Stochastic bifurcations and coherencelike resonance in a self-sustained bistable noisy oscillator

A. Zakharova,[1] T. Vadivasova,[2] V. Anishchenko,[2] A. Koseska,[3] and J. Kurths[1,4]
[1]*Potsdam Institute for Climate Impact Research, Potsdam, Germany*
[2]*Saratov State University, Saratov, Russia*
[3]*University of Potsdam, Potsdam, Germany*
[4]*Humboldt University, Berlin, Germany*
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We investigate the influence of additive Gaussian white noise on two different bistable self-sustained oscillators: Duffing–Van der Pol oscillator with hard excitation and a model of a synthetic genetic oscillator. In the deterministic case, both oscillators are characterized with a coexistence of a stable limit cycle and a stable equilibrium state. We find that under the influence of noise, their dynamics can be well characterized through the concept of stochastic bifurcation, consisting in a qualitative change of the stationary amplitude distribution. For the Duffing-Van der Pol oscillator analytical results, obtained for a quasiharmonic approach, are compared with the result of direct computer simulations. In particular, we show that the dynamics is different for isochronous and anisochronous systems. Moreover, we find that the increase of noise intensity in the isochronous regime leads to a narrowing of the spectral line. This effect is similar to coherence resonance. However, in the case of anisochronous systems, this effect breaks down and a new phenomenon, anisochronous-based stochastic bifurcation occurs.

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I. INTRODUCTION

The investigation of the influence of random forces (noise) on nonlinear dynamical systems is one of the most relevant and intensively developing research directions in nonlinear dynamics. In general, it is well known that noise can induce shift of the bifurcations to different control parameter values, compared to their deterministic counterparts. Moreover, new types of dynamical behavior can be observed in presence of noise, generally referred to as noise-induced effects [1]. On the other hand, the investigation of stochasticity is very important for the understanding of the dynamical features of real systems, since they are inevitably affected by internal and external noise sources. Thus, it is significantly relevant to study changes in the dynamics of nonlinear systems through the concept of stochastic bifurcations [2]. In this direction, large number of investigations has been devoted to stochastic bifurcations and noise-induced transitions [3–14]. In stochastic systems, one can either treat the noise intensity as a bifurcational parameter, but additionally other statistical characteristics of noise (e.g., mean value, spectral width etc.) can be used to track the dynamical changes in the system, and thus, developing and proposing novel ways to control the behavior of nonlinear systems by means of noise.

At this point, the necessity occurs to define a stochastic bifurcation. In general, stochastic bifurcations are characterized with a qualitative change of the stationary probability distribution, e.g., a transition from unimodal to bimodal distribution [1,4,5,7]. Such a change in the distribution law results in a change of other stochastic characteristics of the system that can be observed experimentally as well. Additionally, stochastic bifurcations can also become apparent through the change of stability of trajectories belonging to a certain set with a given invariant measure [8]. The first type of stochastic bifurcations is called *P*-bifurcations (phenomenological bifurcations), whereas the second one-*D*-bifurcations (dynamical bifurcations) [2]. Moreover, the stochastic bifurcations can also consist of two steps: *P*-bifurcation and *D*-bifurcation, separated in the parameter space by a certain bifurcational interval [9–11].

In the present work we study the qualitative change of the stationary probability distribution of amplitude of oscillations in periodic self-sustained oscillators that are characterized with a region of bistability in the deterministic case. This region is bounded by a tangent bifurcation of limit cycles from the one side and a subcritical Hopf bifurcation from the other one. It is a well known fact that Gaussian noise can join basins of different attractors. In this contribution, we study the changes in the bistability region borders which occur. Moreover, in the vicinity of the saddle-node bifurcation of limit cycles, for increasing noise intensities, we observe an effect similar to coherence resonance. We find that for a certain interval of the noise intensity, the spectral line width of oscillations decreases [15]. Our purpose is to investigate the interconnection of this phenomenon and stochastic bifurcations. Additionally, we study the influence of the anisochronicity of oscillations on the spectrum properties of the noisy oscillator as well.

