Traveling and solitary wave solutions to the one-dimensional Gross-Pitaevskii equation

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The evolution of traveling and solitary waves in Bose-Einstein condensates (BECs) with a time-dependent scattering length in an attractive/repulsive parabolic potential is studied. The homogeneous balance principle and the *F*-expansion technique are used to solve the one-dimensional Gross-Pitaevskii equation with time-varying coefficients. We obtained three classes of new exact traveling wave and localized solutions. Our results demonstrate that the BEC solitary wave solutions can be manipulated and controlled by the time-dependent scattering length.

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I. INTRODUCTION

The experimental realization of Bose-Einstein condensates (BECs) in ultracold atomic gases [1–3] has triggered both experimental and theoretical exploration of the properties of Bose gases [4,5]. One of the important aspects in this area is the exploration of nonlinear (NL) properties of matter waves. Localized NL excitations, such as solitons, have been observed in BECs. These studies have stimulated intense research activities on NL atom optics and other areas of condensed matter physics and fluid dynamics.

In the case of NL matter waves, bright solitons (BSs) are expected only for an attractive interaction (the *s*-wave scattering length $\chi(t)>0$), whereas dark solitons (DSs) are expected for a repulsive interaction ($\chi(t)<0$). Recently, experiments have demonstrated that the variation of the effective scattering length (SL), including its sign, can be achieved by utilizing the so-called Feshbach resonance [6,7]. It has been shown in Ref. [8] that the variation of the nonlinearity of the Gross-Pitaevskii (GP) equation via Feshbach resonance provides a powerful tool for controlling the generation of bright and dark soliton trains, starting from the periodic waves.

The construction of exact solutions of NL partial differential equations (PDEs) is one of the essential and most important tasks in NL science. The objective of this paper is to identify traveling wave solutions of the one-dimensional (1D) GP equation, by utilizing the homogeneous balance principle and the *F*-expansion technique, and to extend the analysis to include the solitary wave solutions. Generally speaking, the presence of solitary wave solutions depends on the *s*-wave SL $\chi(t)$ and the trapping potential coefficient $\beta(t)$ [appearing below in Eq. (1)], so that we shall also provide a constraint condition on these coefficients, for exact solution by the present method.

Owing to the importance of the GP equation with time-varying coefficients, there have naturally been attempts at its exact solution before. An early attempt was presented by Xue [9], who obtained some analytical solutions and showed that the BSs can be compressed into desired width and amplitude in a controllable manner by changing the SL and the external potential. Al Khawaja [10] obtained an exact solitonic solution by employing the Darboux transformation method. It required a known "seed" solution of the GP equation in question, to obtain a general exact solution in terms of exponential and trigonometric functions. Atre *et al.* [11] obtained a class of solitary wave solutions to the 1D GP equation by utilizing the self-similar method. They introduced an ansatz solution that, via a Riccati equation, can be mapped into a linear Schrödinger eigenvalue problem. A soliton configuration could be associated with each such solvable problem, expressible in terms of Jacobi elliptic functions (JEFs). Our solutions are also expressed in terms of JEFs, and naturally there exist relations between the solutions reported in [11] and some of the solutions reported here. However, the methods of solution and the applications of solutions are different. Our solution method involves the homogeneous balance principle and the *F*-expansion technique [12–14]. It is a simple systematic procedure for solving PDEs of the NL Schrödinger type, which allows an easy determination of the traveling wave and localized solutions. Another recent account, involving self-similar solutions ("similaritons") obtained by a lens-type transformation, is presented by Wu and Porsezian [15].

The paper is organized as follows. In Sec. II we introduce the homogeneous balance principle and the *F*-expansion technique [12–14]. Applying the solution procedure to the 1D GP we obtain three families of exact solutions. In Sec. III we further investigate the main features of analytical solutions for a few selected SLs $\chi(t)$, obtained by computer simulations. The last section presents a short summary.

II. PERIODIC TRAVELING WAVE SOLUTIONS

The condensate wave function is well described by the GP equation. In the physically important case of cigar-shaped BECs, it is reasonable to reduce the GP equation to a 1D nonlinear Schrödinger equation (NLSE) [11,16]

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\chi(t)u|u|^2+\beta(t)x^2u=0, \tag{1}$$

where the dimensionless time *t* and the coordinate *x* are measured in some convenient units. The *s*-wave SL $\chi(t)$ and the

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TABLE I. Jacobi elliptic functions.

