Analytical description for field-line wandering in strong magnetic turbulence

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We investigate analytically the random walk of magnetic field lines. In previous analytical treatments of field line wandering or random walk, it was assumed that the turbulent magnetic field is much weaker than the mean field. In the present paper, we develop an improved analytical method to describe the stochastic properties of turbulent magnetic fields. This approach is an extension of the standard theory of field line wandering and can be applied to weak as well as to strong magnetic turbulence.

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I. INTRODUCTION

In the theory of field line wandering, we assume a superposition of a mean magnetic field $\vec{B}_0$ and a stochastic/turbulent component $\delta\vec{B}$. Such configurations can be found in various astrophysical contexts such as the solar wind or the interstellar medium. Due to the stochastic component, field lines are not well defined and have to be described using methods of statistical physics. The main aim of the theory of field line random walk (FLRW) is the computation of the field line diffusion coefficient and the field line distribution function. These parameters are also important for describing the interaction between turbulence and charged particles such as cosmic rays (see, e.g., [1–5])

In initial investigations of FLRW, a simple slab model for the turbulence was used (see, e.g., [6]). The slab model is a one-dimensional model for the turbulence, in which the stochastic field depends only on the coordinate along the mean magnetic field $\delta\vec{B}(\vec{x})=\delta\vec{B}(z)$. For such a simplified model, the theory of field line wandering is exact. However, slab turbulence is not a very realistic model for approximating solar wind or interstellar turbulence. In the first case observations have shown that there is a strong perpendicular component (see, e.g., [7]) also known as two-dimensional (2D) modes. In the two-dimensional model, the stochastic field depends only on the two coordinates across the mean magnetic field corresponding to $\delta\vec{B}(\vec{x})=\delta\vec{B}(x,y)$. For small values of the *plasma beta* (which is the ratio of the plasma pressure to the magnetic pressure), the theory of nearly incompressible magnetohydrodynamics (MHD) (see [8]) predicts a collapse in dimensionality making turbulence in the solar wind a superposition of a dominant 2D and a slab component. In the case of interstellar turbulence *in situ* observations are, of course, not available. In this case, however, numerical simulations of turbulence have demonstrated that the slab model is not very accurate (for a review, see [9]).

For improved turbulence models such as slab/2D models or isotropic and anisotropic models, an exact description of FLRW is no longer possible. In this case different theories have been proposed in the past few years. The most prominent examples are:

(i) *Quasilinear Theory.* Jokipii and Parker (see [6]) used a quasilinear theory (QLT) for computing the field line diffusion coefficient. QLT, which is a first-order perturbation theory, assumes that the turbulent magnetic field can be evaluated along unperturbed field lines while a field line diffusion coefficient is computed.

(ii) *Nonlinear Diffusion Theory.* Matthaeus *et al.* (see [10]) developed a nonlinear theory for computing field line diffusion coefficients. This theory is based on the assumption that the field lines are diffusive and obey *Gaussian statistics*. Furthermore, the so-called *Corrsin approximation* has been used [11–13].

(iii) *Generalized Nonlinear Theory.* Shalchi and Kourakis (see [14]) generalized the diffusion theory of Matthaeus *et al.* to allow also for nondiffusive behavior of FLRW. For several turbulence spectra proposed by previous authors, Shalchi and Kourakis (see [15]) found superdiffusive FLRW. Superdiffusion of field lines was also obtained by Zimbardo *et al.* [16,17].

These analytical theories were applied to different turbulence models and physical situations in the previous years (see, e.g., [18–20]). Other authors applied analytical forms of field line diffusion coefficients to the transport of cosmic rays by employing a *Chapman-Kolmogorov approach* (see [1–5],). Recently, Shalchi *et al.* [21] showed that the assumed statistics is less important, confirming the theories of Matthaeus *et al.* [10] and Shalchi and Kourakis [14], in which the *ad hoc* assumption of *Gaussian statistics* has been used.

Another problem that we have ignored in the discussion above is the time dependence of the turbulent field $\delta\vec{B}(\vec{x},t)$. To include such dynamical turbulence effects in the theory of field line wandering is difficult. Therefore, we employ the so-called magnetostatic approximation in the present paper. Time dependent turbulence was employed in the context of cosmic ray propagation studies (see, e.g., [22] for a review). A simple example for a dynamical turbulence model would be the assumption that the turbulence dynamics can be represented by undamped parallel propagating shear Alfvén waves. This model was combined with the standard theory of field line wandering by [3].

The basis of any analytical theory of FLRW is the field line equation

$$\frac{dx}{dz}=\frac{B_x[\vec{x}(z)]}{B_z[\vec{x}(z)]}=\frac{\delta B_x[\vec{x}(z)]}{B_0+\delta B_z[\vec{x}(z)]}, \tag{1}$$

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where we have chosen our system of *Cartesian coordinates*, so that the z axis is aligned along the mean magnetic field ($\vec{B}_0=B_0\vec{e}_z$). The notation $\delta B_i[\vec{x}(z)]$ means that the turbulent field, which depends on all three Cartesian coordinates, is computed along the field line, which is represented by the z-dependent vector $\vec{x}(z)$.