II. NOISE INFLUENCE ON A BISTABLE DUFFING–VAN DER POL OSCILLATOR

The first model under investigation is the classical bistable self-sustained Duffing–Van der Pol oscillator with additive Gaussian white noise

$$\ddot{x} - (\varepsilon + x^2 - x^4)\dot{x} + x + \beta x^3 = \sqrt{2D}n(t), \quad \beta \geq 0, \qquad (1)$$

where $n(t)$ is the normalized source of Gaussian white noise: $\langle n(t)n(t+\tau)\rangle = \delta(\tau)$, $\langle n(t)\rangle = 0$, and D—the noise intensity. In the deterministic case, for $-\frac{1}{8} < \varepsilon < 0$, the system [Eq. (1)] is characterized with a bistable behavior: two attractors are present in the phase plane—a stable focus at the origin and a stable limit cycle. Thus, the bistability region is restricted with a saddle-node bifurcation of cycles at $\varepsilon=-\frac{1}{8}$, and a subcritical Andronov-Hopf bifurcation at $\varepsilon=0$. Additionally, the parameter β defines the anisochronicity of oscillations—for $\beta=0$ the system [Eq. (1)] is isochronous, i.e., the frequency of the oscillations does not depend on their amplitude.

In the quasiharmonic regime, on assumption that noise intensity is small, we introduce change of variables,

$$x(t)=a(t)\cos[t+\varphi(t)], \quad \dot{x}(t)=-a(t)\sin[t+\varphi(t)].$$

After the substitution and averaging the equations of Duffing–Van der Pol system over the period of oscillations, we obtain the following stochastic equations for (the slow, on a scale of $\frac{1}{2\pi}$, variables) the instantaneous amplitude $a(t)$ and phase $\varphi(t)$ [16],

$$\dot{a}=\left(\frac{\varepsilon}{2}+\frac{a^2}{8}-\frac{a^4}{16}\right)a+\frac{D}{2a}+\sqrt{D}n_1(t),$$

$$\dot{\varphi}=\frac{3\beta}{8}a^2+\frac{\sqrt{D}}{a}n_2(t), \tag{2}$$

where $n_1(t)$ and $n_2(t)$ are independent normalized sources of Gaussian white noise. As derived from Eq. (2), the amplitude of the stable cycle for $D=0$ is defined by $a_0=\sqrt{1+\sqrt{1+8\varepsilon}}$. It is worth pointing out that $\dot{a}$ does not depend on φ, allowing us further to develop a probability density for a, rather than a joint density for a and φ.

For $D\neq 0$, the probability density $p(a,t)$ of the instantaneous amplitude satisfies the Fokker-Planck-Kolmogorov equation [17,18],

$$\frac{\partial p(a,t)}{\partial t}=-\frac{\partial}{\partial a}\left[\left(\frac{\varepsilon a}{2}+\frac{a^3}{8}-\frac{a^5}{16}+\frac{D}{2a}\right)p(a,t)\right]+\frac{D}{2}\frac{\partial^2 p(a,t)}{\partial a^2}. \tag{3}$$

Hence, the stationary solution of Eq. (3) is

$$p(a)=Nae^{[-(1/48D)a^2(a^4-3a^2-24\varepsilon)]}, \tag{4}$$

with N being a normalization constant. As far as the instantaneous amplitude in Eq. (2) does not depend on the phase, its shape is identical for isochronous and anisochronous oscillators.

Additionally, the extrema of the distribution [Eq. (4)] are the roots of the equation

$$f(a_m)=a_m^6-2a_m^4-8\varepsilon a_m^2-8D=0, \tag{5}$$

where a_m is the amplitude, corresponding to the extremum of distribution [Eq. (4)] and m is the index number of the extremum. Varying parameters ε and D, the number of the real roots of Eq. (5) changes. This effect can be seen as a type of a stochastic P-bifurcation in the system [Eq. (2)] [2]. In general, the bimodal distribution for the noisy oscillator is similar to the bistability in the deterministic case. However, the bimodality region which exists for $D\neq 0$ and the region of bistability present for $D=0$ do not coincide and in this case the noise intensity acts as the bifurcation parameter. It is

