Ginzburg-Landau equation for dynamical four-wave mixing in gain nonlinear media with relaxation

Svitlana Bugaychuk[1,*] and Robert Conte[2,3,†]

[1]*Institute of Physics, National Academy of Sciences, 46 Prospect Nauki, Kiev 03028, Ukraine*

[2]*LRC MESO, École normale supérieure de Cachan (CMLA) et CEA-DAM 61, Avenue du Président Wilson, F-94235 Cachan Cedex, France*

[3]*Service de physique de l'état condensé (CNRS URA 2464), CEA-Saclay, F-91191 Gif-sur-Yvette Cedex, France*

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We consider the dynamical degenerate four-wave mixing (FWM) model in a cubic nonlinear medium including both the time relaxation of the induced nonlinearity and the nonlocal coupling. The initial ten-dimensional FWM system can be rewritten as a three-variable intrinsic system (namely, the intensity pattern, the amplitude of the nonlinearity, and the total net gain) which is very close to the pumped Maxwell-Bloch system. In the case of a purely nonlocal response the initial system reduces to a real damped sine-Gordon (SG) equation. We obtain a solution of this equation in the form of a sech function with a time-dependent coefficient. By applying the reductive perturbation method to this damped SG equation, we obtain exactly the cubic complex Ginzburg Landau equation (CGL3) but with a time dependence in the loss or gain coefficient. The CGL3 describes the properties of the spatially localized interference pattern formed by the FWM.

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I. INTRODUCTION

The effect of interaction of light and matter in nonlinear optics is very often characterized by a coupling coefficient which reveals a response of the matter. If in addition a mutual mixing of several waves is taken into consideration, one deals with nonuniform spatial (or spatiotemporal) fields. People usually consider the reaction of matter is local on the action of the field. But this is not always the case. In inertial or nonlocal systems the response can be retarded in time or shifted in space. As a result, the beam-coupling coefficient takes a complex value and some phase addition appears between the mixed waves. This can lead to the control of parameters of one beam by guiding the properties of another beam as well as to the formation of stable localized structures (i.e., intensity patterns). In this paper we show rigorously that a nonlinear system describing the degenerate wave mixing in a medium which possesses both a nonlocal response and relaxation is reduced to one nonlinear complex Ginzburg-Landau equation (CGLE). We develop the technique to obtain the cubic CGLE by using the reductive perturbation method for the nonlinear dynamical wave coupling system.

Nonlinear dynamical systems have been studied intensively during the last decennia after localized structures (e.g., solitons) were found in such systems. The CGLE became a widely used physical-mathematical model appearing in many branches of physics, chemistry, and biology, in order to describe various localized structures [1–4]. Moreover the CGLE is considered as the simplest model containing dissipative soliton solutions, which exist in nonequilibrium systems where gain and loss are balanced [2,3,5]. In optics, the dissipative solitons described by the Ginzburg-Landau equation appear for pulsed operation of passively mode-locked lasers as well as for all-optical long-haul soliton transmission lines [2,3,6,7].

The dissipative models which take into account wave interactions have been studied in [8,9], first of all as the envelope of dissipative solitons emitted by an optical parametric oscillator. In [8] the author presents theoretical and experimental studies of stimulated Brillouin backscattering of a continuous pump-wave resulting in backward-traveling solitary pulses in long fiber-ring cavities. Nonlinear optical cavities with three-wave interaction in a nonlinear crystal, when the waves have different frequencies, were considered in Ref. [9]. It was shown that the spatial dissipative solitons can form spontaneously in that case. We consider the cubic CGLE which appears in the problem of dynamical interaction of four waves with the same frequencies in extended nonlocal media. We show that the CGLE is obtained because of a photoinduced nonlocal nonlinear response which includes a time relaxation term in the considered (dissipative) model.

The next feature that we utilize in the model, the nonlocality, reveals itself as an ubiquitous property in many branches of physics, e.g., optics, plasmas, and Bose-Einstein condensates [10,11]. Usually the nonlocal response appears when the nonlinearity is associated with some sort of transport process such as heat conduction in media with thermal response [12], diffusion of molecules or atoms accompanying nonlinear light propagation in atomic vapors [13,14] and charge transport in photorefractive crystals [15,16]. Specific properties of spatial solitons were investigated in nematic liquid crystals, where nonlocal response exists due to reorientation of anisotropic molecules by a propagating beam [17–20]. The nonlocal nonlinearities with formation of dissipative optical solitons for a wide-aperture laser with saturable absorption were studied recently in [21,22].

