# Biased percolation on scale-free networks

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Biased (degree-dependent) percolation was recently shown to provide strategies for turning robust networks fragile and vice versa. Here, we present more detailed results for biased edge percolation on scale-free networks. We assume a network in which the probability for an edge between nodes $i$ and $j$ to be retained is proportional to $(k_i k_j)^{-\alpha}$ with $k_i$ and $k_j$ the degrees of the nodes. We discuss two methods of network reconstruction, sequential and simultaneous, and investigate their properties by analytical and numerical means. The system is examined away from the percolation transition, where the size of the giant cluster is obtained, and close to the transition, where nonuniversal critical exponents are extracted using the generating-functions method. The theory is found to agree quite well with simulations. By presenting an extension of the Fortuin-Kasteleyn construction, we find that biased percolation is well-described by the $q \to 1$ limit of the $q$-state Potts model with inhomogeneous couplings.

## I. INTRODUCTION

In recent years, much attention has been devoted to the study of real-life networks. Such networks may be modeled by points or nodes connected by edges. One feature is the scale-free topology, described by a probability distribution $P(k)$ for the number of edges $k$ of a node, which falls off as a power law $k^{-\gamma}$ for large values of $k$. Most of the investigated cases turn out to have a topological exponent, or "degree exponent," $\gamma$, in the range $2 < \gamma < 3.5$. For $\gamma > 2$, the mean degree $\langle k \rangle$ is finite, and for $\gamma > 3$ also the variance $\langle k^2 \rangle$ is finite. Like fully random Poisson-distributed networks (with a typical scale), also scale-free networks are of "small-world" or even "ultrasmall-world" type [1]. By now, many properties have been revealed and investigated thoroughly: these include degree-degree correlations, clustering, and directedness of the edges in the network [2–5].

Another well-known property of scale-free networks is their resilience against random failure, a robustness caused by the presence of hubs (nodes with very high degree). On the other hand, these hubs may cause the network to be very vulnerable when a targeted attack is performed. In the limit of infinitely large networks, the network is said to be *robust* when even after removing an arbitrary fraction of the edges, there is still a nonzero probability that two randomly chosen nodes are part of a connected cluster. On the other hand, when removing edges from a *fragile* network, a point will be reached when the giant cluster, the one with a size comparable to the network size, is destroyed; this very point is called the percolation threshold. The percolation transition is a genuine phase transition and is normally of second order so that critical exponents can be properly defined [6–8].

Since the first studies of percolation on scale-free networks [9], a lot of work has been done on node percolation [6,9–13], bond percolation [12,14–16], percolation on multitype networks [14–17], clustered networks [18,19], correlated networks [7,10,18,20], directed networks [8,17,20],

degree-dependent edge percolation [14,15,21], and degree-dependent node percolation [13].

The percolation transition has many connections to real systems. For example, it can be related to disease propagation models [14,22–24]. In this analogy, the infection of an individual is represented by the activation of a node of the (social) network. When a giant cluster of active nodes emerges, an epidemic is established. Disease propagation on such networks can be efficiently suppressed by selective vaccination, depending, for example, on the connectedness of each node.

An alternative interpretation of a network with a certain fraction of deactivated edges is in terms of a transport network in which the edges transmit data or deliverables between nodes with a certain transmission probability. This probability depends in general on the degrees of the connected nodes. For example, communication with the highly connected hubs on the internet is in general more efficient. It is, however, also possible that nodes with more edges are less robust. Indeed, in more social terms, friendships involving people who have many acquaintances are more likely to end than friendships between people with few connections. Or, as another example, traffic on a network induces high loads on connections between hubs in the (transport) network, which in turn makes them more vulnerable to failure. Clearly, the resilience of an edge in a real network may depend strongly on the degrees of the nodes it connects.

We study the properties of a network after biased or degree-dependent edge removal. More specifically, we consider networks in which the edge between nodes $i$ and $j$ is *retained* with a probability proportional to its weight

$$w_{ij} = (k_i k_j)^{-\alpha}, \tag{1}$$

where $k_i$ and $k_j$ denote the degrees of nodes $i$ and $j$, respectively, and $\alpha$ is the "bias exponent." By tuning $\alpha$, we can explore three qualitatively different regimes: random failure ($\alpha = 0$), the attack of edges connected to hubs ($\alpha > 0$), and

the depreciation of edges between the least connected nodes ($\alpha < 0$). Henceforth, we call the regime $\alpha > 0$ "centrally biased" (CB). The converse regime, $\alpha < 0$, is termed "peripherally biased" (PB). The allowed range of the bias exponent is given by $\alpha \in [2 - \gamma, 1]$.

A degree-dependence similar to that in Eq. (1) has already been considered in Refs. [25,26], where Ising spin couplings $J_{ij}$ on scale-free networks were taken to be proportional to $w_{ij}$. The motivation for introducing degree-dependent couplings was the observation that for $\gamma \leq 3$ the system is always "ordered" (critical temperature $T_c = \infty$) due to the dominance of the hubs. However, degree-dependent couplings make it possible to compensate high degree with weak interaction (assuming $\alpha > 0$) so that the effect of the hubs can be neutralized. In doing so, it was observed that a network with "interaction exponent" $\alpha$ and degree exponent $\gamma$ has the same critical behavior as a network with interaction exponent zero (uniform couplings $J$) and degree exponent

$$\bar{\gamma} = \frac{\gamma - \alpha}{1 - \alpha}. \qquad (2)$$

In this way, it was possible to "trade interactions for topology" and study the rich mean-field critical behavior, with nonuniversal critical exponents depending on $\gamma$ [5], simply by varying $\alpha$ in a given network with fixed $\gamma$. The same exponent mapping will be recovered in this work in the following sense: at percolation the properties of a network with bias exponent $\alpha$ and degree exponent $\gamma$ are the same as those of a network with bias exponent zero and degree exponent $\bar{\gamma}$, *or* degree exponent $\gamma$, depending on conditions that will be specified.

The significance and potential usefulness of biased depreciation of a network is now becoming clearer. Indeed, it has been shown that networks with $\gamma > 3$ are *fragile* under random failure, while networks with $\gamma < 3$ are *robust* under random removal of edges or nodes [9]. If it should turn out, and under certain conditions this is what we find, that the depreciated network behaves as one in which $\gamma$ is replaced by $\bar{\gamma}$, it becomes possible to control the robustness or fragility of a network systematically by tuning the bias exponent $\alpha$. In other words, a network that is robust under random failure may turn out to be fragile under biased failure, and the other way round. Note that applying bias does not presuppose global knowledge about the network (location of the hubs…) but only requires local information on nodes and their degree.

The exponent equality can be intuitively understood from the following heuristic argument, which is safe to use provided $\alpha > 0$ and $k$ is sufficiently large. Using Eq. (1), one can anticipate that after depreciation of the network, a node with degree $k$ will, on average, have a new degree $\bar{k}$ proportional to $k^{1-\alpha}$. Since all nodes remain in place during the depreciation process, the original degree distribution $P(k)$ changes into a new distribution $\bar{P}(\bar{k})$ after depreciation, the relation between them being

$$P(k)dk = \bar{P}(\bar{k})d\bar{k}. \qquad (3)$$

Using $\bar{k} \propto k^{1-\alpha}$, one directly infers that indeed $\bar{P}(\bar{k}) \propto \bar{k}^{-\bar{\gamma}}$ and the network after depreciation thus acquires degree exponent

$\bar{\gamma}$ and the corresponding percolation properties. A more rigorous proof of this plausible expectation is given in the Appendix.

The paper is organized as follows: in Secs. II A and II B, we introduce random scale-free networks and present two distinct approaches by means of which a network can be reconstructed in a degree-dependent manner. Based on these schemes, we focus in Sec. II C on the degree distribution and the degree-degree correlations of the network after (partial) reconstruction. The percolation threshold is then extracted from these degree characteristics in Sec. III. The theory of generating functions for degree-dependent percolation on random networks will be extensively presented in Sec. IV. In Sec. V, the equivalence of our model with the Potts model is elaborated and using this equivalence and finite-size scaling theory, we arrive at the critical exponents for the percolation transition in Sec. VI. Finally, our results are extensively compared to simulational results in Sec. VII. Our conclusions are presented in Sec. VIII. A summary of part of the results presented here has been reported in Ref. [27].

## II. DEGREE-DEPENDENT PERCOLATION ON RANDOM GRAPHS

This section concerns (maximally) random scale-free networks. These are networks generated with the so-called configuration model, which assumes that the degrees of the nodes in the network are distributed according to a probability $P(k)$, which is taken to be the power law

$$P(k) = Ck^{-\gamma}, \qquad (4)$$

for values of $k$ between the minimal and maximal degrees $m$ and $K$, respectively. $C$ is the normalization constant. In order to ensure a finite mean degree, we take $\gamma > 2$. The graph is then completed by connecting the stubs emanating from all nodes. The probability $P_n(k)$ that a randomly chosen edge leads to a node of degree $k$ must, therefore, be

$$P_n(k) = \frac{kP(k)}{\langle k \rangle}, \qquad (5)$$

where $\langle \cdot \rangle$ denotes the average over the nodes, obtained using probability distribution $P(k)$. The probability distribution $P_n$ is also called the nearest-neighbor degree distribution. *Random networks* are constructed by connecting the earlier mentioned stubs randomly. We do not allow self-connections or multiple connections between nodes and use the method proposed in Ref. [28] to avoid degree-degree correlations in the network. To quantify degree-degree correlations, let us introduce the probability $P(k,q)$ that nodes of degree $k$ and $q$ are connected. If no correlations are present, $P(k,q)$ reduces to

$$P(k,q) = P_n(k)P_n(q) = \frac{kqP(k)P(q)}{\langle k \rangle^2}. \qquad (6)$$

Below and close to the critical point, large and random networks can locally be treated as trees and loops are sparse so that their effect can, to a good approximation, be ignored. The local treelike structure will be used in Sec. IV when the generating-functions method is presented.

We continue with presenting two distinct depreciation methods to study degree-dependent percolation. The statistical edge properties are now being considered, which will allow us in Sec. II C to obtain the statistics of nodal properties.