Solution	c_0	c_2	c_4	$F(\theta)$	$m=0$	$m=1$
1	1	m^2	m^2	$\text{sn}(\theta)$	sin	tanh
2	$1-m^2$	$2m^2-1$	$-m^2$	$\text{cn}(\theta)$	cos	sech
3	m^2-1	$2-m^2$	-1	$\text{dn}(\theta)$	1	sech
4	m^2	m^2	1	$\text{ns}(\theta)$	cosec	coth
5	$-m^2$	$2m^2-1$	$1-m^2$	$\text{nc}(\theta)$	sec	cosh
6	-1	$2-m^2$	m^2-1	$\text{nd}(\theta)$	1	cosh
7	1	$2-m^2$	$1-m^2$	$\text{sc}(\theta)$	tan	sinh
8	1	$2m^2-1$	$-m^2(1-m^2)$	$\text{sd}(\theta)$	cot	cosech
9	$1-m^2$	$2-m^2$	1	$\text{cs}(\theta)$	cos	1
10	$-m^2(1-m^2)$	$2m^2-1$	1	$\text{ds}(\theta)$	sec	1

trapping potential coefficient $\beta(t)$ are assumed functions of time, so that we consider NLSE with variable coefficients in an external parabolic potential (PP).

To obtain the solution of Eq. (1), the complex wave function $u(t,x)$ is written as [17,18] $u(t,x)=A(t,x)e^{iB(t,x)}$, where the amplitude $A(t,x)$ and the phase $B(t,x)$ are real functions of x and t. Substituting $u(t,x)$ into Eq. (1) and setting the real and imaginary parts of the resulting equations to zero, leads to the following set of PDEs,

$$\frac{\partial A}{\partial t}+\frac{1}{2}\left(2\frac{\partial A}{\partial x}\frac{\partial B}{\partial x}+A\frac{\partial^2 B}{\partial x^2}\right)=0, \tag{2a}$$

$$-A\frac{\partial B}{\partial t}+\frac{1}{2}\left[\frac{\partial^2 A}{\partial x^2}-A\left(\frac{\partial B}{\partial x}\right)^2\right]+\chi A^3+\beta x^2 A=0. \tag{2b}$$

We seek traveling wave solutions to Eq. (2), according to the balance principle and the F-expansion technique [12,13]. It is perhaps worth noting that the balance and F-expansion methods, originally developed for 1D systems, have been extended to multidimensional PDEs in [17,18]. The solution of Eqs. (2) is chosen in the following form:

$$A(t,x)=f(t)F(\theta)+g(t)F^{-1}(\theta). \tag{3a}$$

$$\theta=k(t)x+\omega(t), \tag{3b}$$

$$B(t,x)=a(t)x^2+b(t)x+c(t), \tag{3c}$$

where f, g, k,ω a, b, and c are the functions of time, to be determined. The function $F(\theta)$ is one of JEFs, which in general satisfy the following general first- and second-order NL ordinary differential equations: $(dF/d\theta)^2=c_0+c_2F^2+c_4F^4$, and $d^2F/d\theta^2=c_2F+2c_4F^3$, where c_0, c_2, and c_4 are real constants related to the elliptic modulus of JEFs. Substituting Eqs. (3) into Eqs. (2), along with the relations mentioned above, collecting the terms of different powers of x^kF^n $(k=0,1,2;n=0,1,2,3)$ and of $\sqrt{c_0+c_2F^2+c_4F^4}$, and then setting each of the terms equal to zero (the balance principle), we obtain an overdetermined system of 14 first-order differential and algebraic equations for the unknown functions f, g, k,ω a, b, and c. By solving these equations self-consistently with the help of Mathematica, one finds the following solutions:

Case 1: $g=0$, $f=f_0\sqrt{|\chi|}$; we obtain the solution family 1,

$$u(x,t)=f_0\sqrt{|\chi|}F(\theta)e^{iB}. \tag{4a}$$

Case 2: $f=0$, $g=g_0\sqrt{|\chi|}$; we obtain the solution family 2,

$$u(x,t)=\frac{g_0\sqrt{|\chi|}}{F(\theta)}e^{iB}. \tag{4b}$$

Case 3: $f=f_0\sqrt{|\chi|}$; $g=g_0\sqrt{|\chi|}$; we obtain the solution family 3,

$$u(x,t)=\left[f_0F(\theta)+\frac{g_0}{F(\theta)}\right]\sqrt{|\chi|}e^{iB}. \tag{4c}$$