One usually investigates stationary changes of the induced nonlinearity. Our dissipative model includes both gain

*bugaich@iop.kiev.ua

†robert.conte@cea.fr

and relaxation of the nonlinearity in a nonlocal medium. Since we consider the process of wave coupling, the photoinduced nonlocal nonlinearity leads to an effect of energy transfer between waves during their propagation. In this way the nonlocality plays the role of an amplified medium to increase the intensities of one beam at the cost of decreasing the energy of another beam. The energy transfer effect is observed in the dynamical holography when the interacting beams record a dynamical grating, which is shifted from the interference pattern, and the same beams diffract from this grating [23,24]. As a result of this energy transfer both the interference pattern and the spatial distribution of the amplitude of the nonlinearity get a stable localized pattern along the z-longitude direction of the medium [25–27]. We show that the CGLE governs the spatiotemporal dynamics for both values.

Dissipative solitons described by CGLE demonstrate a rich variety of unusual properties [2,28], such as stable periodic pulsations, bounded solitary waves, periodic "explosions," and collapse. All these unique features may find applications in nonlinear wave coupling, in particular in the dynamic holography in media with nonlocal response. Among possible applications in photonics let us mention: (i) holographic interferometers including phase-shifted interferometers; (ii) traps of light (trapping states) in a resonator; (iii) manipulation of pulses having different intensities and durations in order to obtain optical logic elements, all-optical switching, pulse retardation etc., as well as the interaction of pulses not only in bulk materials but with thin nonlinear films, nanomaterials and metamaterials; and many others. During the process the medium should possess a nonlocal nonlinearity, e.g., some kind of transport mechanism; or the medium can have a local nonlinearity but a regime of moving dynamical gratings should be realized.

The paper is organized as follows. In Sec. II we introduce the four-wave mixing (FWM) model and recall the existing results. In Sec. III, we revisit the derivation of the damped sine-Gordon equation and derive a solution to the FWM. Finally, in Sec. IV, we apply the method of multiple scale expansion and find as a result the cubic CGLE. This procedure proves that the FWM as well as the dynamical self-diffraction of waves can be considered as a dissipative nonlinear system containing stable soliton solutions.

II. INTRINSIC SYSTEM OF THE DISSIPATIVE FWM MODEL

The one-dimensional degenerate FWM initial system consists of five partial differential equations, namely, four coupled wave equations for slow variable amplitudes which connect waves 1 and 2 propagating in a forward direction and waves 3 and 4 propagating in a backward direction,

$$\partial_z A_1 = -i\mathcal{E}A_2, \quad \partial_z \bar{A}_2 = i\mathcal{E}\bar{A}_1,$$

$$\partial_z \bar{A}_3 = -i\mathcal{E}\bar{A}_4, \quad \partial_z A_4 = i\mathcal{E}A_3, \tag{1}$$

and the dynamical equation for the medium, which in the simplest case includes only a gain being proportional to the intensity pattern and an exponential relaxation, in the form

$$\partial_t \mathcal{E} = \gamma I_{\mathrm{m}} - \frac{\mathcal{E}}{\tau}. \tag{2}$$

We assume here that the interference pattern is formed by two pairs of copropagating waves

$$I_{\mathrm{m}} = A_1\bar{A}_2 + \bar{A}_3A_4. \tag{3}$$

In Eqs. (1)–(3), $A_j(t,z)$ is the slow variable amplitude of the jth plane wave $E_j(t,z)=A_j(t,z)e^{i(wt-\vec{k}_j\vec{r})}$ and $\mathcal{E}(t,z)$ is the amplitude of the photoinduced nonlinear susceptibility. It must be emphasized that the response constant $\gamma=\gamma_{\mathrm{L}}+i\gamma_{\mathrm{NL}}=|\gamma|e^{ig}$ is complex. The complex value of the coupling coefficient $\mathcal{E}$ is an essential feature for the existence of solitonlike solutions. The interacting waves are connected by the impulse conservation law:

$$\vec{k}_1 - \vec{k}_2 = \vec{k}_4 - \vec{k}_3. \tag{4}$$

We assume the following normalization: all wave amplitudes are normalized by the square root of the total light intensity $I_0=|A_1|^2+|A_2|^2+|A_3|^2+|A_4|^2=\mathrm{const}$, $\mathcal{E}$ is the dimensionless coefficient of the nonlinearity, and z is the dimensionless longitudinal coordinate $z=[k_0^2/(2k_z')]z'$, where k_0 is the amplitude of the wave vector in the free space and z' is the spatial coordinate. We keep the dimension of the time coordinate t in order to display the dependence of the dispersion relation on the time relaxation constant τ. In this way, in order to make Eq. (2) dimensionless, the gain coefficient is normalized by the time relaxation constant τ and has the dimension $[\gamma]=T^{-1}$.