### A. Sequential approach

Our first method, which we call the *sequential approach*, starts from a random network with all $N$ node degrees distributed according to the degree distribution $P(k)$. Initially, all edges are removed and we aim at reintroducing a fraction $f$ of the total number of edges $N_e = \langle k \rangle N / 2$. This is achieved by activating one edge in each time step $t$. Consequently, the probability that the edge between nodes $i$ and $j$ is activated is $w_{ij}/Z_t$ where $Z_t$ is the sum of weights $w_{ij}$ of all nonactivated edges after $t-1$ steps. Thus, the probability $\rho_{ij}(f)$ that an edge between nodes $i$ and $j$ is again present after the reinclusion of a fraction $f$ of the edges, is [29]

$$\rho_{ij}(f) = 1 - \prod_{t=1}^{fN_e}\left(1 - \frac{w_{ij}}{Z_t}\right). \tag{7}$$

For sufficiently large networks, $w_{ij}/Z_t$ is typically small compared to one and Eq. (7) is well-approximated by

$$\rho_{ij}(f) \approx 1 - e^{-D_f w_{ij}}, \tag{8}$$

with the positive parameter $D_f = \sum_{t=1}^{fN_e} Z_t^{-1}$. It can be argued that for a sufficiently narrow distribution of the weights [30],

$$Z_t = \langle w \rangle_e (N_e - t + 1), \tag{9}$$

where $\langle \cdot \rangle_e$ denotes the average over all edges. The following property is readily derived using the weights, Eq. (1), and the distribution $P(k,q)$, Eq. (6):

$$\sqrt{\langle w \rangle_e} = \frac{\langle k^{1-\alpha} \rangle}{\langle k \rangle} = \frac{\gamma - 2}{\gamma - 2 + \alpha}m^{-\alpha}. \tag{10}$$

Using Eq. (9), $D_f$ can be determined, such that for large $N_e$,

$$D_f = -\ln[1-f]/\langle w \rangle_e, \tag{11}$$

and thus [31]

$$\rho_{ij}(f) = 1 - [1-f]^{w_{ij}/\langle w \rangle_e}. \tag{12}$$

It is instructive to consider a few asymptotic regimes of Eq. (12). First, in the case $\alpha = 0$, one recovers the expression for degree-independent percolation $\rho_{ij} = f$ as expected. Second, for arbitrary $\alpha$, we can distinguish the dilute limit and the dense limit in terms of $f$, and find

$$\rho_{ij} \sim fw_{ij}/\langle w \rangle_e \quad \text{when } f \to 0, \tag{13a}$$

$$\rho_{ij} \sim 1 \quad \text{when } f \to 1. \tag{13b}$$

Third, when $w_{ij}/\langle w \rangle_e \ll [-\ln(1-f)]^{-1}$

$$\rho_{ij} \sim -\ln(1-f)w_{ij}/\langle w \rangle_e. \tag{14}$$

We proceed by defining the marginal distribution $\rho_k$ as the mean probability that an edge connected to a node with degree $k$ is present in the network after reconstruction. Thus,

$$\rho_k = \sum_{q=m}^{K} P_n(q)\rho_{kq}, \tag{15}$$

where $\rho_{kq} = 1 - e^{-D_f w_{kq}}$. A good analytic approximation to $\rho_k$ can be obtained by substituting $\rho_{kq}$ into Eq. (15), considering an integral instead of a sum, taking the macroscopic limit $K \to \infty$, and expanding the exponential,

$$\rho_k = 1 - \sum_{n=0}^{\infty} \frac{(-)^n(D_f m^{-\alpha} k^{-\alpha})^n}{(1 + n\alpha/(\gamma - 2))n!}. \tag{16}$$

Alternatively, this result follows straightforwardly from the fact that the marginal distribution involves the incomplete Gamma function [31]. The usefulness of this explicit form can best be appreciated by first considering the range $0 < \alpha \ll \gamma - 2$, for which we obtain the simple analytic result

$$\rho_k \sim 1 - \exp\left\{-\frac{\gamma - 2}{\gamma - 2 + \alpha}D_f m^{-\alpha} k^{-\alpha}\right\}$$

$$= 1 - \exp\{-D_f \sqrt{\langle w \rangle_e} k^{-\alpha}\}. \tag{17}$$

Although this result is strictly only valid for the specified range of $\alpha$ specified above, numerical inspection shows that it is a rather good approximation to the integral representation of the sum Eq. (15) for a wider range of $\alpha$, *including negative values*. In fact, the result is useful in the entire interval of our interest $\alpha \in [2 - \gamma, 1]$. Using the previously obtained approximation to $D_f$, Eq. (11), it can be further simplified to

$$\rho_k \approx 1 - [1-f]^{k^{-\alpha}/\sqrt{\langle w \rangle_e}}. \tag{18}$$

Based on the asymptotic regimes of $\rho_{ij}$, it is also possible to extract the behavior of $\rho_k$. When $f \to 0$, we may use Eq. (13a)

$$\rho_k \sim k^{-\alpha}f/\sqrt{\langle w \rangle_e}. \tag{19}$$

On the other hand, when $\alpha > 0$ and $k \gg k_\times$, where the crossover value for $k$ is given by $k_\times \equiv D_f^{1/\alpha}/m$, we get

$$\rho_k \sim k^{-\alpha}D_f\sqrt{\langle w \rangle_e} \approx -k^{-\alpha}\ln(1-f)/\sqrt{\langle w \rangle_e}. \tag{20}$$

### B. Simultaneous approach

As an alternative to the sequential approach, we introduce now the simultaneous approach. Again, we start from a fully depreciated uncorrelated network with degree distribution $P(k)$. We then visit each edge (between nodes $i$ and $j$) once and activate this edge with probability

$$\rho_{ij} = fw_{ij}/\langle w \rangle_e. \tag{21}$$

In contrast to the sequential approach, $\rho_{ij}$ is now history-independent. Note also that $\langle \rho_{ij} \rangle_e = f$ as it must be. For the marginal distribution $\rho_k$, one finds

$$\rho_k = k^{-\alpha}f/\sqrt{\langle w \rangle_e}, \tag{22}$$

which satisfies $f\rho_{kq} = \rho_k\rho_q$ and is the same as in the $f \to 0$ limit of the sequential approach [Eq. (19)]. However, for each value of $k$ and $q$, the probability $\rho_{kq}$ must be less than or equal to one. This means that Eq. (21) is only well-defined for values of $f$ for which

$$f < f_u \equiv \left( \frac{\langle k^{1-\alpha} \rangle}{\langle k \rangle} \right)^2 \times [\min(m^\alpha, K^\alpha)]^2. \qquad (23)$$

It can be calculated that, in the macroscopic limit, the right-hand side (rhs) of Eq. (23) vanishes when $\alpha < 0$ and, therefore, the simultaneous approach is only meaningful for positive $\alpha$ and provided

$$f < \left( \frac{\gamma - 2}{\gamma - 2 + \alpha} \right)^2. \qquad (24)$$

To reach fractions above this limit in the simulations, we *iterate* the simultaneous approach. The first iteration involves the usual simultaneous approach with $f = f_u$; the second iteration is initialized by considering a new network consisting of all edges that have not been reintroduced during the first sweep. For that network, one calculates the probabilities $w_{ij}/Z_2$ and a new value of $f_u$, which is the minimum of the set $\{\langle w \rangle_e / w_{ij}\}$ where also the average is only over edges of the new network. One then applies the simultaneous approach until the new $f_u$ is reached, after which a third iteration can be initialized if necessary. Such iterations, however, introduce correlations and history-dependence. Note that the sequential approach can be seen as an extreme case of an iterated simultaneous approach in which only one edge is reconstructed in each iteration.

### C. Degree distribution and correlations of the reconstructed network

We now seek to obtain the degree distribution and characterize degree-degree correlations for the network after reconstruction. The following is valid for both the simultaneous and sequential approaches. Henceforth, we adopt the convention that an overbar indicates quantities in the diluted, or depreciated, network.

For the node degree distribution $\bar{P}(\bar{k})$ and the degree-degree correlations embodied in $\bar{P}(\bar{k}, \bar{q})$ of the depreciated network, we can write

$$\bar{P}(\bar{k}) = \sum_{k=\bar{k}}^{K} P(A_k \wedge B_{k \to \bar{k}}), \qquad (25a)$$

$$\bar{P}(\bar{k}, \bar{q}) = \sum_{k=\bar{k}}^{K} \sum_{q=\bar{q}}^{K} P(C_{qk} \wedge B_{q \to \bar{q}} \wedge B_{k \to \bar{k}} \wedge D)/f. \qquad (25b)$$

Here, we introduced the notation for events *A-D*,

(i) $A_k$: a randomly chosen node of the *original* network has degree $k$.

(ii) $B_{k \to \bar{k}}$: the degree of a node goes from $k$ in the *original* to $\bar{k}$ in the *depreciated* network.

(iii) $C_{qk}$: the nodes connected by a randomly chosen edge of the *original* network have degrees $q$ and $k$.

(iv) $D$: the chosen edge has not been removed from the *original* network.

For the node degree distribution, one readily finds

$$\bar{P}(\bar{k}) = \sum_{k=\bar{k}}^{K} P(k) \binom{k}{\bar{k}} \rho_k^{\bar{k}} (1 - \rho_k)^{k - \bar{k}}. \qquad (26)$$

For large values of $k$, i.e., $k \gg k_\times$, and $\alpha > 0$, the probability of retaining a node of degree $k$ falls off as $\rho_k \propto k^{-\alpha}$; this is valid using the sequential approach [see Eq. (20)], as well as the simultaneous one [see Eq. (22)]. Substituting this into Eq. (26) and approximating the binomial distribution in Eq. (26) by a normal distribution, one arrives at

$$\bar{P}(\bar{k}) \propto \bar{k}^{-\bar{\gamma}} \text{ for } \bar{k} \to \infty, \qquad (27)$$

where $\bar{\gamma}$ is defined in Eq. (3). This result, which is proven in the Appendix, confirms the validity of the expectation raised in the Introduction. Note that in case $\alpha = 0$, $\bar{P}(\bar{k}) \propto \bar{k}^{-\gamma}$ as it must be. We introduce now averaging over nodes of the reconstructed network

$$\langle\langle \cdot \rangle\rangle = \sum_{\bar{k}=0}^{K} \bar{P}(\bar{k}) \cdot . \qquad (28)$$

For further purposes, we calculate now the first and second moment of $\bar{P}(\bar{k})$ in terms of the moments of $P(k)$

$$\langle\langle \bar{k} \rangle\rangle = \langle k \rho_k \rangle = f \langle k \rangle, \qquad (29a)$$

$$\langle\langle \bar{k}^2 \rangle\rangle = \langle k \rho_k (k \rho_k - \rho_k + 1) \rangle. \qquad (29b)$$

The degree-degree correlations are embodied in Eq. (25b). This function can be further worked out to yield

$$\bar{P}(\bar{k}, \bar{q}) = \sum_{k=\bar{k}}^{K} \sum_{q=\bar{q}}^{K} P_n(q) P_n(k) \rho_{kq} \times \binom{q-1}{\bar{q}-1} \rho_q^{\bar{q}-1} (1 - \rho_q)^{q-\bar{q}}$$
$$\times \binom{k-1}{\bar{k}-1} \rho_k^{\bar{k}-1} (1 - \rho_k)^{k-\bar{k}} / f. \qquad (30)$$

This can be reduced to

$$\bar{P}(\bar{k}, \bar{q}) = \bar{k}\bar{q} \sum_{k=\bar{k}}^{K} \sum_{q=\bar{q}}^{K} \frac{P(q)P(k)}{(f\langle k \rangle)^2} P(B_{q \to \bar{q}}) P(B_{k \to \bar{k}}) \frac{f \rho_{kq}}{\rho_k \rho_q}. \qquad (31)$$

Equation (31) expresses the degree-degree correlations of a network after degree-dependent depreciation of a fully uncorrelated network. The question that can now be raised is when the depreciated network is also free of correlations, or, when is $\bar{P}(\bar{k}, \bar{q}) = \bar{P}_n(\bar{k})\bar{P}_n(\bar{q})$? It is readily checked that this is true, provided

$$f \rho_{kq} = \rho_k \rho_q. \qquad (32)$$

As Eq. (32) is valid for the simultaneous approach [see Eqs. (22) and (21)], no correlations appear in the reconstructed network (after a single iteration). For the sequential approach, on the other hand, Eq. (32) is generally not satisfied and the reconstructed network will be correlated.