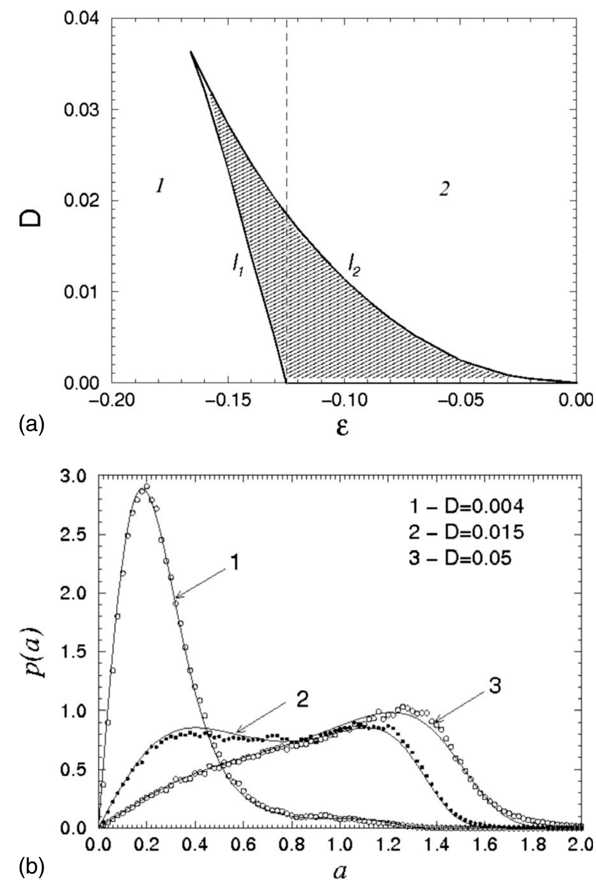

FIG. 1. Stochastic P-bifurcations in the Duffing–Van der Pol oscillator [Eqs. (1) and (2)]. (a) Bifurcation diagram of the system [Eq. (2)] in the parameter plane (ε,D). The tinted region represents the bimodal distribution; lines l_1 and l_2 correspond to the appearance and disappearance of one of the maxima of $p(a)$. The vertical dashed line stands for the bistability border of the deterministic oscillator (region 2 corresponds to bistability in the deterministic model). (b) Stationary amplitude distribution for $\varepsilon=-0.13$ and different values of the noise intensity D. The circles represent numerical computation for the oscillator [Eq. (1)], whereas the solid lines denote the algebraic calculations using formula [Eq. (4)] (the normalization constant N is defined numerically). For numerical integration of the stochastic equations we used the Heun scheme [19].

important to note that for the stationary distribution $p(x,y)$ of the dynamical variables x and y, we do not observe any qualitative transformation with the increase of the noise intensity. The number of maxima of $p(x,y)$ changes only for the lines, corresponding to the saddle node and the subcritical Andronov-Hopf bifurcation of the deterministic system. Since qualitative transformations of $p(a)$ in the presence of noise play an important role in the system's behavior, we denote them as P bifurcations.

Figure 1(a) shows the bifurcation diagram of the system [Eq. (2)], obtained from the analysis of the dependence of Eq. (5) on the parameters ε and D. In the tinted region, the stationary amplitude distribution is bimodal. The lines l_1 and l_2 that bound this region correspond to stochastic P-bifurcations. Increasing the value of D, the bimodality region shifts to smaller values of ε and becomes narrower. If D is increased even further (e.g., for $D>D_{cr}\approx 0.036$), then

P-bifurcations are not observed while varying ε, and the bimodality region does not exist any more.

It is important to note that the averaged model [Eq. (2)] does not reflect the properties of the initial system [Eq. (1)] completely for large values of noise intensities and for nonharmonic regimes. However, within the bifurcation diagram shown in Fig. 1(a), the amplitude distributions calculated by Eq. (4) and obtained numerically for the system [Eq. (1)] coincide significantly well. This leads to the conclusion that the bifurcation diagram [Fig. 1(a)] is related not only to the averaged model, but also to the initial system and does not depend on the parameter of anisochronicity $\beta \geq 0$. Figure 1(b) shows the stationary amplitude distributions for different values of the noise intensity and for a fixed value of the nonlinearity parameter ε ($\varepsilon=-0.13$). In this case, there is only one attractor in the deterministic system: a stable focus at the origin. For small noise intensities [below the line l_1 in Fig. 1(a)], the amplitude distribution has only one maximum situated in the vicinity of zero [curve 1 in Fig. 1(b)]: e.g., the phase trajectory is mainly located in the vicinity of the stable focus, where nonlinear effects can be neglected. Correspondingly, the amplitude distribution is similar to a Rayleigh distribution. When the noise intensity is increased however, the phase trajectory visits more and more frequently the regions far away from the origin, and the nonlinearity of the system becomes increasingly important. At the same time, the trajectory stays longer in the region, where in the deterministic case, for $\varepsilon \geq -\frac{1}{8}$, the stable limit cycle is located. Hence, the distribution evolves with D. For $D \approx 0.005$ [on the line l_1, Fig. 1(a)], a transition from a unimodal to a bimodal distribution occurs [see curve 2 in Fig. 1(b)]. In the case of strong noise intensities however, the trajectory finds itself very rarely in the vicinity of the stable focus, and for $D \approx 0.0202$ [on the line l_2, Fig. 1(a)] the second stochastic bifurcation takes place. This results in the disappearance of a pair of extrema. Consequently, the amplitude distribution becomes unimodal again, but its maximum is shifted toward larger amplitude values [curve 3 in Fig. 1(b)].