Systems (1)–(3) has been considered for the dynamic holography in the case of a purely nonlocal response $\gamma=i\gamma_{\mathrm{NL}}$. Then $\mathcal{E}$ is interpreted as the amplitude of the dynamical grating. As previously found [26,27,29,30], the initial system is then reducible to a damped sine-Gordon (SG) equation, which has a stationary solution in the form of a sech function $|\mathcal{E}|=\gamma C/\cosh[2\gamma Cz-p]$, with C,p as the arbitrary constants. Numerical solutions in the form of periodic oscillations were investigated in [30]. The first experimental observation of localization of the dynamical grating amplitude along the longitudinal coordinate in bulk ferroelectric crystals was made in [27]. For the general case of a complex γ, the general stationary solution was later found in [31], together with, in the dynamical case, the general solution (expressed with elliptic functions) of the reduction $(z,t)\to\sqrt{z}e^{-t/\tau}$ for a purely nonlocal response.

The ten-dimensional system [Eq. (1)–(3)] is invariant under any time-dependent rotation in the space $\{A_1,\bar{A}_2,A_4,\bar{A}_3\}$ which preserves the interference pattern [Eq. (3)]. In a previous work [31], we could remove this five-parameter unessential freedom and obtain the following intrinsic system,

$$\partial_z I_{\mathrm{m}} = -i\mathcal{E}I_{\mathrm{d}}, \quad \partial_z I_{\mathrm{d}} = -2i\bar{\mathcal{E}}I_{\mathrm{m}} + 2i\mathcal{E}\overline{I_{\mathrm{m}}},$$

$$\partial_t \mathcal{E} = \gamma I_{\mathrm{m}} - \frac{\mathcal{E}}{\tau}, \tag{5}$$

admitting the first integral

$$4|I_{\rm m}|^2 + I_{\rm d}^2 = K(t), \quad K \text{ arbitrary}. \tag{6}$$

The real field $I_{\rm d}$,

$$I_{\rm d} = -|A_1|^2 + |A_2|^2 - |A_3|^2 + |A_4|^2, \tag{7}$$

is the relative *net gain*. Therefore the four-wave mixing is characterized by three intrinsic variables: the intensity pattern $I_{\rm m}$, the grating amplitude $\mathcal{E}$, and the relative net gain $I_{\rm d}$.

This intrinsic system [Eq. (5)] is very similar to the pumped Maxwell-Bloch system, an integrable system of nonlinear optics defined as [32]

$$\partial_X \rho = N e, \quad \partial_X \bar{\rho} = N \bar{e},$$

$$\partial_X N = -(\rho \bar{e} + \bar{\rho} e)/2 + 4s,$$

$$\partial_T e = \rho, \quad \partial_T \bar{e} = \bar{\rho}, \tag{8}$$

with s as a real constant (the system is "pumped" when s is nonzero).

When the four-wave mixing model is undamped ($\tau = +\infty$) and has a purely nonlocal response [$\Re(\gamma)=0$], while the Maxwell-Bloch system is unpumped ($s=0$), these two systems can be identified,

$$\frac{1}{\tau} = 0, \quad \Re(\gamma) = 0, \quad s = 0,$$

$$\frac{z}{X} = \frac{t}{T} = \frac{2|\gamma| I_{\rm m}}{\rho} = \frac{2|\gamma| \overline{I_{\rm m}}}{\bar{\rho}} = \frac{|\gamma| I_{\rm d}}{N} = \frac{-2i\mathcal{E}}{e} = \frac{2i\bar{\mathcal{E}}}{\bar{e}}, \tag{9}$$

and in this case the undamped purely nonlocal response four-wave mixing model admits all the solutions of the unpumped complex Maxwell-Bloch system.

III. DERIVATION OF THE DAMPED SINE-GORDON EQUATION

As shown in [26,27,29,30], under some specific assumptions, the system made of the four complex Eqs. (1) can be integrated explicitly. Because we need it later, let us first establish this derivation in full generality.

If one represents the complex amplitudes as

$$A_j = M_j e^{i\varphi_j}, \quad \mathcal{E} = M_e e^{i\varphi_e}, \quad (M_j, M_e, \varphi_j, \varphi_e) \text{ real}, \tag{10}$$

and introduces the notation

$$\Phi_{12} = \varphi_1 - \varphi_2 - \varphi_e + \frac{\pi}{2}, \quad \Phi_{43} = \varphi_4 - \varphi_3 - \varphi_e + \frac{\pi}{2}, \tag{11}$$

system (1) becomes

$$\partial_z M_1 = + M_2 M_e \cos \Phi_{12}, \quad \partial_z \varphi_1 = -\frac{M_2 M_e}{M_1} \sin \Phi_{12},$$

$$\partial_z M_2 = - M_1 M_e \cos \Phi_{12}, \quad \partial_z \varphi_2 = -\frac{M_1 M_e}{M_2} \sin \Phi_{12},$$

$$\partial_z M_4 = - M_3 M_e \cos \Phi_{43}, \quad \partial_z \varphi_4 = +\frac{M_3 M_e}{M_4} \sin \Phi_{43},$$

$$\partial_z M_3 = + M_4 M_e \cos \Phi_{43}, \quad \partial_z \varphi_3 = +\frac{M_4 M_e}{M_3} \sin \Phi_{43}. \tag{12}$$