The following limiting case of the *sequential* approach is interesting: take $\alpha$ positive and consider an edge between

two nodes of large degrees $k$ and $q$ such that $k \gg k_\times$ and $q \gg k_\times$. We may then substitute Eqs. (14) and (20) into Eq. (31). One soon arrives at the result

$$\frac{\bar{P}(\bar{k}, \bar{q})}{\bar{P}_n(\bar{k}) \bar{P}_n(\bar{q})} = -\frac{f}{\ln(1-f)}. \tag{33}$$

Since the rhs is smaller than one, this demonstrates that the sequential approach causes *disassortative mixing* in the depreciated network when $\alpha > 0$. In other words, nodes with large degrees tend to be connected to nodes with small degrees and vice versa. Using simulations, we will present evidence in Sec. VII that such correlations are introduced.

Finally, note also that Eq. (31) reduces to the correct nearest-neighbor degree distribution upon summing over $\bar{q}$

$$\bar{P}_n(\bar{k}) = \frac{\bar{k} \bar{P}(\bar{k})}{f\langle k \rangle}. \tag{34}$$

## III. PERCOLATION THRESHOLD FOR CENTRAL BIAS

Here, we focus solely on centrally biased depreciation ($\alpha > 0$) using the simultaneous approach. One may wonder what happens if centrally biased depreciation is applied to a robust network with $\gamma < 3$ such that the edges between and emanating from hubs are preferentially removed. Since, in that case, $\bar{\gamma} > \gamma$, one may speculate that a robust network may turn fragile and that the threshold for this to occur is $\bar{\gamma} = 3$ instead of the threshold $\gamma = 3$ valid for degree-independent percolation. We will address this question further and conclude that it is indeed so.

On the other hand, if we start from a fragile network ($\gamma > 3$) and apply CB, it is logical that the net remains fragile. Upon removing edges linked to hubs with a larger probability, we are more likely to destroy the coherence of the network. The question can then still be posed how much the percolation threshold of the reconstructed network is shifted.

Our first task now is to calculate the critical fraction at which the network becomes disconnected. According to Molloy and Reed [32], the critical fraction of a *random* network can be found by looking at the average nearest-neighbor distribution. If, upon following a random edge, the attained node has more than two neighbors, the network is said to be percolating, that is, a giant cluster will be present in the network. Note that this criterion is exact for the simultaneous but not for the sequential approach, due to the appearance of degree-degree correlations. In the reconstructed network, the Molloy-Reed criterion reads

$$1 = \frac{\langle\langle \bar{k}(\bar{k}-1) \rangle\rangle}{\langle\langle \bar{k} \rangle\rangle}, \tag{35}$$

or equivalently, using Eq. (29)

$$2\langle k\rho_k \rangle = \langle k\rho_k(k\rho_k - \rho_k + 1) \rangle. \tag{36}$$

Using Eq. (22) for the simultaneous approach, we find the following expression for the critical fraction $f_c$ at percolation [27]:

$$f_c = \frac{\langle k^{1-\alpha} \rangle^2}{\langle k \rangle (\langle k^{2-2\alpha} \rangle - \langle k^{1-2\alpha} \rangle)}. \tag{37}$$

For unbiased depreciation of the network ($\alpha = 0$), this last expression reduces to the well-known formula of random percolation on random networks [9].

Equation (37) allows us now to find out whether the network is robust, or in other words, whether $f_c \to 0$. This vanishing occurs, in the macroscopic limit, when the term $\langle k^{2-2\alpha} \rangle$ diverges and, therefore,

$$\bar{\gamma} \begin{cases} > 3: & \text{the network is fragile,} \\ < 3: & \text{the network is robust.} \end{cases} \tag{38}$$

The scaling relation of $f_c$ as a function of the network size can be found when $\bar{\gamma} < 3$; using $N \propto K^{\gamma-1}$, one finds

$$f_c \propto N^{(\bar{\gamma}-3)/(\bar{\gamma}-1)}. \tag{39}$$

Whether or not a network is robust for the degree-dependent attack is thus not solely a property of the network. Also the exponent $\alpha$ plays a crucial role in the arguments and its effect can be absorbed by using the exponent $\bar{\gamma}$ instead of the exponent before dilution, $\gamma$. In Sec. VI, we will take a closer look at the regime around the percolation threshold and we will find that the same mapping from $\gamma$ to $\bar{\gamma}$ is valid.

## IV. GENERATING-FUNCTIONS APPROACH

We now introduce the generating-functions approach for degree-dependent percolation. By this method, certain properties of the finite clusters in the network are easily obtained; this in turn allows to draw conclusions about the giant cluster. The method is exact if loops in the network can be ignored; since in the macroscopic limit, the average loop sizes in the finite clusters diverge [33], the method turns out to be exact. This will be apparent in Sec. VII when comparing the analytical results with simulations.

### A. Introduction

Generating functions are used in a wide branch of mathematical problems concerning series [34]. A generating function of a series is the power series, which has as coefficients the elements of the series. Applied to the context of percolation problems, this series is taken to be that of the discrete probability distributions characterizing the network under consideration [14,35]. We explain first the general formalism while closely following the approach of Newman [14], which we adapt for degree-dependent edge percolation [36].

The most fundamental generating function is the one that generates the degree distribution of the network

$$G_0(h) = \sum_{k=m}^{K} P(k) e^{-hk}. \tag{40}$$

We also define the generating function for the distribution of residual edges of a node reached upon following a random edge

$$G_1(h) = \sum_{k=m}^{K} P_n(k) e^{-h(k-1)}. \tag{41}$$

The exponent of $e^{-h}$ is $k-1$ because the edge, which is used to reach the node, is not counted.

There is a threefold advantage in working with the generating functions instead of working with the degree distribution itself. First, moments can be obtained easily from the generating functions. For instance, the average degree is given by

$$\langle k \rangle = \sum_{k=m}^{K} k P(k) = -G_0'(0), \tag{42}$$

where $G_0'$ denotes the derivative with respect to $h$. Higher moments can be obtained with higher-order derivatives.

Second, we can benefit from the so-called *powers property* of generating functions: if the distribution of a property $k$ of an object is generated by a function $G(h)$, then the generating function of the sum of $n$ independent realizations of $k$ is $G(h)^n$. For instance, if we randomly choose $n$ nodes in our network, the distribution of the sum of the degrees is generated by $G_0(h)^n$.

Third, the use of generating functions will allow us in Sec. V to highlight the equivalence with the $q \rightarrow 1$ limit of the $q$-state Potts model where the parameter $h$ will play the role of the magnetic field.

### B. Self-consistent equations

We can now define the equivalents of $G_0(h)$ and $G_1(h)$ for the network *after dilution* as $F_0(h)$ and $F_1(h)$

$$F_0(h) = \sum_{\bar{k}=0}^{K} \bar{P}(\bar{k}) e^{-h\bar{k}}, \tag{43a}$$

$$F_1(h) = \sum_{\bar{k}=0}^{K} \bar{P}_n(\bar{k}) e^{-h(\bar{k}-1)}. \tag{43b}$$

Note that the minimal degree in the network after dilution is zero instead of $m$. Equation (43) can be worked out further using Eqs. (26) and (34)

$$F_0(h) = \sum_{k=m}^{K} P(k)(1 - \rho_k + e^{-h}\rho_k)^k, \tag{44a}$$

$$F_1(h) = \sum_{k=m}^{K} \frac{\rho_k P_n(k)}{f}(1 - \rho_k + e^{-h}\rho_k)^{k-1}. \tag{44b}$$

The most interesting quantity for us is the size distribution of the finite clusters, the generating function of which can be readily derived using $F_0$ and $F_1$. Let $H_0$ denote the generating function for the probability that a randomly chosen node belongs to a connected cluster of a given (finite) size. Furthermore, let $H_1$ be the generating function for the probability that upon following a randomly chosen edge to one end, a cluster of a given (finite) size is reached. If the network can

be treated as a tree, these generating functions satisfy the following self-consistency equations [37]:

$$H_1(h) = e^{-h}F_1[H_1(h)], \tag{45a}$$

$$H_0(h) = e^{-h}F_0[H_1(h)]. \tag{45b}$$

Here, the function $F_{0,1}[H_1(h)]$ denotes the function $F_{0,1}$ in which $e^{-h}$ is replaced by $H_1(h)$. The proof of these relations relies on the aforementioned powers property and is expounded in Ref. [14]. The percolation threshold can now be derived with the aid of these functions.