In the presence of additive Gaussian noise there exists only one invariant set of trajectories in the phase space, characterized with a defined stationary probability density over it. Therefore, additive noise destroys dynamical bifurcations which are connected with the change of stability of the invariant sets [2]. We have performed numerically a detailed stability analysis of system [Eq. (1)], which we omit here for brevity. This shows that the largest Lyapunov exponent remains negative for all values of ε and D. Studying the averaged model [Eq. (2)], the same conclusion can be obtained. Moreover, one can distinguish the Lyapunov exponent connected with the phase dynamics from the one characterizing the amplitude behavior: the first one is identical to zero, and the second one (averaged over the stationary distribution [Eq. (4)]) is always negative.

Though the P-bifurcations observed for the amplitude distribution of the Duffing–Van der Pol oscillator [Eqs. (1) and (2)] do not depend on the anisochronicity parameter β, the power spectra of the oscillations are essentially different for the isochronous and the anisochronous cases. The normalized power spectra of the oscillator [Eq. (1)], obtained numerically for different values of noise intensity at $\beta=0$ (isochronous oscillations) and at $\beta=0.5$ (anisochronous oscillations), are shown in Fig. 2.

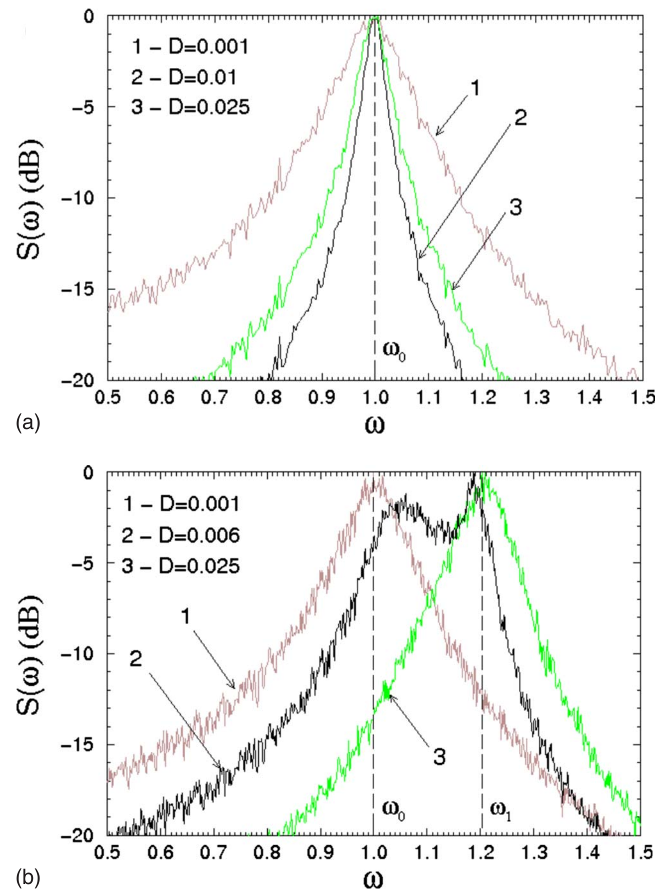

FIG. 2. (Color online) Normalized power spectral density of oscillation in (Eq. (1)) for $\varepsilon=-0.13$ and different values of noise intensity: (a)—in the isochronous case ($\beta=0$) and (b)—in the anisochronous case ($\beta=0.5$).