It is then convenient to introduce the first integrals

$$f_{12}^2(t) = |A_1|^2 + |A_2|^2, \quad f_{43}^2(t) = |A_4|^2 + |A_3|^2, \tag{13}$$

and to compute the z evolution of the two functions

$$v_{12} = |A_1|^2 - |A_2|^2, \quad v_{43} = |A_4|^2 - |A_3|^2. \tag{14}$$

One finds

$$\partial_z v_{12} = + 4 M_1 M_2 M_e \cos \Phi_{12},$$

$$\partial_z v_{43} = - 4 M_4 M_3 M_e \cos \Phi_{43}, \tag{15}$$

and, by elimination of M_j,

$$(\partial_z v_{12})^2 = 4(f_{12}^4 - v_{12}^2) M_e^2 \cos^2 \Phi_{12},$$

$$(\partial_z v_{43})^2 = 4(f_{43}^4 - v_{43}^2) M_e^2 \cos^2 \Phi_{43}. \tag{16}$$

If one defines two functions $u_{12}(z,t)$, $u_{43}(z,t)$ by the relations

$$M_e^2 \cos^2 \Phi_{12} = (\partial_z u_{12})^2, \quad M_e^2 \cos^2 \Phi_{43} = (\partial_z u_{43})^2, \tag{17}$$

the two Eqs. (16) can be integrated explicitly in terms of the two variables u_{12}, u_{43},

$$v_{12} = - f_{12}^2 \cos[2(u_{12} - c_{12})],$$

$$v_{43} = - f_{43}^2 \cos[2(u_{43} + c_{43})], \tag{18}$$

with c_{12} and c_{43} arbitrary functions of t. Basic trigonometry then yields

$$A_1 = + f_{12} \sin(u_{12} - c_{12}) e^{i\varphi_1},$$

$$A_2 = + f_{12} \cos(u_{12} - c_{12}) e^{i\varphi_2},$$

$$A_4 = - f_{43} \sin(u_{43} + c_{43}) e^{i\varphi_4},$$

$$A_3 = + f_{43} \cos(u_{43} + c_{43}) e^{i\varphi_3}. \tag{19}$$

We have not succeeded to similarly integrate the equations for φ_j in Eq. (12) without any additional assumption. Let us therefore assume, as was done in [26,27,30], that these four equations for the spatial evolution of φ_j identically vanish, i.e., that $\sin \Phi_{12} = \sin \Phi_{43} = 0$ and the phases φ_j are independent of z,

$$\partial_z \varphi_j = 0, \quad j = 1,2,3,4,$$

$$\Phi_{12} = n_{12}\pi, \quad \Phi_{43} = n_{43}\pi, \quad n_{12} \text{ and } n_{43} \text{ integers},$$

$$\partial_z \varphi_e = 0, \tag{20}$$

and for convenience let us redefine solution (19) as

$$\mathcal{E} = (\partial_z u) e^{i\varphi_e},$$

$$A_1 = + f_{12} \sin[s_{12}(u - C_{12})] e^{i\varphi_1},$$

$$A_2 = + f_{12} \cos[s_{12}(u - C_{12})]e^{i\varphi_2},$$

$$A_4 = - f_{43} \sin[s_{43}(u + C_{43})]e^{i\varphi_4},$$

$$A_3 = + f_{43} \cos[s_{43}(u + C_{43})]e^{i\varphi_3},$$

$$\Phi_{12} \equiv \varphi_1 - \varphi_2 - \varphi_e + \frac{\pi}{2} = n_{12}\pi, \quad s_{12} = (-1)^{n_{12}},$$

$$\Phi_{43} \equiv \varphi_4 - \varphi_3 - \varphi_e + \frac{\pi}{2} = n_{43}\pi, \quad s_{43} = (-1)^{n_{43}},$$

$$I_{\rm m} = \frac{1}{2}[f_{12}^2 \sin 2(u - C_{12}) - f_{43}^2 \sin 2(u + C_{43})]e^{i(\varphi_e - \pi/2)},$$

$$I_{\rm d} = f_{12}^2 \cos 2(u - C_{12}) + f_{43}^2 \cos 2(u + C_{43}),$$

$$n_{12}, \ n_{43} \in \mathcal{Z}. \tag{21}$$

The last complex equation to be enforced [Eq. (2)] is equivalent to the two real equations

$$\partial_z \partial_t u + \frac{1}{\tau}\partial_z u - K \sin(2u + \alpha) = 0,$$

$$K e^{i\alpha} = \frac{\gamma_{\rm NL} \sin g}{2}(f_{12}^2 e^{-2iC_{12}} - f_{43}^2 e^{2iC_{43}}), \tag{22}$$

$$(\partial_z u)(\partial_t \varphi_e) + (\cot g)K \sin(2u + \alpha) = 0, \quad \gamma = |\gamma| e^{ig}. \tag{23}$$