Equation (1) can be simplified by assuming weak turbulence ($\delta B_i \ll B_0$) or at least a weak parallel component of the turbulent magnetic field ($\delta B_z \ll B_0$). In these cases the fundamental equation in the theory of FLRW reads $dx = dz\,\delta B_x/B_0$. This form has been used in previous linear and nonlinear investigations of field line wandering (see, e.g., [10,14]). In the following we call this approach the weak turbulence approximation (WTA).

It is the purpose of the present paper is to develop an analytical approach for FLRW without assuming a weak turbulent magnetic field. Our approach is based on ideas developed by Shalchi and Dosch [23] for describing cross-field diffusion of charged energetic particles.

II. THEORY OF FIELD LINE WANDERING

A. Fundamental equations

By defining the parameters [24]

$$D_x(z)=\frac{dx(z)}{dz}, \quad D_y(z)=\frac{dy(z)}{dz}, \tag{2}$$

the field line equation [see Eq. (1)] can be written as

$$\delta B_i[\vec{x}(z)] = D_i(z)\{B_0+\delta B_z[\vec{x}(z)]\}, \tag{3}$$

with $i=x,y$ and the field line vector $\vec{x}(z)$. The ij component of the field line diffusion tensor κ_{ij} can be computed by employing the well-established *Taylor-Green-Kubo (TGK) formulation* (see [25–27]),

$$\kappa_{ij}=\int_0^\infty dz \langle D_i(z)D_j^*(0)\rangle. \tag{4}$$

From Eq. (3) we can derive

$$\begin{aligned}\langle \delta B_i(z)\delta B_j^*(0)\rangle &= \langle D_i(z)D_j^*(0)\rangle B_0^2 + \langle D_i(z)D_j^*(0)\delta B_z(z)\delta B_z^*(0)\rangle \\ &\approx \langle D_i(z)D_j^*(0)\rangle B_0^2 + \langle D_i(z)D_j^*(0)\rangle \\ &\quad \times \langle \delta B_z(z)\delta B_z^*(0)\rangle. \end{aligned} \tag{5}$$

Here, we assumed that *third-order correlations* are zero and that *fourth-order correlations* can be approximated by a product of two *second-order correlations* as suggested by Matthaeus *et al.* [28]. Note that $\delta B_i(z)$ denotes $\delta B_i[\vec{x}(z)]$ (the magnetic field has to be evaluated along a field line). The validity of the approximation used in Eq. (5) has not been explored theoretically. However, in order to develop an analytical theory of FLRW, this approximation is necessary. Previously this approximation has been used in cosmic ray diffusion theory and the results of the corresponding theory were compared with computer simulations (see, e.g., [22] for a review). According to such comparisons Eq. (5) provides a good approximation.

Equation (5) can be combined with the TGK formula [Eq. (4)] to yield

$$\kappa_{ij}=\int_0^\infty dz \frac{\langle \delta B_i(z)\delta B_j^*(0)\rangle}{B_0^2+\langle \delta B_z(z)\delta B_z^*(0)\rangle}. \tag{6}$$

Although mathematically complicated, Eq. (6) can be seen as a fundamental equation in the theory of FLRW. In the following, we will evaluate Eq. (6) analytically and numerically.

B. Application of Corrsin's independence hypothesis

A shown above, one has to know the magnetic correlation function $\langle \delta B_i(z)\delta B_j^*(0)\rangle$ in order to compute the diffusion coefficient [see, e.g., Eq. (6)]. In the present paper we describe two possibilities to compute this fundamental function. The first, which is described in the present paragraph, is the standard approach which is used in the theory of FLRW. The approach relies on the so-called *Corrsin approximation*. An alternative method is presented in the next section.

To replace magnetic correlation functions one can use

$$\begin{aligned}\langle \delta B_i(z)\delta B_j^*(0)\rangle &= \int d^3k \int d^3k' \langle \delta B_i(\vec{k})\delta B_j^*(\vec{k}') e^{i[\vec{k}\cdot\vec{x}(z)-\vec{k}'\cdot\vec{x}(0)]}\rangle \\ &\approx \int d^3k \int d^3k' \langle \delta B_i(\vec{k})\delta B_j^*(\vec{k}')\rangle \\ &\quad \times \langle e^{i[\vec{k}\cdot\vec{x}(z)-\vec{k}'\cdot\vec{x}(0)]}\rangle \\ &\approx \int d^3k P_{ij}(\vec{k})\langle e^{i\vec{k}\cdot[\vec{x}(z)-\vec{x}(0)]}\rangle. \end{aligned} \tag{7}$$

