the energy spectrum. The latter is known to change between integrable and nonintegrable systems: one sharp difference in spectral properties is revealed through the smearing out of cusps noted above [Fig. 4(b)]. More generally, the measurement of a Loschmidt echo could be used to probe the difference in energy level statistics between integrable (Poisson) and nonintegrable (Wigner-Dyson) systems.

Local observables are also likely to behave differently in integrable and nonintegrable systems, e.g., decay of *local* observables to a thermal distribution is expected in the nonintegrable case. The dynamical crossover to nonintegrable behavior was observed here as a rapid increase in EE and the destruction of cusps in the Loschmidt echo. Accessing the long-time regime and approach to equilibrium in the nonintegrable case is a major challenge for numerical methods.

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