In all the cases $k=k_0\chi$; $\omega=-k_0b_0\int\chi^2dt+\omega_0$; $b=b_0\chi$; and $c=\frac{1}{2}(c_2k_0^2-b_0^2)\int\chi^2dt$. Also $a=-\frac{1}{2\chi}\frac{d\chi}{dt}$, so that $B=-\frac{1}{2\chi}\frac{d\chi}{dt}x^2+b_0\chi x+\frac{1}{2}(c_2k_0^2-b_0^2)\int\chi^2dt$, and $\theta=k_0(\chi x-b_0\int\chi^2dt-x_0)$, where $x_0=-\omega_0/k_0$. Here and in what follows the symbols with the subscript 0 are used to denote the initial values of the corresponding parameter functions at the initial time $t=0$. Essentially, the solution families 1 and 2 are equivalent to each other, because both F and $1/F$ are JEFs themselves. The chirp function $a(t)$ is directly related to SL. On the other hand, the SL function $\chi(t)$ is expressed in terms of the trapping potential coefficient $\beta(t)$. This relation can be conveniently understood as an integrability condition on Eq. (1),

$$\beta=\frac{1}{\chi^2}\left(\frac{d\chi}{dt}\right)^2-\frac{1}{\chi}\frac{d^2\chi}{dt^2}. \tag{5}$$

The form of the solutions depends on what JEFs are utilized. Table I lists some of the JEFs that may appear in the solutions. As long as one chooses the constants c_0, c_2, and c_4, according to the relations listed in Table I and substitutes the appropriate $F(\theta)$ into Eqs. (4), one obtains the exact periodic traveling wave solutions to the 1D GP equation. The elliptic modulus m varies between 0 and 1. When $m\to 0$, JEFs degenerate into the trigonometric functions, and the periodic traveling wave solutions become the periodic trigonometric solutions. When $m\to 1$, JEFs degenerate into the hyperbolic functions, and the periodic traveling wave solutions become the solitary wave solutions.

TABLE II. Some single-JEF solitary solutions.

Solution type	Single-JEF solitary wave	θ	B
Bright solitary (BS) wave	$u_1=f_0\sqrt{\lvert\chi\rvert}\,\sec h(\theta)e^{iB(x,t)}$	$\theta=k_0(\chi x-b_0\int\chi^2dt-x_0)$	$B=-\frac{1}{2\chi}\frac{d\chi}{dt}x^2+b_0\chi x+\frac{1}{2}(k_0^2-b_0^2)\int\chi^2dt$
Dark solitary (DS) wave	$u_2=f_0\sqrt{\lvert\chi\rvert}\,\tanh(\theta)e^{iB(x,t)}$	$\theta=k_0(\chi x-b_0\int\chi^2dt-x_0)$	$B=-\frac{1}{2\chi}\frac{d\chi}{dt}x^2+b_0\chi x+\frac{1}{2}(k_0^2-b_0^2)\int\chi^2dt$

III. SOME CHARACTERISTICS OF THE SOLITARY WAVE SOLUTIONS

By definition, solitons preserve their form while propagating; they are localized, and after interacting with other solitons emerge from collisions unchanged (except for a possible phase shift) [19]. They can also periodically change the width and the peak maximum, in which case they are known as the soliton breathers. Since we do not consider here interactions between our solitary waves, we do not claim them to be true solitons; the task of their interactions will be accomplished elsewhere. Nevertheless, the connection of GP equation with NLSE, which admits soliton solutions, suggests that our solitary wave solutions are genuine solitons as well. Still, we refer to our solutions only as solitary wave solutions and report their existence as $m\to 1$. Trains of soliton solutions in terms of JEFs, as well as bound states of solitons in 1D GP equation have been discussed in [11,20]. It is seen that the form of localized solutions in GP equation is controlled by the SL $\chi(t)$. Obviously, when $\chi(t)=$const, $\beta(t)$ is zero, and Eq. (1) becomes the standard NLSE [21], which is an integrable system. Such a system supports both bright and dark soliton solutions. When $\chi(t)$ is a function of time, the 1D GP equation with variable coefficients is usually nonintegrable, and the solutions must be found numerically or perturbatively, from suitably chosen ansatz solutions. Regardless of what $\chi(t)$ is, as long as the condition (5) is satisfied, the GP equation is an integrable system, and Eq. (1) contains solitary wave solutions. In Table II we exhibit the most interesting single-JEF solitary wave solutions. The system can be controlled by choosing different SLs. To display some unique features of these exact solitary wave solutions, we choose the SL $\chi(t)$ in terms of the trigonometric or exponential functions.