Furthermore, in the isochronous case, an effect similar to coherence resonance is observed. In particular, for a certain noise intensity (close to the central region of the bimodal amplitude distribution) the spectrum width becomes minimal [Fig. 2(a)]. This effect was initially observed experimentally in an optical bistable oscillator [15] and referred to as coherence resonance. However, this term is not entirely correct, since the mechanism of the present phenomenon is principally different from the mechanism of classical coherence resonance (CR), as defined in [20]. We point out here the differences: for small noise intensities, the trajectory of the system [Eq. (1)] spends most of the time in the vicinity of the equilibrium point. It is known from linear analysis in general, that the spectrum of small oscillations near an equilibrium point has the form of a Lorentzian, whose width at half-maximum is defined by $|\varepsilon|$. The spectrum width of the oscillations in the vicinity of the limit cycle, on the other hand, is determined by the noise intensity [16] and can be smaller than $|\varepsilon|$. Accordingly, we observe narrowing of the spectral line as a result of the change of the amplitude distribution with the increase of noise intensity. For $\varepsilon < -1/8$ (i.e., outside the bistability region of the deterministic oscillator) such kind of behavior can be found only after the stochastic bifurcation on line l_1. The narrowest spectrum then is not observed for small, but for noise of intermediate intensity. A similar explanation of the observed effect was given in [15] as well, but no presence of *P*-bifurcations for the amplitude distribution was established there. Moreover, it was assumed that the location of the spectral maximum does not depend on the noise intensity. We show here, however, that these assumptions hold true only in the case of isochronous oscillations.

The power spectrum of the anisochronous oscillator [Eq. (1)] for increasing noise intensities, behaves differently in comparison to the isochronous case [Fig. 2(b)]. In particular, for small D it has only one maximum at frequency $\omega_0=1$ (as for the isochronous oscillator). Such a spectrum corresponds to the rotation of the trajectory in the vicinity of the equilibrium point at the origin. With the increase of D, however, there appears a second maximum at a different frequency ω_1, which corresponds to the rotation of the trajectory on a limit cycle. Thus, in some interval of D there are two spectral maxima. Further on, for larger D, the first maximum disappears and there remains only the maximum at frequency ω_1. The width of the first spectral line does not change essentially, but the width of the second one grows with the increase of D. (The evolution of the power spectrum for anisochronous oscillations is described in detail in Sec. III.) We denote this phenomenon *anisochronous-based stochastic bifurcation (ASB)*. Consequently, in the case of anisochronous oscillations, the effect we observe is substantially different than the CR.

III. NOISE INFLUENCE ON A SYNTHETIC GENE OSCILLATOR

Next we analyze a system demonstrating self-sustained oscillations, namely, a paradigmatic mathematical model of a synthetic gene oscillator, as discussed in [21],

$$\dot{x}(t) = \frac{1 + x^2 + \alpha\sigma x^4}{(1 + x^2 + \sigma x^4)(1 + y^4)} - \gamma_x x + \sqrt{2D}n(t),$$

$$\tau_y \dot{y}(t) = \frac{1 + x^2 + \alpha\sigma x^4}{(1 + x^2 + \sigma x^4)(1 + y^4)} - \gamma_y y, \qquad (6)$$

The dimensionless system presented here describes the evolution of concentrations of the two constituent proteins $x(cI)$ and $y(lac)$, where the time scale for y is defined by τ_y, which at the same time is a design parameter (for detailed explanation of the system see [21]). α represents the degree to which the transcription rate is increased, σ is the affinity for a dimer, and γ_x and γ_y characterize the degradation rates. The noise term $n(t)$ models the contribution of random fluctuations and is a Gaussian white noise with zero mean. We assume that the deterministic equations provide a reasonable description of the system's dynamics, whereas the noise term represents the inevitable fluctuations in living systems. It is considered that the noise intensity D is rather small, not exceeding the order of 10^{-4}; hence, a sufficient motivation to use Gaussian noise and Langevin equations. The numerical integrations are performed using standard techniques for stochastic differential equations ([19]).