Several macroscopic quantities can be easily identified in the *depreciated* network [5]. For example, we define $\mathcal{P}_\infty$ as the probability that a node belongs to the giant cluster, $\mathcal{L}_\infty$ as the edge probability for being in the giant cluster and $\mathcal{S}$ as the average cluster size of finite clusters

$$\mathcal{P}_\infty = 1 - H_0(0), \tag{46a}$$

$$\mathcal{L}_\infty = 1 - [H_1(0)]^2, \tag{46b}$$

$$\mathcal{S} = -H_0'(0). \tag{46c}$$

Moreover, the degree distribution of nodes in the giant cluster varies as

$$\bar{P}_{gc}(\bar{k}) \propto \{1 - [H_1(0)]^{\bar{k}}\}\bar{P}(\bar{k}), \tag{47}$$

from which it follows that the degree distribution of the finite clusters varies as $\bar{P}_{fc}(\bar{k}) \propto [H_1(0)]^{\bar{k}}\bar{P}(\bar{k})$. In case there are both finite clusters and a giant cluster, the degree distributions have the asymptotic behavior $(\bar{k} \rightarrow \infty)$

$$\bar{P}_{gc}(\bar{k}) \sim \bar{k}^{-\bar{\gamma}}, \tag{48a}$$

$$\bar{P}_{fc}(\bar{k}) \sim e^{-\bar{k}/\lambda}, \tag{48b}$$

with $\lambda = -\ln[H_1(0)]$. In other words, in the presence of a giant cluster, only the degree distribution of the giant cluster falls off with a power law with exponent $\bar{\gamma}$. The average cluster size (46c) in the diluted network, on the other hand, can be further worked out by differentiating Eqs. (45) with respect to $x$

$$\mathcal{S} = 1 + \frac{f\langle k \rangle}{1 + F_1'(0)}. \tag{49}$$

Hence, the average cluster size diverges when

$$1 = -F_1'(0). \tag{50}$$

This is yet another way of writing the Molloy-Reed criterion Eq. (35).

### C. Full derivation of self-consistent equations

We prove now that the self-consistent Eqs. (45) are only valid in case no correlations are introduced in the reconstructed network, or, when $f\rho_{kq} = \rho_k\rho_q$, as is valid for the simultaneous method only. Here we will give a precise deri-

vation of the self-consistent Eqs. (45), thereby taking into account the degree-dependence of the functions $H_1$.

Let us first look at the generating function $\hat{H}_1^{q \to k}(h)$ for the probability that an edge, which connects nodes of degree $q$ and $k$, branches out in a cluster of a given edge number along the node of degree $k$. It is readily derived that $\hat{H}_1^{q \to k}$ satisfies the equation

$$\hat{H}_1^{q \to k}(h) = e^{-h} \left( 1 + \sum_{\hat{k}=m}^{K} P_n(\hat{k}) \rho_{k\hat{k}} [\hat{H}_1^{k \to \hat{k}}(h) - 1] \right)^{k-1}.$$
(51)

We proceed by defining $H_1^q(h) = \sum_k P_n(k) \rho_{qk} \hat{H}_1^{q \to k}(h) / \rho_q$, such that we arrive at a set of self-consistent equations for each value of $q$

$$H_1^q(h) = e^{-h} \sum_{k=m}^{K} \frac{P_n(k) \rho_{qk}}{\rho_q} (1 + \rho_k [H_1^k(h) - 1])^{k-1}.$$
(52)

Note that, as derived in Sec. II C, no correlations are induced during depreciation when $f\rho_{kq} = \rho_k \rho_q$. In that case, this equation reduces to Eq. (45a). After solving Eq. (52) with respect to $H_1^k$ for all values of $k$, we can also calculate

$$H_0(h) = \sum_{k=m}^{K} e^{-h} P(k) \{ 1 + \rho_k [H_1^k(h) - 1] \}^k.$$
(53)

Again, this expression reduces to Eq. (45b) in case of correlation-free depreciation when $H_1^k$ is independent of $k$.

Below the percolation transition, a trivial solution exists: $H_1^k(0) = 1$. This solution, however, turns unstable at the percolation threshold. The threshold value may be derived by linearization of $H_1^k(0)$ around its equilibrium value: $H_1^k(0) = 1 - \varepsilon_k$ with $\varepsilon_k \ll 1$. The percolation criterion is then

$$\varepsilon_q = \sum_{k=m}^{K} \frac{P_n(k) \rho_{qk} \rho_k}{\rho_q} (k-1) \varepsilon_k.$$
(54)

Again, in absence of correlations in the network, that is, when the criterion $f\rho_{qk} = \rho_k \rho_q$ is satisfied, this reduces to the earlier encountered Molloy-Reed criterion of Eq. (36). Equation (54) is the criterion for the percolation threshold for correlated systems, such as the one created using the sequential method. However, solving Eq. (54) to obtain $f_c$ constitutes a rather difficult task.

### D. Original and depreciated network

We show now that another approach exists by which one easily derives the self-consistent equations characterizing the network. This method is closer to the one followed in other works.

Let us call $R^{i \to j}$ the probability that an edge in the network does not lead to a vertex connected via the remaining edges to the giant component (infinite cluster) and $\rho_{ij}$ the probability that the edge between nodes $i$ and $j$ is active. Then, following the edge along node $j$, one finds:

$$R^{i \to j} = 1 - \rho_{ij} + \rho_{ij} \prod_{z=1,\dots,k_j-1} R^{j \to z}.$$
(55)

This type of equation was already obtained for the $q \to 1$ limit of the $q$-state Potts model [38]. Using the tree approximation, we can rewrite everything as a function of the degrees of the nodes

$$R^{q \to k} = 1 - \rho_{qk} + \rho_{qk} \left[ \sum_{\hat{k}=m}^{K} P_n(\hat{k}) R^{k \to \hat{k}} \right]^{k-1}.$$
(56)

This self-consistent set of equations is equivalent to the ones that we obtained in Eq. (52). Indeed, after the transformation $R^{q \to k} - 1 = \rho_{qk}(\hat{R}^{q \to k} - 1)$, we find

$$\hat{R}^{q \to k} = \left[ 1 + \sum_{\hat{k}=m}^{K} P_n(\hat{k}) \rho_{k\hat{k}} (\hat{R}^{k \to \hat{k}} - 1) \right]^{k-1},$$
(57)

which is exactly the same as Eq. (51) for $h=0$.

The purpose of this derivation is to show that our self-consistent Eq. (51) is in agreement with Eq. (56). For degree-independent $R$ and $\rho$, an equation similar to Eq. (56) appears frequently in the literature. The difference between Eq. (51) and Eq. (57) is that $\hat{H}_1$ is normalized with respect to the depreciated network, whereas $\hat{R}$ is normalized with respect to the original network.

## V. EQUIVALENCE WITH THE POTTS MODEL

There exists an equivalence between edge percolation and the $q \to 1$ limit of the $q$-state Potts model [39]. This connection was first worked out by Fortuin and Kasteleyn in Ref. [40]. Although initially used for lattice models, the connection was very general and is valid for any network [21,40]. Moreover, their proof can easily be generalized to incorporate edge-dependent coupling constants and edge-dependent removal into the Potts model and the percolation model, respectively. We explain here in more detail this equivalence and reformulate our percolation problem as a spinlike problem, which will allow us to derive critical exponents and compare them with the ones obtained for the Potts model. We will also find support for our simple scaling relation using exponent $\bar{\gamma}$ [Eq. (3)], as was already encountered in studies concerning degree-dependent Ising interactions on scale-free networks [25,26]. Note that the Ising model [41,42] and the Potts model [38,43], together with their critical properties, were already studied on scale-free networks.

The Potts model can be seen as a generalization of the Ising model in which each site $i$ has a spin $\sigma_i$. In the Potts model, these spins can take $q$ distinct values $0, \dots, q-1$ and the Potts Hamiltonian is

$$\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} \delta_{\sigma_i, \sigma_j} - h k_B T \sum_i \delta_{\sigma_i, 0}.$$
(58)

Here, $\langle ij \rangle$ indicates nearest-neighbor sites $i$ and $j$, $J_{ij}$ is the coupling constant and $\delta$ the Kronecker delta function. Note that the Ising model corresponds to the $q=2$ Potts model. The Fortuin-Kasteleyn theorem states now that the free en-

ergy of the $q \to 1$ limit of the $q$-state Potts model is the same as the "free energy" of the percolating network where the latter is the generating function of the cluster size distribution function $n_s$

$$\mathcal{F}(f,h) = \left\langle \sum_s n_s e^{-hs} \right\rangle. \tag{59}$$

Here, the average is performed over all networks in which the probability to retain the edge between nodes $i$ and $j$ is $\rho_{ij}$. The parameter $\rho_{ij}$ in the percolation problem corresponds in the following way to parameters of the Potts model [44,45]

$$\rho_{ij} \quad \leftrightarrow \quad 1 - e^{-J_{ij}/k_B T}. $$

From this relation, we can immediately identify the probability $\mathcal{P}_\infty$ for a node to be in the infinite cluster and the average cluster size $\mathcal{S}$, earlier introduced in Eq. (46). As we are interested in the behavior near criticality, we introduce

$$\epsilon = f - f_c, \tag{60}$$

and obtain

$$\mathcal{P}_\infty(\epsilon) = 1 + \left. \frac{\partial \mathcal{F}}{\partial h} \right|_{h=0}, \tag{61a}$$

$$\mathcal{S}(\epsilon) = \left. \frac{\partial^2 \mathcal{F}}{\partial h^2} \right|_{h=0}. \tag{61b}$$

Note also that $\mathcal{F}(\epsilon,0)$ gives the total number of finite clusters.

## VI. SCALING THEORY AND CRITICAL EXPONENTS

In the following section, we introduce finite-size scaling in order to find critical exponents near the percolation transition. In order to solve the scaling relation, we use a Landau-like theory which we derive from the exact relations (45). We follow closely the approach presented in Refs. [46–48] for finite-size scaling in systems with dimensions above the upper critical dimension. However, we will find that the forms of the Landau-like theories of Refs. [43,48] were too limited for studying the percolation transition in case the distribution function has a very fat tail, that is when $2 < \gamma < 3$.

According to finite-size scaling, the free energy $\mathcal{F}$ of a large but finite network with $N$ nodes close to criticality can be written in the general form [49]

$$\mathcal{F}(\epsilon,h) = N^{-1}\mathbb{F}(\epsilon N^{1/\nu_\epsilon}, h N^{1/\nu_h}), \tag{62}$$

where $\mathbb{F}$ is a well-behaved function. The variable $\epsilon N^{1/\nu_\epsilon}$ originates from the existence of a "correlation number" $N_\xi$ (instead of a correlation length), which scales as $N_\xi \propto \epsilon^{-\nu_\epsilon}$ such that the first variable of $\mathbb{F}$ can be rewritten as $(N/N_\xi)^{1/\nu_\epsilon}$ [46,47]. It is then obvious that close to criticality, the (singular part of the) free energy scales as

$$\mathcal{F}(\epsilon,0) \propto \epsilon^{\nu_\epsilon}, \tag{63a}$$

$$\mathcal{F}(0,h) \propto h^{\nu_h}. \tag{63b}$$

As a second scaling ansatz, we assume that, in the macroscopic limit, the scaling of the cluster size distribution $n_s$ can be written as [50]

$$n_s(\epsilon) = s^{-\tau}\mathbb{G}(\epsilon s^\sigma), \tag{64}$$

where again $\mathbb{G}$ is a well-behaved function. The form of $n_s$ one usually has in mind is $n_s \propto s^{-\tau} e^{-s/s^*}$ [14], which is essentially a damped power law with cutoff $s^*$, valid for large cluster sizes. At criticality, very large clusters arise, caused by the diverging cutoff $s^*$ according to $s^* \propto \epsilon^{-1/\sigma}$ such that $n_s(0) \sim s^{-\tau}$.