Here, we have employed a *Fourier representation* of the turbulent field, the Corrsin approximation [11–13], assumed homogeneous turbulence, and have introduced the magnetic correlation tensor in the wave vector space $P_{ij}(\vec{k}) = \langle \delta B_i(\vec{k})\delta B_j^*(\vec{k})\rangle$. The vector $\vec{k}$ is the turbulence wave vector. The *Corrsin approach* has been used in the nonlinear theory of FLRW developed by Matthaeus *et al.* [10]. It should be noted, however, that the same approximation has already been used by Lerche [29] who called this approach a *random-phase approximation* since it is assumed that the amplitudes $\delta B_i(\vec{k})$ are uncorrelated from the phases $\exp(i\vec{k}\cdot\vec{x})$. As far as we have been able to ascertain no detailed investigation of this assumption has ever been given—despite its extensive use. However, several authors compared analytical results which are based on the Corrsin approximation with computer simulations (see, e.g., [18]). According to this work the Corrsin approximation works very well in the theory of field line wandering.

The characteristic function in Eq. (7) can be replaced by

$$\langle e^{i\vec{k}\cdot[\vec{x}(z)-\vec{x}(0)]}\rangle = \cos(k_\parallel z) e^{-\kappa_\perp k_\perp^2 |z|}. \tag{8}$$

Equation (8) is valid for *Gaussian statistics* [30] and for diffusive FLRW. The parameter $\kappa_\perp$ denotes the field line diffusion coefficient in axisymmetric turbulence with $\kappa_\perp = \kappa_{xx}=\kappa_{yy}$ and $\kappa_{xy}=\kappa_{yx}=0$. By combining Eqs. (7) and (8) we can easily derive

$$\langle \delta B_i(z)\delta B_j^*(0)\rangle = \int d^3k P_{ij}(\vec{k})\cos(k_\parallel z)e^{-\kappa_\perp k_\perp^2|z|}. \tag{9}$$

Equation (9) in combination with Eq. (6) allows the calculation of a field line diffusion coefficient. In order to apply this formulation, one has to specify the properties of the correlation tensor $P_{ij}(\vec{k})$. The general form of this tensor for axisymmetric turbulence has been discussed by Matthaeus and Smith [31]. For vanishing magnetic helicity these authors derived

$$P_{lm}(\vec{k}) = A(k_\parallel,k_\perp)\left[\delta_{lm} - \frac{k_l k_m}{k^2}\right]. \tag{10}$$

The form Eq. (10) takes into account the solenoidal constraint and can be applied for arbitrary (but axisymmetric) turbulence. The function $A(k_\parallel,k_\perp)$ is the turbulence wave spectrum which has to be specified.

C. Recovery of the standard theory in the limit of weak turbulence

In the following, we show how the standard theory of FLRW can be derived from Eq. (6) by assuming weak turbulence. In cases, in which the parallel turbulent field is much weaker than the mean field ($\delta B_z \ll B_0$), Eq. (6) becomes

$$\kappa_{ij} \approx \frac{1}{B_0^2}\int_0^\infty dz \langle \delta B_i(z)\delta B_j^*(0)\rangle. \tag{11}$$

With Eq. (9) and by assuming axisymmetry, we derive

$$\begin{aligned}\kappa_\perp &\approx \frac{1}{B_0^2}\int_0^\infty dz\langle \delta B_x(z)\delta B_x^*(0)\rangle \\ &\approx \frac{1}{B_0^2}\int d^3k P_{xx}(\vec{k})\int_0^\infty dz\, \cos(k_\parallel z)e^{-\kappa_\perp k_\perp^2 z} \\ &\approx \frac{1}{B_0^2}\int d^3k P_{xx}(\vec{k})\frac{k_\perp^2\kappa_\perp}{(k_\perp^2\kappa_\perp)^2 + k_\parallel}. \end{aligned}\tag{12}$$

Here, we assumed convergence of the wave-number integral. However, there is a whole family of wave spectra, for which the integral is not convergent. These cases correspond to superdiffusion of magnetic field lines (see, e.g., [15,20]). Formula (12) corresponds to the diffusion theory proposed by Matthaeus *et al.* (1995). As demonstrated, we can derive this theory from the more general Eq. (6) by assuming $\delta B_z \ll B_0$. In the next section, we drop this assumption.

III. ANALYTICAL AND NUMERICAL RESULTS FOR AN ARBITRARY TURBULENCE STRENGTH

A. Simple models for magnetic correlation functions

In contrast to the previous paragraph, we now use a different approach for replacing the magnetic correlation functions. The correlation functions which enter Eq. (6) are the correlation functions along the magnetic field line. Therefore, we can write

$$\langle \delta B_i(z)\delta B_j^*(0)\rangle = \int dx\int dy\langle \delta B_i(\vec{x})\delta B_j^*(\vec{0})\rangle f_{FL}(x,y;z). \tag{13}$$

For the field line distribution function $f_{FL}(x,y;z)$ we can assume again a *Gaussian distribution* and diffusive FLRW,

$$f_{FL}(x,y;z) = \frac{1}{4\pi|z|\kappa_\perp}e^{-(x^2+y^2)/(4\kappa_\perp|z|)}. \tag{14}$$