If $\partial_t \varphi_e \neq 0$, the ODE (23) (with t as a parameter) integrates as

$$\cos(2u + \alpha) = + \tanh 2\left\{\frac{K(t)\cot g}{\partial_t \varphi_e}[z - z_0(t)]\right\},$$

$$\sin(2u + \alpha) = - \operatorname{sech} 2\left\{\frac{K(t)\cot g}{\partial_t \varphi_e}[z - z_0(t)]\right\}, \tag{24}$$

then Eq. (22) restricts this solution to

$$\varphi_e = -\frac{\cot g}{\tau}(t - t_0), \quad \partial_z u = - K\tau \operatorname{sech} 2K\tau(z - z_0), \tag{25}$$

$$\cos(2u + \alpha) = - \tanh 2K\tau(z - z_0),$$

$$\sin(2u + \alpha) = - \operatorname{sech} 2K\tau(z - z_0), \tag{26}$$

in which K, t_0, z_0 are arbitrary constants. This solution can also be viewed as the general solution of the reduction $I_{\rm m}/\mathcal{E}$=complex constant of the intrinsic system [Eq. (5)],

$$\forall \tau, \ \gamma: \begin{cases} I_{\rm d} = -\dfrac{2K}{|\gamma| \sin g}\tanh 2K\tau(z - z_0) \\ \mathcal{E} = - i|\gamma|(\sin g)\tau I_{\rm m} \\ \quad = - e^{-i(\cot g)(t - t_0)/\tau} K\tau \operatorname{sech} 2K\tau(z - z_0), \end{cases} \tag{27}$$

in which the wave number K is arbitrary. Very similar to [31] [Eq. (23)], this solution is however new and it depends on both space and time.

If $\partial_t \varphi_e = 0$, then γ must be purely imaginary

$$\partial_t \varphi_e = 0, \quad \cos g = 0, \tag{28}$$

this defines the already investigated damped sine-Gordon equation.

The result of the above computation can be summarized as follows. Under the three assumptions that the phases of each A_j are independent of z, the phase of $\mathcal{E}$ is constant, and γ is purely imaginary, one obtains a solution of systems (1)–(3) represented as Eq. (21), in terms of the real solution u of a damped sine-Gordon Eq. (22) (with $\sin g = 1$). The representation [Eq. (21)] displays the invariance $(1,2,3,4,\partial_z,u) \to (4,3,2,1,-\partial_z,-u)$ and depends on six arbitrary real functions of t $(f_{12}, f_{43}, C_{12}, C_{43}, \varphi_1 + \varphi_2, \varphi_4 + \varphi_3)$ and one arbitrary real constant (the phase φ_e).

IV. $\Re(\gamma) = 0$: FROM REAL DAMPED SINE-GORDON TO CGL3

It is a classical result [33] that the nonlinear Schrödinger equation (NLS) can be derived from the sine-Gordon equation by a reductive perturbation method, see details in, e.g., [34]. When applied to the real damped sine-Gordon Eq. (22), this method yields a complex cubic Ginzburg-Landau equation which we now derive. Consider the damped sine-Gordon Eq. (22)

$$E \equiv \partial_t \partial_z u + \frac{1}{\tau}\partial_z u - K(t)\sin[2u + \alpha(t)] = 0, \tag{29}$$

in which $u(z,t)$, $K(t)$, $\alpha(t)$, and τ are real.

Following the classical derivation of NLS from the sine-Gordon equation [33,34], we define a multiscale expansion in which u is of order ε, while $K(t)$ is of order one,

$$u(z,t) + \frac{\alpha(t)}{2} = \varepsilon \sum_{j=0}^{+\infty} \varepsilon^j \varphi_j(z, \varepsilon z, \ldots, \varepsilon^k z, \ldots, t, \ldots, \varepsilon^k t, \ldots),$$

$$K(t) = \sum_{j=0}^{+\infty} \varepsilon^j K_j(\varepsilon^j t, \ldots, \varepsilon^k t, \ldots), \quad \tau = \text{unchanged},$$

$$E = \varepsilon \sum_{j=0}^{+\infty} \varepsilon^j E_j, \tag{30}$$

and after renaming the scaled independent variables as

$$\varepsilon^k z = Z_k, \quad \varepsilon^k t = T_k, \tag{31}$$

one requires each coefficient E_j to vanish.