Using the analogy with the Potts model, we define the usual critical exponents $\alpha$, $\gamma_p$, $\beta$, and $\delta$ for the percolation problem as

$$\mathcal{F}(\epsilon,0) \sim \epsilon^{2-\alpha}, \tag{65a}$$

$$\mathcal{P}_\infty(\epsilon) \sim \epsilon^\beta, \tag{65b}$$

$$\mathcal{S}(\epsilon) \sim \epsilon^{-\gamma_p}, \tag{65c}$$

$$\left. \frac{\partial \mathcal{F}}{\partial h} \right|_{\epsilon=0} + 1 \sim h^{1/\delta}. \tag{65d}$$

Using the scaling forms of Eqs. (62) and (64), standard techniques provide us with exponent relations by which all critical exponents can be related to $\nu_h$ and $\nu_\epsilon$. One arrives at [49]

$$\beta = \nu_\epsilon(1 - \nu_h^{-1}), \tag{66a}$$

$$\gamma_p = \nu_\epsilon(2\nu_h^{-1} - 1), \tag{66b}$$

$$\alpha = 2 - 2\beta - \gamma_p, \tag{66c}$$

$$\sigma = (\beta + \gamma_p)^{-1}, \tag{66d}$$

$$\tau = 2 + \beta(\beta + \gamma_p)^{-1}, \tag{66e}$$

$$\delta = (\beta + \gamma_p)/\beta, \tag{66f}$$

$$\nu_f = \beta + \gamma_p. \tag{66g}$$

The last exponent $\nu_f$ is related to the usual fractal dimension $d_f$ in the same way that $\nu_\epsilon$ is related to the usual dimension $d$ and quantifies how the cluster size $s$ scales with $\epsilon$ close to criticality, i.e., $s \propto \epsilon^{-\nu_f}$. Note that the cutoff size $s^*$ scales like $s$, since $\nu_f = 1/\sigma$.

The problem we are left with now is to find the scaling exponents $\nu_h$ and $\nu_\epsilon$ for percolation on scale-free networks. This can be done in an exact way since we know the equation of state from Eq. (45) as a function of the order parameter

$$\psi(\epsilon,h) = 1 - H_1(\epsilon,h). \tag{67}$$

As we are merely interested in the behavior near the transition where $\epsilon \ll 1$, $h \ll 1$ and $\psi \ll 1$, we can expand Eq. (45a). For the case $\bar{\gamma} > 3$, we find the form [6]

TABLE I. Critical Exponents

| | $2 < \bar{\gamma} < 3$ | $3 < \bar{\gamma} < 4$ | $\bar{\gamma} > 4$ |
|---|---|---|---|
| $\beta$ | $\dfrac{1}{3 - \bar{\gamma}}$ | $\dfrac{1}{\bar{\gamma} - 3}$ | $1$ |
| $\tau$ | $\dfrac{2\bar{\gamma} - 3}{\bar{\gamma} - 2}$ | $\dfrac{2\bar{\gamma} - 3}{\bar{\gamma} - 2}$ | $5/2$ |
| $\sigma$ | $\dfrac{3 - \bar{\gamma}}{\bar{\gamma} - 2}$ | $\dfrac{\bar{\gamma} - 3}{\bar{\gamma} - 2}$ | $1/2$ |
| $\alpha$ | $-\dfrac{3\bar{\gamma} - 7}{3 - \bar{\gamma}}$ | $-\dfrac{5 - \bar{\gamma}}{\bar{\gamma} - 3}$ | $-1$ |
| $\gamma_p$ | $-1$ | $1$ | $1$ |
| $\delta$ | $\bar{\gamma} - 2$ | $\bar{\gamma} - 2$ | $2$ |
| $\nu_f$ | $\dfrac{\bar{\gamma} - 2}{3 - \bar{\gamma}}$ | $\dfrac{\bar{\gamma} - 2}{\bar{\gamma} - 3}$ | $2$ |

$$h = -c_1 \epsilon \psi + c_2 \psi^2 + \ldots + c_s \psi^{\bar{\gamma} - 2} + \ldots, \tag{68}$$

in which all $c_i$ as well as the coefficient of the singular term, $c_s$, are positive constants. This equation also follows from minimization of the free energy:

$$\mathcal{F}(\epsilon, h) \propto -h\psi - \bar{c}_1 \epsilon \psi^2 + \bar{c}_2 \psi^3 + \ldots + \bar{c}_s \psi^{\bar{\gamma} - 1} + \ldots, \tag{69}$$

with respect to the order parameter $\psi$.

We distinguish two cases now. First, when $4 < \bar{\gamma}$, we know that $f_c$ is finite and the relevant part of the equation of state for $\psi$ becomes

$$h = -c_1 \epsilon \psi + c_2 \psi^2. \tag{70}$$

Solving for $\psi$, and substitution into Eq. (69) one then simply finds that the free energy scales as

$$\mathcal{F}(\epsilon, 0) \propto \epsilon^3, \tag{71a}$$

$$\mathcal{F}(0, h) \propto h^{3/2}. \tag{71b}$$

In other words, when $4 < \bar{\gamma}$, we find that $\nu_\epsilon = 3$ and $\nu_h = 3/2$. From these two exponents, and using Eq. (66), we list all other exponents in the last column of Table I. Note that, as expected, these exponents agree with the usual mean-field results for percolation [35,49].

Second, when $3 < \bar{\gamma} < 4$, the relevant part of Eq. (68) reduces to

$$h = -c_1 \epsilon \psi + c_s \psi^{\bar{\gamma} - 2}, \tag{72}$$

from which follows that

$$\mathcal{F}(\epsilon, 0) \sim \epsilon^{(\bar{\gamma} - 1)/(\bar{\gamma} - 3)}, \tag{73a}$$

$$\mathcal{F}(0, h) \sim h^{(\bar{\gamma} - 1)/(\bar{\gamma} - 2)}. \tag{73b}$$

Therefore, we come to

$$\nu_\epsilon = \frac{\bar{\gamma} - 1}{\bar{\gamma} - 3} \quad \text{and} \quad \nu_h = \frac{\bar{\gamma} - 1}{\bar{\gamma} - 2}, \tag{74}$$

and we obtain all other exponents as given in the second column of Table I.

Lastly, in case $2 < \bar{\gamma} < 3$, the critical fraction $f_c = 0$. Again, we expand Eq. (45a) using the small parameters $\epsilon \ll 1$, $|h| \ll 1$, and $\psi \ll 1$. The equation of state for $\psi$ and the associated free energy (69) become

$$h = -c_s(\epsilon \psi)^{\bar{\gamma} - 2} + \psi + \ldots, \tag{75a}$$

$$\mathcal{F}(\epsilon, h) \propto \epsilon(-h\psi + \bar{c}_s \epsilon^{\bar{\gamma} - 2} \psi^{\bar{\gamma} - 1} + \psi^2/2) + \ldots, \tag{75b}$$

where $c_s > 0$. It follows that

$$\mathcal{F}(\epsilon, 0) \sim \epsilon^{(\bar{\gamma} - 1)/(3 - \bar{\gamma})}, \tag{76a}$$

$$\mathcal{F}(\epsilon, h)|_{\epsilon \to 0} \sim |h|^{(\bar{\gamma} - 1)/(\bar{\gamma} - 2)}. \tag{76b}$$

In the last expression, the limit $\epsilon \to 0$ is only taken in the free energy and $h$ is taken small and negative such that $\psi$ is still positive. Therefore,

$$\nu_\epsilon = \frac{\bar{\gamma} - 1}{3 - \bar{\gamma}} \quad \text{and} \quad \nu_h = \frac{\bar{\gamma} - 1}{\bar{\gamma} - 2}. \tag{77}$$

The other exponents are listed in the first column of Table I. It must be noted here that in practice, the exponents of Eq. (77) may be impossible to find with the configurational model in case we start from a robust network ($2 < \gamma < 3$). This stems from the fact that a structural cutoff for the maximally allowed degree $K$ must be introduced to obtain an uncorrelated network. Such cutoff can be of the form $K \sim N^{1/\omega}$ with $\omega \in [2, \infty[$. However, it is well-known that the cutoff affects the critical exponents [51]. Indeed, performing the averages in Eq. (37) with use of the cutoff, one readily obtains $f_c \propto N^{(1 - \alpha)(\bar{\gamma} - 3)/\omega}$ [cf. Equation (39)] and, therefore,

$$\nu_\epsilon = -\frac{\omega}{(1 - \alpha)(\bar{\gamma} - 3)}. \tag{78}$$

In case we start from a fragile network ($\gamma > 3$), $\omega$ equals $\gamma - 1$ and the $\nu_\epsilon$ of Eq. (77) is retrieved. Therefore, degree-dependent edge removal may be used as a tool to observe critical exponents in the delicate regime $2 < \bar{\gamma} < 3$, *without* the use of a structural cutoff [51]. In our simulations as presented in Sec. VII, we take $\omega = 2$ in case $\gamma < 3$.

The exponents obtained for the case $\bar{\gamma} > 3$ reduce to the ones for *node* percolation in the limit $\alpha = 0$ when $\bar{\gamma} = \gamma$ [6]. However, in the limit $\alpha = 0$ the exponents in the first column do not coincide with those given in Refs. [6,52].