To compute field line diffusion coefficients we also have to specify the magnetic correlation function. To ensure mathematical tractability and for simple analytical estimations we employ the following forms (again we have assumed axisymmetry with respect to the mean magnetic field):

$$\begin{aligned}\langle \delta B_x(\vec{x})\delta B_x^*(\vec{0})\rangle &= \delta B_x^2 e^{-z^2/l_\parallel^2}e^{-(x^2+y^2)/l_\perp^2}, \\ \langle \delta B_z(\vec{x})\delta B_z^*(\vec{0})\rangle &= \delta B_z^2 e^{-z^2/l_\parallel^2}e^{-(x^2+y^2)/l_\perp^2}.\end{aligned}\tag{15}$$

A more realistic description of turbulence can be achieved by replacing the magnetic correlations by using the *Fourier transformation approach* described in Eqs. (7)–(9). In this case one has to specify the wave spectrum and other turbulence properties. It would be straightforward to combine this approach with Eq. (6). In the present paper, however, we abstain from such calculations for the sake of mathematical tractability.

The model defined in Eq. (15) describes the decorrelation of the magnetic fields with increasing distance. The parameters $\delta B_x^2 = \delta B_y^2$ and δB_z^2 describe the total strength of the turbulent magnetic fields in the different directions with respect to the mean field. $l_\parallel$ and $l_\perp$ are the correlation lengths along and across the mean magnetic field.

From Eq. (15) we can obtain models used previously:

(i) *Isotropic turbulence.* The isotropic model can be obtained from Eq. (15) by setting $\delta B_x^2 = \delta B_y^2 = \delta B_z^2 = \delta B^2/3$ and $l_\parallel = l_\perp$.

(ii) *Slab turbulence.* The slab model can be obtained from Eq. (15) by setting $\delta B_z^2 = 0$ and $l_\perp = \infty$.

(iii) *Two-dimensional turbulence.* The two-dimensional model can be obtained by setting $\delta B_z^2 = 0$ and $l_\parallel = \infty$. It should be noted that the assumption $\delta B_z^2 = 0$ is part of the standard two-dimensional model used in Matthaeus *et al.* [10] and in other articles. However, two-dimensional models with $\delta B_z^2 \neq 0$ can also be formulated.

B. Field line diffusion coefficients for an isotropic distribution of the magnetic fields

By combining Eqs. (14) and (15) with Eq. (13) we find for the correlation functions along the magnetic field lines the following forms

$$\langle \delta B_x[\vec{x}(z)]\delta B_x^*[\vec{x}(0)]\rangle = \frac{\delta B_x^2 e^{-z^2/l_\parallel^2}}{1 + 4|z|\kappa_\perp/l_\perp^2},$$

$$\langle \delta B_z[\vec{x}(z)]\delta B_z^*[\vec{x}(0)]\rangle = \frac{\delta B_z^2 e^{-z^2/l_\parallel^2}}{1+4|z|\kappa_\perp/l_\perp^2}. \tag{16}$$

We can simplify Eqs. (16) if we assume an isotropic field distribution corresponding to $\delta B_x^2=\delta B_y^2=\delta B_z^2=\delta B^2/3$ [32]. By combining Eqs. (16) and (6), and employing the integral transformation $\xi=z/l_\parallel$, we find

$$K=\frac{1}{3}B^2\int_0^\infty d\xi \frac{e^{-\xi^2}}{1+4KL^2\xi+\frac{1}{3}B^2e^{-\xi^2}}. \tag{17}$$

Here, we have used $B=\delta B/B_0$, $L=l_\parallel/l_\perp$, and the dimensionless diffusion coefficient $K=\kappa_\perp/l_\parallel$. The field line diffusion coefficient is controlled by two parameters, namely, the strength of the turbulent field with respect to the mean field B and the ratio of the two correlation length scales L. Equation (17) is a nonlinear integral equation, which is solved in the next section numerically and for some cases analytically.

C. Special cases/previous results

For special parameter regimes we can simplify Eq. (17) as follows.

(i) *The limit* $B\to 0$. In this case, Eq. (17) becomes

$$K\approx\frac{B^2}{3}\int_0^\infty d\xi\frac{e^{-\xi^2}}{1+4KL^2\xi}. \tag{18}$$

This case corresponds to the WTA used in previous investigations. In this special case Eq. (18) can be multiplied by L^2 to give

$$KL^2=\frac{R^2}{3}\int_0^\infty d\xi\frac{e^{-\xi^2}}{1+4KL^2\xi}. \tag{19}$$

In this limit the diffusion coefficient KL^2 depends only on one independent parameter, namely,

$$R\equiv BL=\frac{\delta B}{B_0}\frac{l_\parallel}{l_\perp}. \tag{20}$$

The parameter R, which is the well-known *Kubo number*, controls the solution of Eq. (19) and separates the quasilinear from the nonlinear regime (see Figs. 1–4 of the present paper).