The zeroth order,

$$L\varphi_0=0, \quad L\equiv \partial_{T_0}\partial_{Z_0}+\frac{1}{\tau}\partial_{Z_0}-2K_0(T_0,\ldots), \tag{32}$$

admits the plane-wave-type complex solution

$$\varphi_0=A(Z_1,Z_2,T_1,T_2,\ldots)e^{i[qZ_0-F(T_0,\ldots)]}, \tag{33}$$

in which the complex constant q and the complex function F obey the dispersion relation

$$i\frac{q}{\tau}+q\frac{\partial F}{\partial T_0}-2K_0=0. \tag{34}$$

Since K_0 may depend on T_0, it is convenient to introduce the primitive Q_0 of K_0 and to represent the dispersion relation by its integrated form

$$F=-i\frac{T_0}{\tau}+2\frac{Q_0}{q}, \quad \frac{\partial Q_0}{\partial T_0}=K_0. \tag{35}$$

The physical solution of Eq. (32) is then chosen as the real part of the above complex plane wave

$$\varphi_0=A(Z_1,Z_2,T_1,T_2,\ldots)e^{\Phi_0}+\text{c.c.},$$

$$\Phi_0=iqZ_0-\frac{T_0}{\tau}-2i\frac{Q_0}{q}. \tag{36}$$

The first-order equation, which defines the evolution of φ_1,

$$L\varphi_1=-G_1e^{\Phi_0}-\overline{G_1}e^{\bar{\Phi}_0},$$

$$G_1\equiv-\frac{2iK_0}{q}\frac{\partial A}{\partial Z_1}+iq\frac{\partial A}{\partial T_1}-2\left(K_1-\frac{\partial Q_0}{\partial T_1}\right)A, \tag{37}$$

requires the vanishing of G_1 in order to avoid φ_1 to diverge. This defines two complex conjugate linear partial differential equations (PDEs) for $A(Z_1,T_1)$ and $\bar{A}$, and the solution of this first order is

$$\varphi_1=0,$$

$$A=a(Z_1-v_gT_1,Z_2-v_gT_2,T_2,\ldots)e^{\Phi_1},$$

$$\Phi_1=-2i\frac{Q_1-Q_0}{q}, \quad v_g=-\frac{2K_0}{q^2}, \quad \frac{\partial Q_1}{\partial T_1}=K_1, \tag{38}$$

in which the complex function of integration a is to be determined.

Since the group velocity v_g is generically complex, let us introduce the two complex conjugate independent variables X_1,Y_1,

$$X_1=Z_1-v_gT_1, \quad Y_1=\overline{X_1}=Z_1-\bar{v}_gT_1. \tag{39}$$

The second-order equation similarly defines the evolution of φ_2,

$$L\varphi_2=-\frac{4}{3}K_0(e^{3(\Phi_0+\Phi_1)}a^3+e^{3(\overline{\Phi_0}+\overline{\Phi_1})}\bar{a}^3)-qG_2e^{\Phi_0+\Phi_1}$$
$$-\overline{qG_2}e^{\overline{\Phi_0}+\overline{\Phi_1}}, \tag{40}$$

$$G_2\equiv i\frac{\partial a}{\partial T_2}-2\frac{iK_1}{q^2}\frac{\partial a}{\partial X_1}+\frac{2K_0}{q^3}\frac{\partial^2 a}{\partial^2 X_1}+4\frac{K_0}{q}e^{2\Re(\Phi_0+\Phi_1)}|a|^2a$$
$$-\frac{2}{q}\left(K_2-\frac{\partial Q_1}{\partial T_2}\right)a, \tag{41}$$

and the cancellation of the secular terms requires G_2 to vanish, which defines two complex conjugate nonlinear PDEs for $a(X_1,T_2)$ and $\bar{a}(Y_1,T_2)$ and yields the value

$$\varphi_2=\frac{2\tau K_0}{24\tau K_0-9iq}ae^{3(\Phi_0+\Phi_1)}+\text{c.c.} \tag{42}$$

Therefore, under the reductive perturbation method, the damped sine-Gordon Eq. (22) generically yields the complex PDE $G_2=0$ Eq. (41), in which K_0,K_1,K_2 depend on T_2.

In the pure sine-Gordon limit $1/\tau=0$, $K(t)=k_0=\text{const}$, with q real, one checks that the PDE $G_2=0$ reduces to the nonlinear Schrödinger equation,

$$\frac{1}{\tau}=0, \quad K(t)=k_0, \quad q \text{ real},$$

$$i\frac{\partial a}{\partial T_2}+\frac{2k_0}{q^3}\frac{\partial^2 a}{\partial^2 X_1}+4\frac{k_0}{q}|a|^2a=0. \tag{43}$$

In the generic case (q complex), the PDE $G_2=0$ Eq. (41) would be identical to the cubic complex Ginzburg-Landau equation (CGL3) if its coefficients were independent of T_2. Let us therefore try to get rid of this dependence on T_2 by performing the transformation

$$a(X_1,T_2)=\psi(\xi,\eta)e^{\lambda(T_2)},$$

$$\xi=X_1-f_1(T_2), \quad \eta=f_2(T_2), \tag{44}$$

in which the complex functions f_1, λ and the real function f_2 can be freely chosen. The best one can achieve is to concentrate the dependence on T_2 in only one coefficient, e.g., the gain or loss term. Then the functions of the transformation are the following:

$$\frac{df_1}{dT_2}=-2\frac{K_1}{q^2}, \quad \frac{df_2}{dT_2}=K_0,$$

$$\lambda=\frac{2i}{q}\int K_1dT_1-2i\frac{\Re(q)}{|q|^2}\int K_2dT_2. \tag{45}$$