Close to the percolation transition, it is possible, with the use of Eqs. (46) to calculate the average degree in the giant cluster. We can identify this as

$$\lim_{f \to f_c} \frac{f N_e (1 - (H_1)^2)}{N(1 - H_0)} = \frac{(f_c + \epsilon)\langle k \rangle 2N}{N\langle k \rangle (f_c + \epsilon)} = 2. \tag{79}$$

This, however, must not be confused with the Molloy-Reed criterion, which states that the average degree of a *neighboring site in the entire network* has degree two.
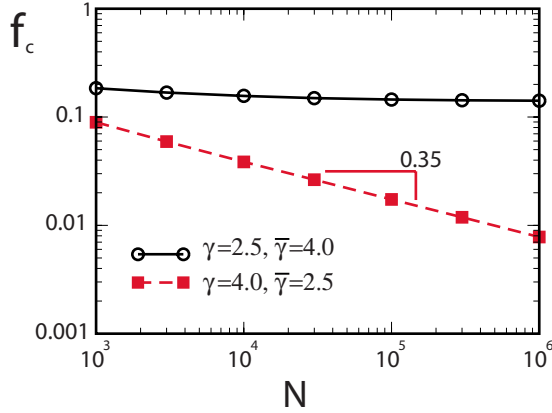
FIG. 1. (Color online) The critical fraction at percolation $f_c$ as a function of the network size $N$. To compute this critical fraction we average over $10^4$ network realizations for each set of parameters and apply for each network the percolation process 100 times. No cutoff was introduced for the maximal node degree and network reconstruction was done with the sequential approach.

In sum, we have now calculated the most important critical exponents for a percolation process. Extra support for our critical exponents comes from scaling relations and the connection with the Potts model.

There is one feature that appears in all the calculated exponents: the only dependence on $\alpha$ arises through the exponent $\bar{\gamma}$. Random percolation on a network with degree exponent $\bar{\gamma}$ gives the same critical exponents as percolation with bias exponent $\alpha$ on a network with exponent $\gamma$. This equivalence was found before for degree-dependent interactions on scale-free networks [25,26]. It is not surprising that the same behavior appears both for edge percolation and for degree-dependent interactions, since both can be linked with the Fortuin-Kasteleyn construction.

## VII. COMPARISON WITH NUMERICAL RESULTS

In this section, we test the previously derived analytical results using simulations. The networks are generated using the uncorrelated configurational model which was introduced in Sec. II. Each simulation involves three free parameters: the degree exponent $\gamma$ of the network, the minimal node degree $m$ and the number of vertices $N$. Unless mentioned otherwise, we set $m=1$. In the configuration model, the degrees of the nodes are determined initially from the discrete degree distribution [53] and then the connections are assigned at random. To obtain an uncorrelated network, the maximal degree is set to $\sqrt{N}$ when $2 < \gamma < 3$ [28]. In some simulations (see Figs. 1 and 2) no degree cutoff was imposed. If $\gamma \geq 3$, the maximal degree is simply $N-1$. Both the sequential and the simultaneous approach are implemented.

### A. Sequential approach

In this section, we discuss simulation results concerning the sequential approach. Most of these results can also be found in Ref. [27].
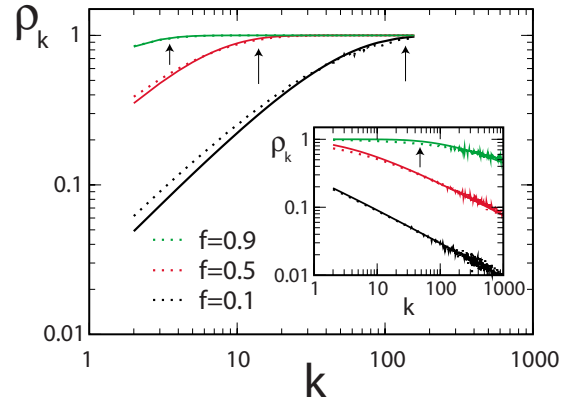


FIG. 2. (Color online) The probability $\rho_k$ to retain a node after depreciation [see Eq. (15)] as a function of its original degree $k$ using the sequential approach. The main panel shows results for networks with $\gamma=4$ submitted to PB with $\alpha=-1$. The dots indicate the simulation results for ten network realizations and ten percolation routines. The continuous lines are the best fit to the data of Eq. (17). The inset shows the same but for a network with $\gamma=2.5$ subjected to CB with $\alpha=0.5$. The arrows indicate the crossover value $k_\times$. The continuous lines are the best fits to the data of Eq. (17). In both cases, $m=2$ and $N=10^5$. No cutoff was introduced for the maximal node degree. From top to bottom $f=0.9$, 0.5, 0.1.

### 1. Scaling of the critical point

First, we investigate the finite-size behavior of the critical fraction $f_c$ of nodes. The results for the sequential approach are shown in Fig. 1. For networks with $\gamma=2.5$ submitted to CB with an effective value $\bar{\gamma}=4$ (continuous black line), we observe that the critical fraction $f_c$ converges to a finite value as $N$ grows, confirming the conjecture that a robust network may turn fragile under CB. In the opposite case, a network with $\gamma=4$ submitted to PB with an effective $\bar{\gamma}=2.5$ (dashed red line), has a critical fraction that decays with the vertex number $N$ as a power law, $f_c \sim N^{-1/\nu_\epsilon}$. The best fit to the data in this case results in $1/\nu_\epsilon=0.35 \pm 0.02$, consistent with the value 1/3 expected from Eq. (39). This result shows that a fragile network under PB will behave in the same fashion as a robust network with a degree distribution controlled by $\bar{\gamma}$ under random failure [54]. Note that this simulation result confirms Eq. (39), although this equation was derived for the simultaneous approach. Indeed, to deduce Eq. (39), correlations in the diluted network are neglected. These simulation results indicate that this is an acceptable approximation.

### 2. Properties of the diluted network

During our theoretical discussion of the sequential approach, certain properties of the diluted network became apparent, such as a cross-over behavior as a function of the degree $k$ for the mean edge preservation probability $\rho_k$. The probability $\rho_k$ can easily be inferred from our simulations by calculating the ratio of the new node degree $\bar{k}$ to the old node degree $k$ for each node and averaging over all nodes with the same degree. The result is shown in Fig. 2 in which the expected crossover behaviors are marked by arrows and the continuous lines are fits to the data of Eq. (17). When CB is
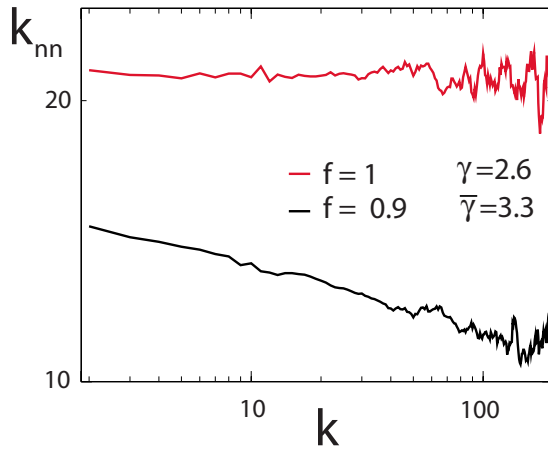
FIG. 3. (Color online) Mean nearest-neighbor degree $k_{nn}$ as a function of the degree of a node $k$, both for the original network (upper, red line) and a network where only a fraction $f=0.9$ of the links is present (lower, black line). Clearly correlations which give rise to disassortative mixing emerge in the depreciated network. We used an original network constructed with the uncorrelated configuration model with $\gamma=2.6$, $m=2$, and $N=10^5$, which is diluted using sequential biased percolation with $\alpha=0.3$, such that $\bar{\gamma}=3.3$. The result was obtained with two network realizations, on both of them the percolation process was applied four times. To reduce the noise level, the mean-neighbor degree is averaged over eight successive values.

applied (inset), a crossover between a regime with $\rho_k \approx 1$ to a decreasing power law is found.

As a second characteristic of diluted networks, the emergence of correlations in the diluted network is discussed. The theoretical result of Eq. (33) suggests disassortative mixing in the diluted network in case central bias is applied. To observe these correlations in the simulations, the mean nearest-neighbor degree is calculated as a function of the node degree. The result of our simulation is shown in Fig. 3. As expected, no correlations are present in the original network (top curve in red, $f=1$). However, with 10% of its edges removed, the mean nearest-neighbor degree in the diluted net clearly decreases as the degree of the node increases (bottom curve in black). This is a clear indication of disassortative mixing after sequentially removing a certain fraction of edges and this confirms our theoretical prediction.

### B. Simultaneous approach

#### 1. Comparison with theory and sequential approach

This section deals with the iterated simultaneous approach as introduced at the end of Sec. II B. To examine the percolation transition, we search for the probability to belong to the largest cluster, $\mathcal{P}_\infty$, as a function of the fraction $f$ of included edges. Results are given in Figs. 4–6.

For random edge removal, it is shown in Fig. 4 that the simultaneous approach coincides with the sequential approach; in that case, only one iteration is necessary to attain $f=1$. When, on the other hand, $\alpha>0$, the simultaneous approach can only be used up to a certain value of $f_u$, smaller than one [see Eq. (24)]. However, the iterative procedure can
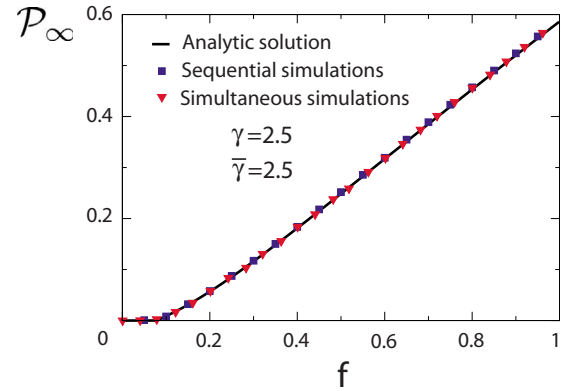


FIG. 4. (Color online) The probability that a node belongs to the giant cluster, $\mathcal{P}_\infty$, as a function of the fraction of retained links $f$ for unbiased percolation ($\alpha=0$). We compare the sequential approach simulations (blue squares), the iterative simultaneous approach simulations (red triangles), and the theory of generating functions (black line). We used an original network with $\gamma=2.5$ and $N=10^5$. The analytical results up to the fraction $f_u=1$ are obtained with the generating-functions theory. All results are in very good mutual agreement.

be used until all links are included. In general, we can include over 98% of the links with a relatively small number of weight recalculations or iterations. For instance, in case a network with $\gamma=2.5$ is subjected to percolation with $\alpha=0.2$, one finds that $f_u=0.51$ and 15 iterative steps are necessary to reach $f=0.98$.