(ii) *The limit* $KL^2\to 0$. In this case, Eq. (17) becomes (see, e.g., [33])

$$K\approx\frac{1}{3}B^2\int_0^\infty d\xi\frac{e^{-\xi^2}}{1+\frac{1}{3}B^2e^{-\xi^2}}\approx-\frac{\sqrt{\pi}}{2}Li_{1/2}\left(-\frac{B}{3}\right), \tag{21}$$

with the *polylogarithm function* $Li_n(z)$ [34]. This formula corresponds to the quasilinear result which can be obtained be setting $K=0$ on the left land side of Eq. (17). Within QLT, the diffusion coefficient K depends only on the relative strength of the turbulence B. For completeness we note that QLT is exact in the limit $L\to 0$ corresponding to slab turbulence.

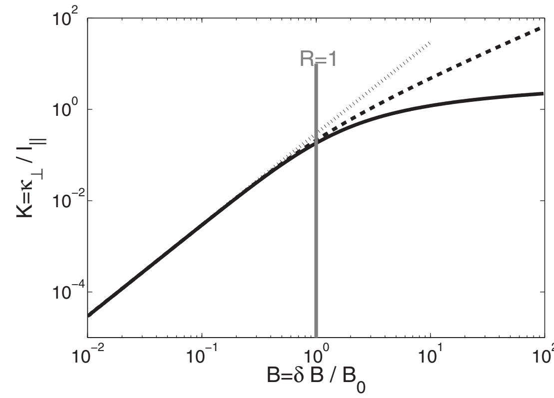

FIG. 1. The field line diffusion coefficient $K=\kappa_\perp/l_\parallel$ versus the strength of the turbulent magnetic field with respect to the mean field $B=\delta B/B_0$ for $L=l_\parallel/l_\perp=1$ corresponding to isotropic turbulence. Shown is the nonlinear result obtained by solving Eq. (17) numerically (solid line) and the analytical result (dotted line) obtained by employing SQLT $(B,K\to 0)$. The analytical result for this limit is given by Eqs. (22) and (23). The weak turbulence result without the quasilinear approximation is also shown (dashed line). Also shown is the value $R=1$ for the Kubo number, which separates the linear and nonlinear regimes within the WTA.

(iii) *The limit* $B\to 0$, $KL^2\to 0$. In this case, Eq. (17) becomes

$$K\approx\frac{B^2}{3}\int_0^\infty d\xi e^{-\xi^2}\approx\frac{\sqrt{\pi}}{6}B^2, \tag{22}$$

which can be written as

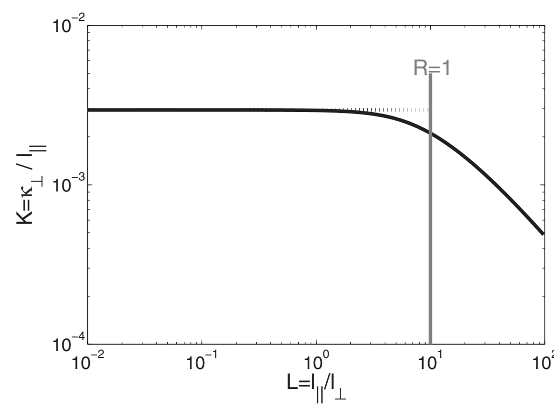

FIG. 2. The field line diffusion coefficient $K=\kappa_\perp/l_\parallel$ versus the ratio of the turbulence correlation lengths $L=l_\parallel/l_\perp$ for weak turbulence $B=\delta B/B_0=0.1$. Shown is the nonlinear result obtained by solving Eq. (17) numerically (solid line) and the analytical result (dotted line) obtained by employing SQLT $(B,K\to 0)$. The analytical result for this limit is given by Eqs. (22) and (23). The weak turbulence result without the quasilinear approximation is also shown (dashed line) but is in perfect coincidence with the solid line. Also shown is the value $R=1$ for the Kubo number, which separates the linear and nonlinear regimes within the WTA.

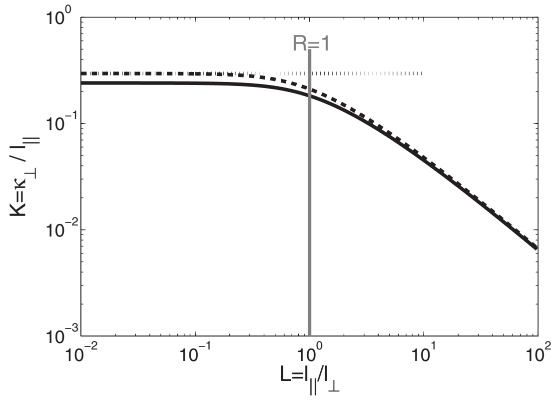

FIG. 3. The field line diffusion coefficient $K=\kappa_\perp/l_\parallel$ versus the ratio of the turbulence correlation lengths $L=l_\parallel/l_\perp$ for intermediate turbulence $B=\delta B/B_0=1$. Shown is the nonlinear result obtained by solving Eq. (17) numerically (solid line) and the analytical result (dotted line) obtained by employing SQLT ($B,K\to 0$). The analytical result for this limit is given by Eqs. (22) and (23). The weak turbulence result without the quasilinear approximation is also shown (dashed line). Also shown is the value $R=1$ for the Kubo number which separates the linear and nonlinear regimes within the WTA.