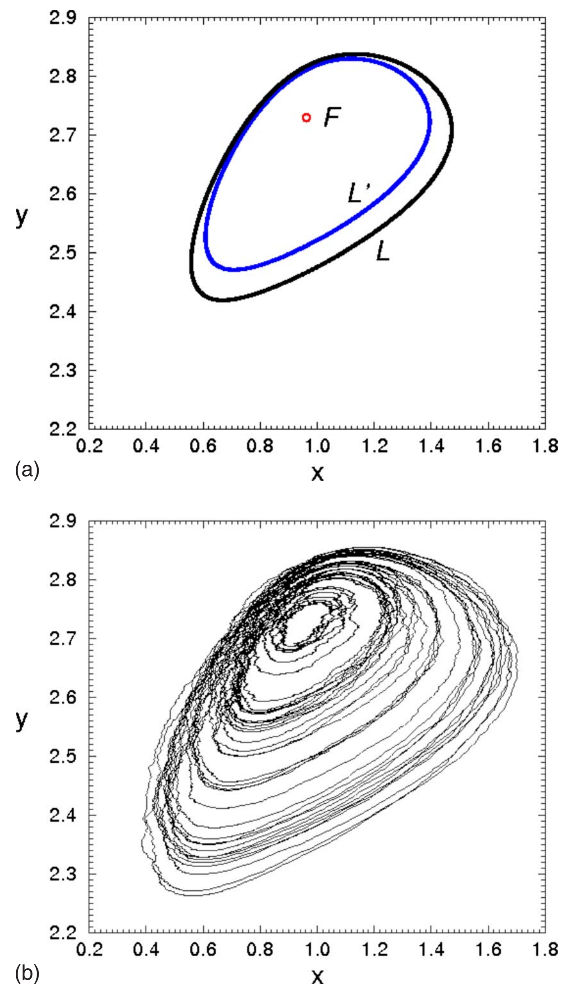

FIG. 3. (Color online) Phase trajectories of the system [Eq. (6)]: (a)—in the deterministic case ($D=0$) in the region of bistability for $\gamma_y \approx 0.037\,01$ (F—stable focus, L—stable limit cycle, L'—unstable limit cycle); (b)—in the presence of noise with intensity $D=0.000\,02$ near the border of bistability for $\gamma_y \approx 0.037\,02$.

The inherent stochasticity of biochemical processes, which depend on relatively infrequent molecular events involving a small number of molecules, is an essential source of internal noise in biochemical systems. Additionally, fluctuations originating from random variations of one or more externally set control parameters act as external noise, which makes the consideration of the effect of noise on the dynamic of genetic network unavoidable ([22,23]). It is noteworthy to mention that our goal here is to study general properties of bistable oscillators under the influence of noise. Therefore, we investigate here the simplest case where noise influences the dynamical behavior of the oscillator through one of the genes, although introducing a stochastic term to the second equation as well does not qualitatively change the obtained results (results not shown here).

In the deterministic case, for certain set of the parameters (see [21] for their values), bistability is observed in the system [Eq. (6)]: there are two attractors—a stable focus F and a stable limit cycle L, separated in the phase plane by an unstable limit cycle L' [Fig. 3(a)]. For $\gamma_y \approx 0.037\,01 \pm 10^{-6}$ a tangent bifurcation of cycles L and L' takes place. In

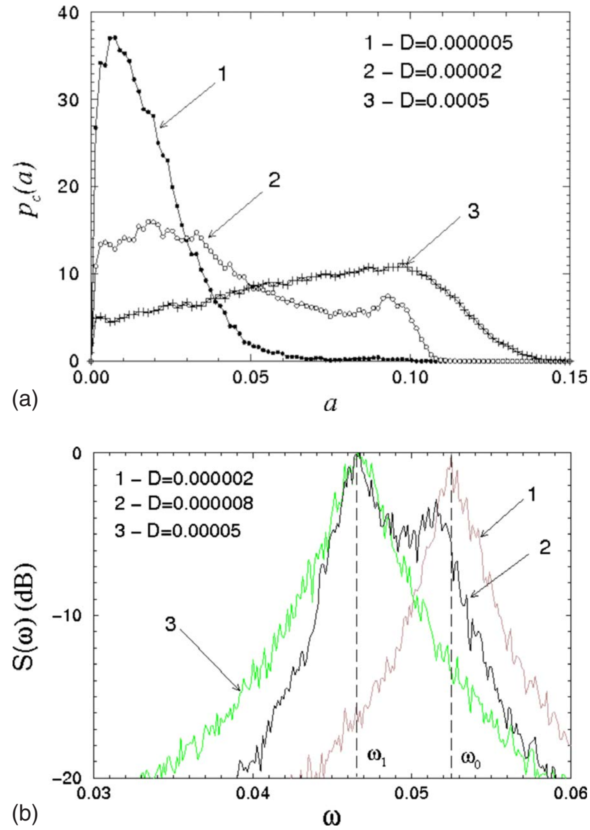

FIG. 4. (Color online) (a) Stationary amplitude distributions $p_c(a)$ (under the condition $\cos(\Phi(t)=0\pm0.001)$. (b) Normalized power spectra in system [Eq. (6)] for $\gamma_y=0.037\,02$ and different values of noise intensity D.

the calculations performed further, we have chosen $\gamma_y=0.037\,02$, at which in the deterministic system [Eq. (6)] there exists only one attractor, the stable focus F. In the stochastic case there is only one invariant set of trajectories in the phase space [Fig. 3(b)].