First, we discuss the BS u_1. From Table II it can be concluded: (1) the amplitude of the solitary wave is proportional to $\sqrt{\chi}$; the beam width inverses with χ, but the shape of the solitary wave does not change as it propagates, so the total power $(N=\int_{-\infty}^{+\infty}|u_1|^2dx=2f_0^2/k_0)$ is conserved. When we increase the absolute value of SL, the solitary solution u_1 becomes compressed. (2) The center position of the solitary wave can be expressed as $x_c=|(b_0\int\chi^2dt+x_0)/\chi|$; it satisfies the following equation of motion: $d^2x_c/dt^2+2\beta(t)x_c=0$. This equation means that the pulse is located at x_c; it behaves like an oscillator with variable spring constant that moves in time t under the influence of PP $\beta(t)x^2$. According to the equation of motion, we can select an appropriate SL χ (that is, the external PP) to control the solitary wave motion. For the DS u_2, we obtain similar results. In the next subsection we discuss the representative examples from Table II, to illustrate the movement of BEC solitary waves under different SLs and external PPs.

A. Bright solitary waves

A few experimental examples of BS matter waves are provided in [22,23]. In order to understand physical significance of BS from Table II, we present some of its characteristics. In particular, taking the SL $\chi=\chi_0\cos(t)$, we obtain the intensity of BS:

$$|u(x,t)|^2=f_0^2|\chi_0\cos(t)|\sec h^2(\theta), \qquad (6)$$

where $\theta=k_0[\chi_0\cos(t)x-\frac{1}{2}\chi_0^2b_0(t+\sin t\cos t)-x_0]$. Figure 1 shows the propagation of the pulse in time t. It is periodic; the BS executes periodic oscillations of a breather, the pulse amplitude displays a cyclic change.

As another example, we examine the formation and dynamics of NL excitations in the presence of periodic SL in

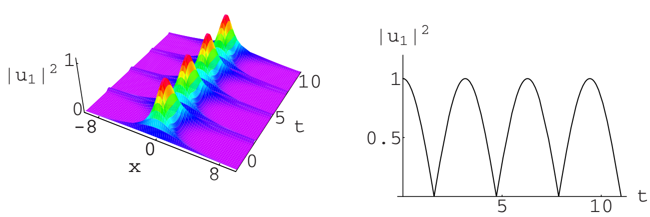

FIG. 1. (Color online) Bright breather solitary wave with the cosine scattering length. The parameters are: $\chi=\chi_0\cos(t)$, $\chi_0=1$, $f_0=k_0=1$, $b_0=0$, $x_0=0$. Left: optical intensity distribution. Right: amplitude variation at the position $x=0$, as a function of time.

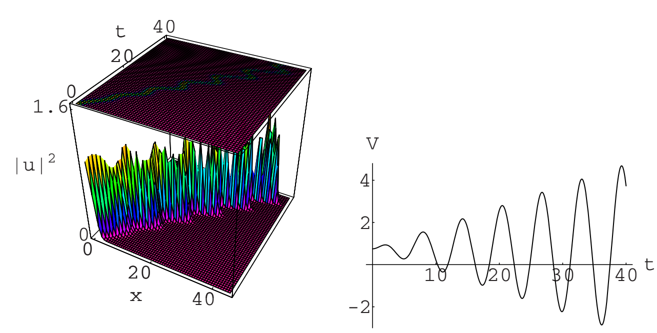

FIG. 2. (Color online) Stable oscillations of a bright breather solitary wave, in the case $\chi=1+\chi_0\sin(t)$, $\chi_0=0.1$, $f_0=k_0=b_0=1$, $x_0=0$. Left: amplitude of the wave. Right: velocity of the pulse center.

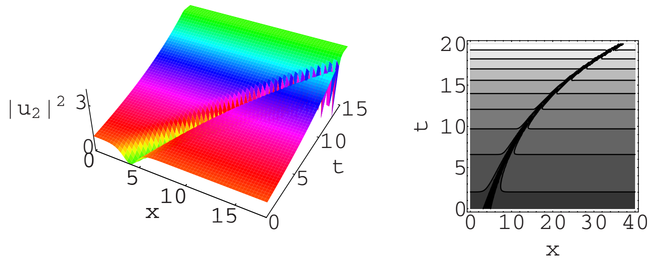

FIG. 3. (Color online) Dynamics of the dark solitary wave compression, as given by Eq. (8). The parameters are given as follows: $\chi=\chi_0 e^{2\beta_0 t}$, $\beta_0=0.05$, $f_0=k_0=b_0=\chi_0=1$, $x_0=0$. Left: intensity distribution as a function of t. Right: View from the above.