$$\kappa_\perp = \frac{\sqrt{\pi}}{6} l_\parallel \left(\frac{\delta B}{B_0}\right)^2. \tag{23}$$

The formula can be derived by employing QLT ($KL^2\to 0$) in combination with the WTA ($B\to 0$) [35]. In this very simple case, the field line diffusion coefficient depends linearly on the parallel correlation length and the relative magnetic energy of the turbulence. This case corresponds to the standard quasilinear theory (SQLT) derived earlier (see [6]).

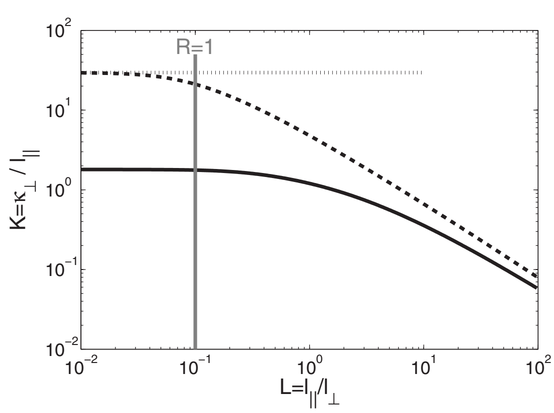

FIG. 4. The field line diffusion coefficient $K=\kappa_\perp/l_\parallel$ versus the ratio of the turbulence correlation lengths $L=l_\parallel/l_\perp$ for strong turbulence $B=\delta B/B_0=10$. Shown is the nonlinear result obtained by solving Eq. (17) numerically (solid line) and the analytical result (dotted line) obtained by employing SQLT ($B,K\to 0$). The analytical result for this limit is given by Eqs. (22) and (23). The weak turbulence result without the quasilinear approximation is also shown (dashed line). Also shown is the value $R=1$ for the Kubo number, which separates the linear and nonlinear regimes within the WTA.

D. Numerical results for field line diffusion coefficients

Equation (17) can be evaluated numerically exactly by using an iteration method. Figure 1 shows the field line diffusion coefficient versus the turbulence strength B for an isotropic model ($L=1$). Figures 2–4 show the field line diffusion coefficient versus the parameter L for different values of the turbulence strength B. Large values of L correspond to two-dimensional turbulence and small values of L to slablike turbulence.

In Fig. 1, it is shown that SQLT works very well for weak turbulence as expected. For strong turbulence, however, we find that QLT as well as the WTA are not very accurate and the results of the present article have to be used. As shown in Figs. 2–4, QLT is also inaccurate for large values of L corresponding to two-dimensional turbulence. The WTA works very well for turbulence that is not too strong.

E. Influence of the Kubo number

In Eq. (19) we identified the *Kubo number* since it controls transport in the weak turbulence limit ($\delta B_z \ll B_0$). The influence of the Kubo number on the transport of field lines and particles has been discussed by other authors (see, e.g., [16,36–40]). In Fig. 1 we assumed $L=1$ and, thus, $R\equiv B$. Therefore, the plot also shows the dependence of the dimensionless diffusion coefficient on the Kubo number R. Within SQLT we can use Eq. (23) which corresponds to $K\sim R^2$. If we drop QLT, and employ only the WTA we have to use Eq. (19). According to Fig. 1 we find for large values of the Kubo number the relation $K\sim R$. For small Kubo numbers the quasilinear regime holds.

If we drop all approximations and employ Eq. (17), we no longer find the simple dependence on the Kubo number as in the WTA. Instead, the dimensionless field line diffusion coefficient depends on two parameters (e.g., on R and B or L and B). According to direct numerical solutions of the field line equation (see [17]) the Kubo number is not the only parameter to determine the value of the diffusion coefficient. The value of the anistropy ratio L also plays an important role. Our analytical theory of FLRW agrees with this conclusion.

F. Comparison with the percolative scaling

In general one could assume that the scaling of the field line diffusion coefficient is given by

$$K = \alpha B^\beta L^\gamma \equiv \alpha R^\beta L^{\gamma-1}. \tag{24}$$

Although this formula is not always correct [see, e.g., Eq. (21)], is can be used to represent the diffusion coefficient for several cases. One example is the standard quasilinear result of Eq. (23). By comparing Eqs. (24) and (23) we find $\alpha=\sqrt{\pi}/6$, $\beta=2$, and $\gamma=0$ for quasilinear transport.