Since for this system [Eq. (6)] the instantaneous amplitude $a(t)$ depends on the rotation angle of the radius-vector connecting the state of equilibrium F with a point of the phase trajectory (i.e., on a reduced phase $\Phi(t)\in[0;2\pi]$), it is more appropriate to consider here a conditional amplitude distribution for a predetermined value of $\Phi(t)$, than the averaged value approach used before. Thus, we calculate now the stationary amplitude distributions $p_c(a)$, corresponding to the condition $\cos(\Phi(t))=0\pm0.001$ (i.e., $\Phi(t)\approx\pi/2$, where $\Phi(t)$ is measured with respect to x), and the power spectra of $x(t)$.

Figure 4(a) shows the stationary amplitude distributions $p_c(a)$ for different noise intensities. The obtained curves are not smooth because of statistical errors and uncertainty in defining the predetermined value of the phase $\Phi(t)$. The latter is particularly large for small amplitudes, explaining the fact that the obtained dependences $p_c(a)$ do not fall into the vicinity of zero when $a\to0$. Nevertheless, the distributions $p_c(a)$ are qualitatively similar to those obtained for oscillator [Eq. (1)] in the quasiharmonic approach. For increased noise intensity, we observe here again stochastic P-bifurcations: transition from a unimodal [curve 1 in Fig. 4(a)] to a bimodal distribution (curve 2) and back again to a unimodal one (curve 3).

Changes of the power spectrum for increasing noise intensity, obtained for [Eq. (6)], verifies the anisochronous character of oscillations. And instead of CR-like effect we observe here ASB [Fig. 4(b)]. The frequency of the oscillations in the vicinity of the unstable focus F is different from that in the region distant from it. In the case of small noise intensities, there is one maximum at frequency $\omega_0\approx0.052$ [curve 1 in Fig. 4(b)], characteristic for the rotation in the vicinity of the focus F. With the increase of D, a second maximum appears, with frequency $\omega_1\approx0.047$ corresponding to the rotation faraway from F, and close to the main frequency of the limit cycle L. If D is increased further, the trajectory spends progressively less time in the vicinity of F, the maximum at the frequency ω_0 gradually disappears, and the width of the remaining line at the frequency ω_1 grows [curves 2 and 3 in Fig. 4(b)]. Except for the disposition of the corresponding spectral lines ($\omega_0>\omega_1$), the evolution of the spectrum with the increase of the noise intensity is, in general very similar to the one observed for the anisochronous oscillator [Eq. (1)] [Fig. 2(b)].

IV. DISCUSSION

Studying oscillators in the regime of bistability, we find a strong sensitivity to the influence of additive noise. This becomes apparent through the occurrence of stochastic P-bifurcations, represented through a qualitative change of the stationary amplitude distribution. An analog to the deterministic bistability, in a noisy oscillator is a regime characterized by a bimodal amplitude distribution, which can be observed outside the region of the bistability of the deterministic system. Hence, stochastic P-bifurcations corresponding to the appearance and disappearance of one of the maxima of the distribution of the amplitudes $p(a)$, serve as borders of the bimodality region. In these investigations, the noise intensity represents a bifurcation parameter: very strong noise intensities cause the disappearance of the bimodal distribution region.

Additionally, the transformation of the distribution law connected with P-bifurcations results in a change of the power spectrum. In the isochronous oscillator, we observe a nonmonotone dependence of the spectral line width on the noise intensity, similar to what is observed for coherence resonance. However, if the oscillations are nonisochronous, there occurs a power redistribution between both lines in the spectrum and we cannot speak about a CR-like effect any longer.

The described effects of noise influence on a bistable oscillator are found for two essentially different models and were partially observed earlier experimentally [15]. Thus, we can state that the results are not model specific, but rather valid in general for bistable systems. Additionally, these effects are quite robust to the characteristics of additive noise: our numerical studies revealed P-bifurcations and a CR-like effect in the isochronous oscillator [Eq. (1)] in case of narrow-band noise with restricted amplitude. However this case requires further investigation.

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