The final CGLE is

$$i\frac{\partial\psi}{\partial\eta}+\frac{2}{q^3}\frac{\partial^2\psi}{\partial^2\xi}+\frac{4}{q}e^{-2T_0/\tau-2\Im(q)Z_0}|\psi|^2\psi+2i\frac{\Im(q)}{|q|^2}\frac{K_2}{K_0}\psi=0. \tag{46}$$

in which the coefficient K_2/K_0 depends on η and the other coefficients are complex constants. Under the condition that K_2/K_0 be independent of η, the above PDE (46) is then identical to the CGL3 equation.

We want to emphasize here that the CGLE [Eq. (46)] includes only the longitudinal space coordinate ξ for the variable ψ. It does not contain any transverse spatial coordinates.

We thus obtain the CGLE for describing the dynamics of the FWM in a nonlocal medium with a dissipative term, where the dependent variable is the envelope of the potential u. With the definition $|\mathcal{E}|=\partial_z u$ and the multiscale expansion $|\mathcal{E}|=\varepsilon\Sigma_{j=0}^{+\infty}\varepsilon^j\mathcal{E}_j$, one can obtain the expression which connects the value u with the envelope of the spatial distribution of the nonlinearity,

$$\mathcal{E}_0=\frac{\partial\phi_0}{\partial Z_0}=iq\psi(\xi,\eta)e^{i(qZ_0-2/qQ_0-2\Re(q)/|q|^2Q_2+iT_0/\tau)}+\text{c.c.} \tag{47}$$

Taking into account Eq. (27) which connects $\mathcal{E}$ and the intensity field $I_{\rm m}$, we obtain that both the nonlinearity spatial shape and the time behavior of the intensity pattern are the same spatiotemporal distribution, where the magnitudes $I_{\rm m}$ and $\mathcal{E}$ only differ by a constant. The complex sign "i" means that the functions $I_{\rm m}$ and $\mathcal{E}$ have a relative shift in the spatial coordinate.

Thus the CGLE (46) together with Eqs. (47) and (27) describe the spatiotemporal dynamics for the physical field $I_{\rm m}$ (the interference pattern), the parameter of the phase transition $|\mathcal{E}|$ and the potential u.

V. CONCLUSION

We have obtained the complex Ginzburg-Landau equation from the nonlinear systems of the dynamical four-wave mixing that includes degenerate wave coupling in a cubic nonlinear medium which has both nonlocal and relaxation response. The obtained CGLE is just the cubic one when the response is purely nonlocal, i.e., there is the energy transfer only between the interacting waves but no phase transfer. In this case the initial FWM system is reduced to a damped sine-Gordon equation containing the first derivative on the spatial longitudinal coordinate z. We show that by applying the reductive perturbation method, the real damped sine-Gordon equation reduces to the cubic CGLE except for a loss or gain coefficient dependent on time. The cubic CGLE describes the dynamics of the formation of localized states (intensity patterns) along longitude z direction in bulk nonlinear medium.

The interest is to apply the reductive perturbation method to the generic system with the complex response. We show the initial generic complex FWM is reduced to the intrinsic system, which has three dependent variables (I_m, $\mathcal{E}$ are complex ones, and I_d is real). The intrinsic system has a form very similar to the complex Maxwell-Bloch system. It coincides completely with the Maxwell-Bloch system when at the same time the response is purely nonlocal the time relaxation is absent. In optics there exists an example of reduction in the Maxwell-Bloch system to the CGLE, which describes the formation of transverse mode structures in lasers [35], but they are derived in a high-order approximation.

Till nowadays a number of solutions of the CGLE have been found including stable localized patterns [4], i.e., dissipative solitons. These solutions may find applications in the dissipative FWM system. They have great potential for practical use in photonics by applying wave-coupling with a nonlocal medium.