Isichenko [38] predicted the existence of a percolative regime with $\beta=0.7$ and $\gamma=-1.3$. The percolative scaling should be valid for high *Kubo numbers* and its existence has been investigated by several authors (see, e.g., [41–43]). In Fig. 5 we have computed the parameter γ for fixed B in the limit of large values of L. This limit should correspond to the

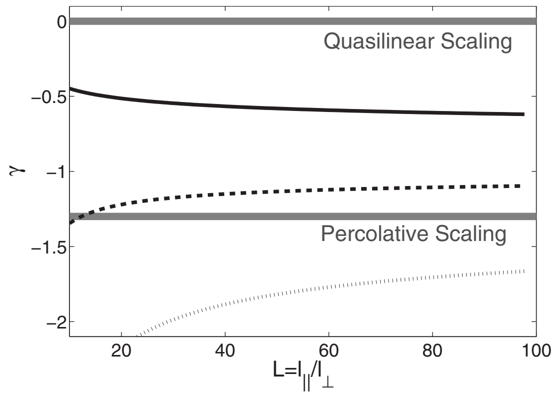

FIG. 5. The parameter γ which is defined as the exponent in the field line diffusion coefficient $K=\alpha B^{\beta} L^{\gamma}$ versus the parameter L. We have considered weak turbulence with $B=0.1$ (dotted line), intermediate turbulence with $B=1$ (dashed line), and strong turbulence with $B=10$ (solid line). Also shown are the standard quasilinear result with $\gamma=0$ and the percolative scaling with $\gamma=-1.3$.

limit of large *Kubo numbers*. Our result predicts values for γ between -0.5 and -1.7. Although our results are close to the percolative scaling for certain values of B, we cannot find the exact percolative scaling. The reason for our different result could be the application of the Corrsin approximation and the approximation of *fourth-order correlations* used in Eq. (5).

IV. SUMMARY AND CONCLUSION

In the present paper, we have revisited the problem of FLRW. For any turbulence model except the case of pure slab fluctuations one has to employ a quasilinear or nonlinear theory for describing the field line statistics. In previous investigations the assumption of weak turbulence ($\delta B_z \ll B_0$) was employed to simplify the field line equations. The WTA has been used in quasilinear as well as nonlinear calculations of field line diffusion coefficients.

In the present paper, we have generalized the standard nonlinear theory of FLRW developed previously (see, e.g., [10,14]). The approach no longer relies on the WTA since the restriction $\delta B_z \ll B_0$ is no longer used in the theory. By combining the fundamental and general formula (6) with Eq. (9) one can compute field line diffusion coefficients. The result depends on the magnetic correlation tensor $P_{ij}(\vec{k})$. For simplicity we combined Eq. (6) with Gaussian models for the decorrelation of magnetic fields [see Eq. (15)] to derive a nonlinear integral equation for FLRW [see Eq. (17)]. From this general formula we derived the quasilinear limit, the weak turbulence limit, and the combination of these two limits. If only the WTA is applied, the solution for the diffusion coefficient depends only on the Kubo number R. In the general case (e.g., for strong turbulence) this simple dependence on the Kubo number cannot be seen.

We have also computed field line diffusion coefficients by solving Eq. (17) numerically. Figures 1–4 show the field line diffusion coefficients and their dependence on the turbulence strength $B=\delta B/B_0$ and the ratio of the two correlation lengths $L=l_{\|}/l_{\perp}$. As expected, the previous result based on the WTA can be confirmed for weak turbulence ($\delta B \ll B_0$). Furthermore, QLT is valid for small values of the parameter L corresponding to slablike turbulence. For strong turbulence previous approaches (QLT as well as WTA) do not agree with our new result. Furthermore, the solution is controlled by two independent parameters, namely B and L instead of one parameter R as in the weak turbulence limit considered previously. We have also explored the scaling of the diffusion coefficient by assuming $K \sim L^{\gamma}$. As expected we found a result which disagrees with the quasilinear scaling ($\gamma=0$). Furthermore we have compared our result with the percolative scaling predicted by Isichenko [38]. This comparison is shown in Fig. 5.

The improved description of FLRW developed here provides a useful tool for computing field line diffusion coefficients and for improving our understanding of fundamental properties of magnetic turbulence and charged particles which experience scattering by interaction with turbulence.

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[1] G. M. Webb, G. P. Zank, E. Kh. Kaghashvili, and J. A. le Roux, Astrophys. J. **651**, 211 (2006).
[2] A. Shalchi and I. Kourakis, Astron. Astrophys. **470**, 405 (2007).
[3] A. Shalchi, R. C. Tautz, and R. Schlickeiser, Astron. Astrophys. **475**, 415 (2007).
[4] G. M. Webb, J. A. le Roux, G. P. Zank, E. Kh. Kaghashvili, and G. Li, AIP Conf. Proc. **1039**, 246 (2008).
[5] A. Shalchi, G. M. Webb, J. A. le Roux, and G. P. Zank, Astrophys. Space Sci. **321**, 197 (2009).
[6] J. R. Jokipii and E. N. Parker, Astrophys. J. **155**, 799 (1969).
[7] W. H. Matthaeus, M. L. Goldstein, and D. A. Roberts, J. Geophys. Res. **95**, 20673 (1990).
[8] G. P. Zank and W. H. Matthaeus, Phys. Fluids A **5**, 257 (1993).
[9] A. Lazarian, A. Beresnyak, A. H. Yan, M. Opher, and Y. Liu, Space Sci. Rev. **143**, 387 (2009).
[10] W. H. Matthaeus, P. C. Gray, D. H. Pontius, Jr., and J. W. Bieber, Phys. Rev. Lett. **75**, 2136 (1995).
[11] S. Corrsin, in *Atmospheric Diffusion and Air Pollution*, edited