The results of such iterative simultaneous approach can be found in Fig. 5. It is immediately clear that the sequential and the simultaneous no longer coincide. This is not surprising since the definitions of edge retaining probabilities $\rho_{ij}$ are different for the sequential and the simultaneous approaches when $\alpha \neq 0$. Furthermore, the sequential approach introduces correlations which are absent in the simultaneous approach. Note, however, that the difference between the two approaches is not a mere consequence of the appearance of correlations in the sequential approach. Indeed, if only the
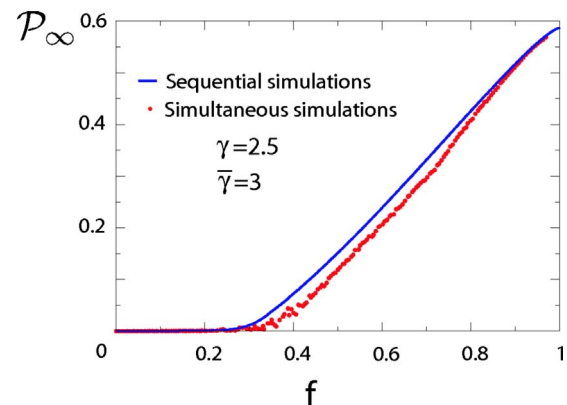


FIG. 5. (Color online) The probability that a node belongs to the giant cluster, $\mathcal{P}_\infty$, as a function of the fraction of retained links $f$ for biased percolation with $\alpha=0.25$. We compare the sequential (blue line) and simultaneous (red dots) approach simulations. We used an original network with $\gamma=2.5$ and $N=10^5$.
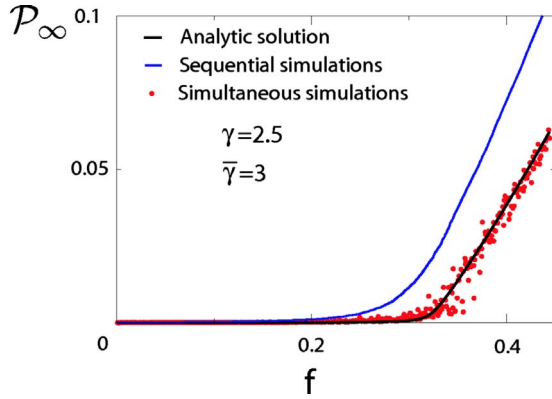
FIG. 6. (Color online) The probability that a node belongs to the giant cluster, $\mathcal{P}_\infty$, as a function of the fraction of retained links $f$ for biased percolation with $\alpha=0.25$. We compare the sequential approach simulations (upper blue line), the iterative simultaneous approach simulations (red dots) and the theory of generating functions (lower black line). We used an original network with $\gamma=2.5$ and $N=10^5$. The analytical results up to the fraction $f_u=0.44$ are obtained with the generating-functions theory. This figure presents more detail of the critical region of Fig. 5.

(dissassortative) correlations were present, the sequential approach should have a larger critical fraction $f_c$ than the simultaneous approach [7]; yet, we find the inverse to be true as evidenced in Fig. 6. At the point at which the weights are recalculated, the giant cluster probability $\mathcal{P}_\infty$ of the iterative simultaneous approach shows kinks as a function of $f$. For instance, a conspicuous kink appears at $f_u=0.69$ in Fig. 6. The iterative simultaneous and the sequential method approach each other as $f \rightarrow 1$ and coincide at $f=1$.

Note that the critical fraction $f_c$ that can be extracted from Figs. 4–6 is nonzero although it is expected to be zero for $2 \leq \bar{\gamma} \leq 3$. This is a consequence of the finite-size effects which cause $f_c$ to scale with the system size according to Eq. (39) (see also Fig. 1).

Although there are differences between the sequential and the simultaneous approach, the differences are clearly not very large. The largest *relative* deviations occur around $f_c$ while the largest *absolute* deviations are situated around the lowest $f_u$ and are typically not more than 10%. Although we find the critical fraction $f_c$ of the simultaneous approach to be always larger than the one of the sequential approach, both values deviate by less than 10%. We conclude that the simultaneous and the sequential approach do differ, but the differences are not large and both approaches are qualitatively similar.

Analytical results for the probability of the largest cluster $\mathcal{P}_\infty$ in the regime $f<f_u$ can be obtained by solving Eq. (45a) numerically for $H_1(1)$ which is then introduced in Eq. (45b). For random edge removal ($\alpha=0$), $f_u=1$ and thus a theoretical result is available for all $f$ values. Moreover, also for other values of $\alpha$ the generating-functions theory calculation (see black line in Fig. 6) agrees well with simulation results throughout the entire reconstruction process. The theoretical model is thus justified by the simulations.

### 2. Properties of the diluted network

We end the section with an overview of the properties of the diluted network. The focus lies again on the appearance
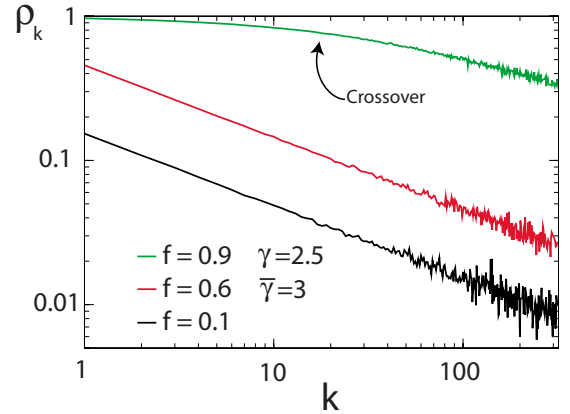


FIG. 7. (Color online) The probability $\rho_k$ to retain a node after depreciation [see Eq. (15)] as a function of its original degree $k$ for different values of $f$ and using the iterated simultaneous approach. From bottom to top, we included 10%, 60%, and 90% of the links. We used a network with $\gamma=2.5$, $\alpha=0.5$ and $N=10^5$ and averaged over ten network realizations. Results are averages over ten percolation simulations. Since $f_u=0.44$, the first regime can be reached in a single sweep and thus no cross-over emerges for $f=0.1$. The crossover (indicated by arrow) becomes apparent for $f=0.9$ when ten iterative steps are performed. This figure should be compared with the inset of Fig. 2 where the sequential approach was used.

of correlations and on the cross-over. In the simultaneous approach (with $f<f_u$), no crossover can appear for the node retaining probability $\rho_k$. Indeed, expression (22) is exact and predicts a decreasing power law for $\rho_k$, that is when $f<f_u$. However, this expression is only valid as long as no recalculation of the weights is performed. As soon as we iterate the simultaneous approach, the results for the simultaneous and sequential methods start approaching each other. Since the sequential approach contains a crossover, we expect the appearance of a crossover in the iterative simultaneous approach. Analogous arguments apply to the appearance of correlations in the diluted network.

The emergence of a crossover in the iterative simultaneous approach is indeed found and shown in Fig. 7. When 10% of the edges are included, no cross-over at all appears. This is consistent with our arguments since we can simply include this fraction of edges in one sweep. Also, in the second sweep, no crossover appears. However, after 10 recalculations of the weights, the crossover is undoubtedly present. Once again, our theoretical arguments are verified. Furthermore, the appearance of correlations is confirmed by our simulation results as evidenced in Fig. 8. Indeed, disassortative correlations are apparent only for large values of $f$, after several iterations have been performed. Note also that these correlations disappear only very slowly upon approach of the point $f=1$ where no correlations are present (see Fig. 3).

### VIII. CONCLUSIONS

We have performed a detailed study of biased percolation on scale-free networks with degree exponent $\gamma$ (with $\gamma>2$) and shown that it is possible to tune a robust network fragile
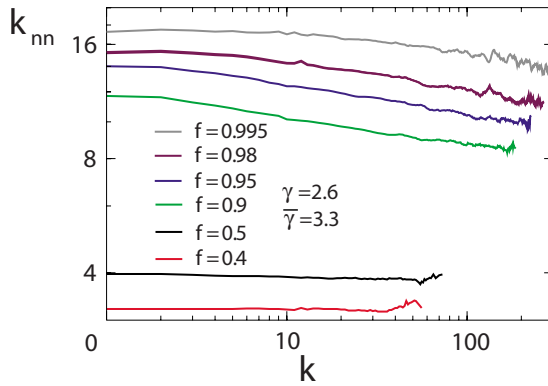
FIG. 8. (Color online) Emergence of correlations in the diluted network in the *iterative* simultaneous approach. This figure shows the mean nearest-neighbor degree in the diluted network. We used an original network constructed with the uncorrelated configuration model with $\gamma=2.6$, $m=2$, and $N=10^5$ which is diluted using the iterative simultaneous approach to biased percolation with $\alpha=0.3045$, such that $\bar{\gamma}=3.3$. The diluted networks contain, from bottom to top, 40 (red), 50 (black), 90 (green), 95 (blue), 98 (purple), and 99.5 (gray) percent of the edges of the original network. To obtain such a network, respectively, 1, 2, 5, 8, and 12 iterations have been made. The result was obtained with 10 network realizations, on all of them, the percolation process was applied 10 times. To reduce the noise level, the mean nearest-neighbor degree was averaged over eight successive values.

and vice versa. Biased percolation involves degree-dependent removal of edges, more specifically, we assumed that the probability to retain an edge is proportional to $(k_i k_j)^{-\alpha}$ with $k_i$ and $k_j$ the degrees of the attached nodes. For $\alpha>0$ the bias is *central* since links between highly connected nodes are preferentially depreciated, while the converse, *peripheral bias*, corresponds to $\alpha<0$. Our most important result is that, at percolation, the properties of a network with bias exponent $\alpha$ and degree exponent $\gamma$ are the same as those of a network with bias exponent zero and degree exponent $\bar{\gamma}=(\gamma-\alpha)/(1-\alpha)$, *or* degree exponent $\gamma$, depending on the sign and the range of $\alpha$. Let us first elaborate on this main result, in the light of the present work and recapitulating arguments presented in the preliminary report [27].

For $\alpha>0$ (with the restriction $\alpha<1$), the new degree exponent $\bar{\gamma}>\gamma$ governs the critical properties of the network that results when the percolation threshold is reached after biased depreciation. The exponent $\bar{\gamma}$ controls the large-degree behavior of the new degree distribution. This new $\bar{P}(\bar{k})$ is not simply scale-free but *asymptotically* scale-free. There is a cross-over value $k_\times$, so that for $\bar{k}<k_\times$ the exponent $\gamma$ is dominant and for $\bar{k}>k_\times$ the exponent $\bar{\gamma}$ takes over. For $\bar{\gamma}>3$, the biased depreciation process will reach the percolation threshold at a finite fraction of retained edges. The network is then fragile under central bias, regardless of whether the network is fragile ($\gamma>3$) or robust ($\gamma<3$) under random removal. For $\bar{\gamma}<3$ the biased depreciation will (in an infinite system) not reach a percolation point since the critical fraction of retained edges, $f_c$, is zero. The network, which is robust for random removal, remains robust under

centrally biased removal. However, for a finite system $f_c$ is small but finite and scales with system size in a manner governed by the exponent $\bar{\gamma}$, whereas the scaling properties of $f_c$ for *random* removal are governed by $\gamma<\bar{\gamma}$. We have shown, by analytic proof that $\bar{\gamma}$ governs the percolation critical behavior for the case $\alpha>0$. Also our numerical results support this conclusion.