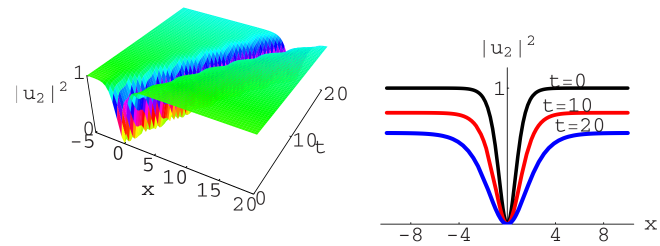

FIG. 4. (Color online) Broadening evolution of a dark solitary wave, with the parameters: $\chi=\chi_0 e^{-\alpha t}$, $\chi_0=1$, $f_0=k_0=1$, $x_0=0$. Left: intensity distribution for $\alpha=0.02$, $b_0=1$. Right: cross sections at $t=0,10,20$ from top to bottom, for $\alpha=0.04$, $b_0=0$.

the attractive interaction regime. A number of condensate profiles emerge in the attractive regime, depending on the nature of SL. Specifically, we take the SL $\chi=1+\chi_0\sin(t)$, where $0<\chi_0<1$. From the integrability condition (5), we find the coefficient of the external PP: $\beta(t)=-\chi_0[\chi_0+\sin(t)+\chi_0\cos^2(t)]/[2+2\chi_0\sin(t)]^2$, from which we obtain the solitary wave intensity, using Table II,

$$|u(x,t)|^2=f_0^2(1+\chi_0\sin t)\sec h^2(\theta), \qquad (7)$$

where

$$\theta=k_0\Big[(1+\chi_0\sin t)x-b_0\Big(2\chi_0+t+\tfrac{1}{2}\chi_0^2 t-\tfrac{1}{2}(4+\chi_0\sin t)\chi_0\cos t\Big)-x_0\Big].$$

In Fig. 2 we display the evolution (left) and the velocity (right) of a BS wave in time, as given by Eq. (7), with the parameters $\chi_0=0.1$, $f_0=k_0=b_0=1$, $x_0=0$. As seen in Fig. 2, the amplitude of the solitary wave displays periodic oscillations. Because the external PP changes from attractive to repulsive alternately, the velocity of the pulse center executes periodic oscillations, and the magnitude of the velocity is also increasing gradually.

B. Dark solitary waves

Early experimental observations of DS matter waves are provided in [24]. By utilizing the solutions reported in Table II, we demonstrate that the manipulation of SL can be used to compress a dark solitary wave of BECs into an arbitrary peak matter density. It has been reported [25] that a change in the SL can also lead to the splitting of solitons and the generation of new solitons. For simplicity, we assume that the solitary wave moves under a repulsive PP, namely $\beta=2\beta_0^2$ (β_0 is a positive constant). From the integrability condition (5), we find the SL $\chi(t)=\chi_0 e^{2\beta_0 t}$. In particular, the propagation speed of the solitary wave is increasing. With the conditions above, the solitary wave intensity can be written in the following form

$$|u(x,t)|^2=f_0^2 e^{2\beta_0 t}\tanh^2(\theta), \qquad (8)$$

where $\theta=k_0(e^{2\beta_0 t}x+1-\frac{b_0}{4\beta_0}e^{4\beta_0 t}-x_0)$.

For a better understanding, Eq. (8) is plotted in Fig. 3, which depicts the dynamics of the controlled 1D GP system in a repulsive PP. As one can see from Fig. 3, with the increasing time, the solitonary wave displays an increase in the peak value and compression in the width. As a result, one can obtain a pulse with an arbitrary peak matter density. In the present model these tendencies continue unabated, however in a real BEC system, such tendencies are suppressed (and the 1D model becomes invalid). Note that in the repulsive background potential the solitary wave is accelerating and propagating away, instead of oscillating, as in an attractive PP. The possibility of compressing the soliton of BECs into an arbitrary peak matter density experimentally could provide a tool for investigating the range of validity of the 1D GP equation [26,27].

Figure 4 shows how the intensity of DS wave changes with time, when the SL is $\chi=\chi_0 e^{-0.02t}$ and β_0 negative, for different b_0. As seen, because SL decays as an exponential, the amplitude of DS decays also, and the beam width increases. So, one can select an appropriate SL, to make the solitary wave broaden in a predictable fashion.

IV. CONCLUSIONS

We have determined different classes of exact traveling wave solutions of the one-dimensional Gross-Pitaevskii equation using the homogeneous balance principle and the F-expansion technique. It is noted that the changes in the scattering length in time can be effectively used to control the BEC matter waves of solitary type. In particular, we have shown the effect of pulse compression on both bright and dark solitary waves in the presence of an attractive/repulsive parabolic potential. Our results demonstrate that the BEC solitary waves can be manipulated and controlled by the time-dependent atomic scattering length.

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