by F. Frenkiel and P. Sheppard (Academic, New York, 1959), Vol. 6.
[12] Y. Salu and D. C. Montgomery, Phys. Fluids **20**, 1 (1977).
[13] W. D. McComb, *The Physics of Fluid Turbulence* (Oxford University Press, New York, 1990), p. 594.
[14] A. Shalchi and I. Kourakis, Phys. Plasmas **14**, 092903 (2007).
[15] A. Shalchi and I. Kourakis, Phys. Plasmas **14**, 112901 (2007).
[16] G. Zimbardo, P. Veltri, G. Basile, and S. Principato, Phys. Plasmas **2**, 2653 (1995).
[17] G. Zimbardo, P. Veltri, and P. Pommois, Phys. Rev. E **61**, 1940 (2000).
[18] D. Ruffolo, P. Chuychai, and W. H. Matthaeus, Astrophys. J. **644**, 971 (2006).
[19] B. Weinhorst, A. Shalchi, and H. Fichtner, Astrophys. J. **677**, 671 (2008).
[20] A. Shalchi and B. Weinhorst, Adv. Space Res. **43**, 1429 (2009).
[21] A. Shalchi, J. A. le Roux, G. M. Webb, and G. P. Zank, J. Phys. A: Math. Theor. **42**, 345501 (2009).
[22] A. Shalchi, *Nonlinear Cosmic Ray Diffusion Theories, Astrophysics and Space Science Library* (Springer-Verlag, Berlin, 2009), Vol. 362.
[23] A. Shalchi and A. Dosch, Phys. Rev. D **79**, 083001 (2009).
[24] In the present paper, we also allow $\delta B_z \geq B_0$ in certain applications. This case, corresponding to strong magnetic turbulence is problematic since one can obtain $D_i(z) \to \infty$. As shown in the following paragraphs, this singularity no longer occurs in the theory and, therefore, definition (2) seems to be unproblematic. However, we keep in mind that for such strong turbulence the theory of FLRW could become inaccurate.
[25] G. I. Taylor, Proc. London Math. Soc. **s2-20**, 196 (1922).
[26] M. S. Green, J. Chem. Phys. **19**, 1036 (1951).
[27] R. Kubo, J. Phys. Soc. Jpn. **12**, 570 (1957).
[28] W. H. Matthaeus, G. Qin, J. W. Bieber, and G. P. Zank, Astrophys. J. **590**, L53 (2003).
[29] I. Lerche, Astrophys. Space Sci. **23**, 339 (1973).
[30] Shalchi *et al.* [21] showed that the assumed statistics is less important. Even in cases where the field line distribution is *non-Gaussian*, Eq. (8) should be a good approximation.
[31] W. H. Matthaeus and C. Smith, Phys. Rev. A **24**, 2135 (1981).
[32] The following calculations could also be done without this approximation. In this case a further parameter, namely, the ratio $\delta B_x / \delta B_z$ would control the value of the diffusion coefficient.
[33] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 2000).
[34] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1974).
[35] Eq. (22) can also be derived from Eq. (21) by using the representation (see, e.g., [34]) $Li_n(z) = \Sigma_{k=1}^{\infty} z^k k^{-n}$ for small arguments corresponding to $B \ll 1$.
[36] B. B. Kadomtsev and O. P. Pogutse, *Plasma Physics and Controlled Nuclear Fusion Research*, Proceedings of the Seventh International Conference, Innsbruck, Austria, 23–30 August 1978 (International Atomic Energy Agency, Vienna, 1979), Vol. 1, pp. 649–662, discussion on pp. 662–663, Paper No. A80-17751 05-75.
[37] M. B. Isichenko, Plasma Phys. Controlled Fusion **33**, 795 (1991).
[38] M. B. Isichenko, Plasma Phys. Controlled Fusion **33**, 809 (1991).
[39] G. Zimbardo and P. Veltri, Phys. Rev. E **51**, 1412 (1995).
[40] P. Pommois, G. Zimbardo, and P. Veltri, Phys. Plasmas **5**, 1288 (1998).
[41] M. Vlad, F. Spineanu, J. H. Misguich, and R. Balescu, Phys. Rev. E **58**, 7359 (1998).
[42] P. Castiglione, J. Phys. A **33**, 1975 (2000).
[43] A. V. Milovanov, Phys. Rev. E **63**, 047301 (2001).