For $\alpha<0$ (with the restriction $2-\gamma<\alpha$), the critical behavior at percolation is more subtle. Peripherally biased removal is less destructive than random depreciation and it is possible that a network that is fragile under random failure becomes robust when peripheral bias is applied. Noting that $\bar{\gamma}<\gamma$, it is obvious that robustness is preserved for networks with $\gamma<3$. Conversely, fragility persists for sure when $3<\bar{\gamma}$. However, it is not obvious what to expect when $\bar{\gamma}<3<\gamma$. The behavior of the new degree distribution $\bar{P}(\bar{k})$ for $\bar{k}>k_\times$ is, for $\alpha<0$, controlled by the exponent $\gamma$, so it would seem that the properties of the network under random failure are simply not affected by peripheral bias. However, a finite-size scaling analysis at criticality reveals that the cross-over value $k_\times$ is larger than the maximal degree in the network, implying that the new degree distribution will be controlled by $\bar{\gamma}$ instead of $\gamma$, *provided* $2-\gamma<\alpha<3-\gamma$ (we assume $\gamma>3$ since this discussion only makes sense for networks fragile under random failure). We conclude that sufficiently strong peripheral bias can turn a fragile network robust, and numerical evidence supports this conclusion. On the other hand, for $3-\gamma<\alpha<0$, it is not clear whether the network stays fragile under peripherally biased failure when $\bar{\gamma}$ drops below 3, which happens for $\alpha<(3-\gamma)/2$. This problem is still largely open to future investigation.

Two distinct approaches by means of which a network can be reconstructed in a degree-dependent manner, the sequential and the simultaneous approach, have been introduced to perform the edge removal process. For the sequential approach, we obtained a very useful *analytic approximation* to the marginal distribution $\rho_k$, which is the mean probability that an edge connected to a node with degree $k$ is present in the network after reconstruction. This analytic form clearly features the cross-over value $k_\times$ which plays a crucial role in the network properties. The simultaneous approach, which is a simpler scheme useful for $\alpha>0$ and for edge number fractions below a value dependent on $\gamma$ and $\alpha$, can be iterated so as to provide an alternative to the sequential approach (for $\alpha>0$). The iterations introduce a history-dependence and lead to the emergence of $k_\times$, rendering both reconstruction methods qualitatively similar.

For both approaches, the new degree distributions have been calculated and the degree-degree correlations emerging in the depreciated network have been characterized, by means of standard combinatorial methods. The main finding as regards the correlations is that the sequential approach causes *disassortative mixing* in the depreciated network when $\alpha>0$.

For the simultaneous reconstruction approach, the exact (since correlation-free) percolation threshold $f_c$ is derived for central bias ($\alpha>0$) as a function of (noninteger) moments of the degree distribution, for $\bar{\gamma}>3$. On the other hand, for $\bar{\gamma}<3$ the exact finite-size scaling law for the vanishing of $f_c$ is

obtained. These results fully demonstrate the validity of our exponent mapping (2) for central bias.

A *generating-functions* approach is introduced for degree-dependent edge percolation, extending previous work on random percolation. This approach allows to obtain the size distribution of finite clusters close to the percolation transition as well as other critical properties. If the network can be treated as a tree, which is valid for all finite clusters, the generating functions satisfy self-consistency equations. We have derived the extensions of these equations for degree-dependent percolation, allowing for *correlated* networks, and have shown that they reduce to the original equations provided no correlations are present. We have also derived the criterion for the percolation threshold for degree-dependent percolation and have shown that it reduces to the familiar Molloy-Reed criterion when correlations are absent. Further, our self-consistency equations reduce, for random percolation, to equations frequently encountered in the literature. Our generating-functions formalism extends known results on random percolation to biased percolation, which may involve correlated networks. In the following, however, we draw further conclusions for the statistical properties (including critical exponents) of *uncorrelated* networks only.

Using the equivalence between the $q \to 1$ limit of the Potts model and edge percolation, we have shown that critical exponents for our biased percolation problem can be obtained from the Potts model free energy by extending this equivalence to inhomogeneous (edge-dependent) couplings in the Potts model and edge-dependent removal probabilities in percolation. The generating-functions approach has been combined with the extension of the Fortuin-Kasteleyn construction for the Potts model and with *finite-size scaling* in order to extract the critical exponents of the percolation transition, for uncorrelated networks. We have found that the critical exponents are functions of $\bar{\gamma}$, assuming that the degree distribution after depreciation is governed by degree exponent $\bar{\gamma}$, asymptotically for large degree. For $\bar{\gamma}$, we obtain critical exponents that reduce to literature values of random percolation simply by substituting $\bar{\gamma} \to \gamma$. However, in the more delicate regime $2 < \bar{\gamma} < 3$ this correspondence is not satisfied. A critical assessment of this discrepancy is not given here, but left to future scrutiny. We conclude that, in all cases, the only way in which the bias exponent $\alpha$ enters in the critical exponents of the percolation transition, is through the new degree exponent $\bar{\gamma}$.

Furthermore, we have used numerical simulations to study the properties of the network after depreciation and near the percolation transition. We verified that robust networks can turn fragile under centrally biased failure and that fragile networks can turn robust under (sufficiently) strong peripherally biased failure, using the sequential approach. Although correlations are introduced in this approach, the results agree well with the predictions for uncorrelated networks. Also, the cross-over behavior of the new degree distributions was tested and found to agree well with the analytical expectations. As regards correlations introduced by the sequential approach, we have been able to verify the occurrence of disassortative mixing predicted theoretically for $\alpha > 0$.

The critical properties at percolation were checked by simulations using the (iterated) simultaneous approach and also compared with results obtained by simulations using the sequential approach. Specifically, we have found that for biased percolation the sequential and the (iterated) simultaneous approach give rise to different results. In particular, the size of the giant cluster predicted by the generating-functions theory agrees very well with the simulations for the (iterated) simultaneous approach. Nevertheless, the differences are often small and we may conclude that both methods are qualitatively similar. Finally, we have also provided evidence for the theoretically expected appearance of cross-over effects and degree-degree correlations for the (iterated) simultaneous approach. Overall, we conclude that good agreement has been found between simulations and theory.

## ACKNOWLEDGMENTS

## APPENDIX

There exists a more formal way to prove that the degree distribution in the diluted net satisfies $\bar{P}(\bar{k}) \propto \bar{k}^{-\bar{\gamma}}$ for large degrees $\bar{k}$ when central bias is applied.

We start from the degree distribution in the diluted network $\bar{P}(\bar{k})$, which was calculated in Eq. (26). For large values of $k$, i.e., $k \gg k_\times$, and $\alpha > 0$, the probability of retaining a node of degree $k$ falls off as $\rho_k \propto k^{-\alpha}$; this is valid using the sequential approach [see Eq. (20)], as well as the simultaneous one [see Eq. (22)].

If both $k\rho_k$ and $k(1 - \rho_k)$ are large, the binomial distribution can be approximated by a normal distribution with mean $k\rho_k$ and variance $k\rho_k(1 - \rho_k)$. The latter condition is always true if we apply CB, since then $1 - \rho_k \approx 1$ for large $k$ as edges between the most connected nodes are almost certainly removed. Since $k\rho_k \propto k^{1-\alpha}$, the first requirement holds only if $\alpha < 1$.

Inserting the normal distribution with variance $k\rho_k(1 - \rho_k) \approx C_0 k^{1-\alpha}$, with $C_0$ a constant, in Eq. (26) and approximating the sum by an integral yields

$$\bar{P}(\bar{k}) \propto \int_{\bar{k}}^{\infty} dk \ k^{-\gamma + (\alpha - 1)/2} \exp\left(-\frac{[\bar{k} - C_0 k^{1-\alpha}]^2}{2 C_0 k^{1-\alpha}}\right). \quad (80)$$

Now, we introduce the auxiliary variable $u \equiv k / \bar{k}^{1/(1-\alpha)}$ and rewrite the integral as follows:

$$\bar{P}(\bar{k}) \propto \bar{k}^{(1-\gamma)/(1-\alpha) - 1/2} \int_{\bar{k}^{-\alpha/(1-\alpha)}}^{\infty} du \ u^{-\gamma + (\alpha - 1)/2}$$

$$\times \exp\left(-\bar{k} \frac{[1 - C_0 u^{1-\alpha}]^2}{2 C_0 u^{1-\alpha}}\right). \quad (81)$$

For $\bar{k} \to \infty$, the integrand has only nonvanishing values in a

neighborhood $\Delta u \approx 1/\sqrt{\bar{k}}$ around $u_c = C_0^{-1/(1-\alpha)}$. If $0 < \alpha < 1$, the lower bound of integration vanishes for large $\bar{k}$. Thus, $u_c$ certainly lies in the domain of integration and the integral can simply be approximated by $C_1/\sqrt{\bar{k}}$ with $C_1$ a constant. After some trivial power counting, we arrive at

$$\bar{P}(\bar{k}) \propto \bar{k}^{-(\gamma-\alpha)/(1-\alpha)}. \tag{82}$$

Thus, we obtain the anticipated behavior for CB. The exponent $\bar{\gamma}$ controls the decay of the degree distribution in the diluted network at large $\bar{k}$.

The case $\alpha = 1$ can be studied using a Poisson distribution approximation for the binomial factors, which results in $\bar{P}(\bar{k})$ being a Poisson-type degree distribution $\bar{P}(\bar{k}) \sim \bar{k}^{1-\gamma}/\bar{k}!$ for large $k$. Thus, $\alpha \leq 1$ is a natural restriction, because the scale-free behavior is destroyed if stronger CB is applied.

We still have to examine the situation for $\alpha = 0$. Then, the lower bound of the integral becomes 1, while $u_c = 1/C_0$. The constant $C_0$ is nothing but the fraction of edges preserved after the depreciation process. Thus, $C_0 < 1$ and $u_c$ also lies in the integration domain. Thus, the previous arguments apply as well to the random node removal process. We conclude that indeed, $\bar{P}(\bar{k}) \sim \bar{k}^{-\bar{\gamma}}$ for $0 \leq \alpha < 1$, which proves the intuitive conjecture given in the Introduction.

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