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[1] I. S. Aranson and L. Kramer, Rev. Mod. Phys. **74**, 99 (2002).
[2] *Dissipative Solitons*, edited by N. Akhmediev and A. Ankiewicz, Lecture Notes in Physics Vol. 661 (Springer, Berlin, 2005), p. 448.
[3] *Dissipative Solitons: From Optics to Biology and Medicine*, edited by N. Akhmediev and A. Ankiewicz, Lecture Notes in Physics Vol. 751 (Springer, New York, 2008).
[4] W. van Saarloos, Phys. Rep. **386**, 29 (2003).
[5] N. N. Rosanov, *Spatial Hysteresis and Optical Patterns* (Springer, Berlin, 2002).
[6] N. Akhmediev, J. M. Soto-Crespo, and G. Town, Phys. Rev. E **63**, 056602 (2001).
[7] J. D. Ania-Castañón, V. Karalekas, P. Harper, and S. K. Turitsyn, Phys. Rev. Lett. **101**, 123903 (2008).
[8] C. Montes, in *Dissipative Solitons: From Optics to Biology and Medicine*, edited by N. Akhmediev and A. Ankiewicz, Lecture Notes in Physics Vol. 751 (Springer, New York, 2008), p. 221.
[9] S. Coulibaly, C. Durniak, and M. Taki, in *Dissipative Solitons: From Optics to Biology and Medicine*, edited by N. Akhmediev and A. Ankiewicz, Lecture Notes in Physics Vol. 751 (Springer, New York, 2008), p. 261.
[10] A. Parola, L. Salasnich, and L. Reatto, Phys. Rev. A **57**, R3180 (1998).
[11] V. M. Pérez-García, V. V. Konotop, and J. J. García-Ripoll, Phys. Rev. E **62**, 4300 (2000).
[12] C. Rotschild, O. Cohen, O. Manela, M. Segev, and T. Carmon, Phys. Rev. Lett. **95**, 213904 (2005).
[13] D. Suter and T. Blasberg, Phys. Rev. A **48**, 4583 (1993).
[14] S. Skupin, M. Saffman, and W. Królikowski, Phys. Rev. Lett. **98**, 263902 (2007).
[15] A. A. Zòzulya and D. Z. Anderson, Phys. Rev. A **51**, 1520 (1995).
[16] Z. Xu, Y. V. Kartashov, and L. Torner, Phys. Rev. Lett. **95**, 113901 (2005).
[17] W. Królikowski, O. Bang, J. J. Rasmussen, and J. Wyller, Phys. Rev. E **64**, 016612 (2001).
[18] C. Conti, M. Peccianti, and G. Assanto, Phys. Rev. Lett. **91**, 073901 (2003).
[19] A. A. Minzoni, N. F. Smyth, A. L. Worthy, and Y. S. Kivshar, Phys. Rev. A **76**, 063803 (2007).
[20] W. Hu, S. Ouyang, P. Yang, Q. Guo, and S. Lan, Phys. Rev. A **77**, 033842 (2008).
[21] E. Ultanir, G. I. Stegeman, D. Michaelis, C. H. Lange, and F. Lederer, in *Dissipative Solitons*, edited by N. Akhmediev and A. Ankiewicz, Lecture Notes in Physics Vol. 661 (Springer,

Berlin, 2005), p. 37.

[22] N. N. Rosanov, S. V. Fedorov, and A. N. Shatsev, in *Dissipative Solitons: From Optics to Biology and Medicine*, edited by N. Akhmediev and A. Ankiewicz, Lecture Notes in Physics Vol. 751 (Springer, New York, 2008), p. 93.

[23] *Photorefractive Materials and Their Applications*, edited by P. Günter and J.-P. Huignard, Topics in Applied Physics Vols. 61 and 62 (Springer Verlag, Heidelberg, 1988).

[24] *Photorefractive Effects, Materials, and Devices*, edited by P. Delayer, C. Denz, L. Mager, and G. Montemezzani, Trends in Optics and Photonics Series Vol. 87 (Optical Society of America, Washington D.C., 2003).

[25] J. H. Hong and R. Saxema, Opt. Lett. **16**, 180 (1991).

[26] M. Jeganathan, M. C. Bashaw, and L. Hesselink, J. Opt. Soc. Am. B **12**, 1370 (1995).

[27] S. Bugaychuk, L. Kóvacs, G. Mandula, K. Polgár, and R. A. Rupp, Phys. Rev. E **67**, 046603 (2003).

[28] E. N. Tsoy, A. Ankiewicz and N. Akhmediev, Phys. Rev. E **73**, 036621 (2006).

[29] A. A. Zòzulya and V. T. Tikhonchuk, Phys. Lett. A **135**, 447 (1989).

[30] A. Błędowski, W. Królikowski, and A. Kujawski, J. Opt. Soc. Am. B **6**, 1544 (1989).

[31] R. Conte and S. Bugaychuk, J. Phys. A: Math. Theor. **42**, 192003 (2009).

[32] S. P. Burtsev, V. E. Zakharov, and A. V. Mikhailov, Teor. Mat. Fiz. **70**, 227 (1987) [Theor. Math. Phys. **70**, 323 (1987)].

[33] T. Taniuti and N. Yajima, J. Math. Phys. **10**, 1369 (1969).

[34] T. Dauxois and M. Peyrard, *Physics of Solitons* (Cambridge University Press, Cambridge, 2006).

[35] K. Staliünas, Phys. Rev. A **48**, 1573 (1993).