Exact multisolitonic solutions and their interactions in a (3+1)-dimensional system

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(Received 13 January 2009; revised manuscript received 14 August 2009; published 15 October 2009)

Taking the approach via a special variable separation, some features of the (3+1)-dimensional multisolitonic solutions, including the embedded soliton, the taperlike soliton, the plateau-type soliton, and the rectangle soliton, were revealed in this study thanks to the intrusion of the appropriate boundary conditions and/or their initial qualifications. Some physical properties, such as the spatiotemporal evolution, wave form structure, and interactive phenomena with or without the background waves of multisolitons are discussed, especially in the two-soliton case. It is found that different interactive behaviors of solitary waves take place under different parameter conditions of collision in this system. It is verified that the elastic interaction phenomena exist in this (3+1)-dimensional integrable model. Furthermore, in other types of nonlinear systems, the abundant (3+1)-dimensional multisolitonic solutions were also investigated.

DOI: 10.1103/PhysRevE.80.046603 PACS number(s): 05.45.Yv

I. INTRODUCTION

The development of the nonlinear wave theory clarifies the role of the "soliton" in various systems [1]. In general, solitons are stable and the interactions between them only affect the phase shifts. Therefore, solitons are regarded as the fundamental structures of the nonlinear integrable systems. In the past years, the (1+1)-dimensional solitons and solitary wave solutions have been studied extensively both in their theoretical and experimental aspects [2]. In the (2+1)-dimensional system, abundant stable localized excitations have also been reported using some significant integrable models, such as the Davey-Stewartson equation, the Nizhnik-Novikov-Vesselov (NNV) equation, the asymmetric NNV equation, the Broer-Kaup-Kupershmidt system, and the general $(N+M)$-component Ablowitz-Kaup-Newell-Segur system, in the field of nonlinear physics [3,4].

From a symmetry study of the (2+1)-dimensional integrable model, a quite rich symmetry structure were identified in comparison with the lower dimensions [5]. This fact indicates that more work can be done regarding the soliton structures and their interactions among the solitons of the higher dimensions nonlinear models. It is worthy to note here that the previous studies demonstrated that there were more localized excitations in (2+1) dimensions than those in (1+1) dimensions [3]. Because of the difficulties to find out the exact solutions with physical significance, information on excitations in (3+1)-dimensional integrable systems has been very limited so far. Given that the real physical space time is in (3+1) dimensions, its localized excitations have attracted a great attention of many mathematicians and physicists for years although only insignificant progress has been made in this direction [6,7]. Moreover, when saying that a model is integrable, one should emphasize two important facts. The first fact is that we should point out in what special sense(s) the model is integrable. For instance, we say a model is Painlevé integrable if it possesses the Painlevé property, and a model is either Lax or IST (inverse scattering transformation) integrable if it has a Lax pair or it can be solved by the IST approach. An integrable model under some special cases may not be integrable under other cases. For instance, some Lax integrable models may not be Painlevé integrable [8]. The second fact is that for the general solution of a higher-dimensional integrable model, for instance, a Painlevé integrable model, it possesses some characteristics with lower-dimensional *arbitrary* functions. The facts implicate that lower-dimensional solutions may be used to construct exact solutions of some higher-dimensional integrable models. In other words, the exotic behavior of integrable models may propagate along the characteristics of the lower- and higher-dimensional solutions.

Motivated by the reasons above, our laboratory has invested considerable efforts in the subject by taking the following (3+1)-dimensional Virasoro integrable model:

$$u_{xxt} + au_{xxx}u_{yz} + bu_{xxy}u_{xz} + cu_{xy}u_{xxz} + du_{xx}u_{xzy} + eu_{xxxyz} = 0, \tag{1}$$

where a, b, c, d, and e are arbitrary constants. It may be worthy to point out here that Eq. (1) was originally proposed by Lin *et al.* through the means of the realizations of the generalized centerless Virasoro-type symmetry algebra [6]. As an alternative expression, Eq. (1) could be considered as the (3+1)-dimensional integrable extension of the (2+1)-dimensional breaking soliton equation

$$v_{xt} = (b+d)v_x v_{xy} + (a+c)v_y v_{xx} + v_{xxxy} \tag{2}$$

for $z=x, u_x=v$. The main characteristic feature of the breaking soliton equations is that the spectral parameter possesses the so-called breaking behavior. Frankly speaking, the spectral value of breaking soliton equations may become a multivalued function under the defined conditions. Hence, the solution of these equations may also turn into multivalued. In the literature, Eq. (1) has been used to describe the wave interactions on the surface of the sea and it has been studied via an extended homogeneous balance method [9].

In order to get more meaningful solutions from Eq. (1), the equation could be rewritten as the following potential form:

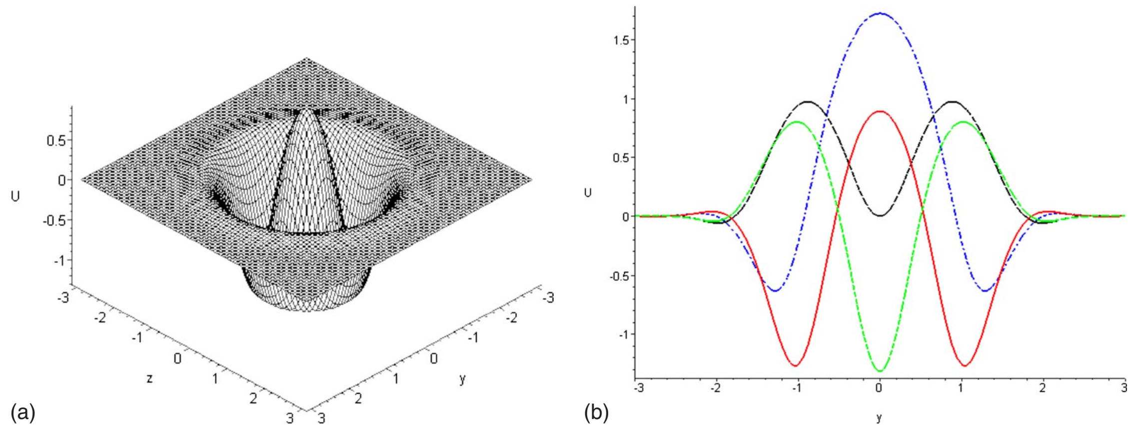

FIG. 1. (Color online) (a) Profile of the embedded solitons for the field v given by Eq. (5a) with the conditions (8a) and (8b). (b) A sectional view related to (a) at $z=0$ and $t=-4$: solid red line, $t=-2$: dotted and dashed blue line, $t=0$: dashed black line, and $t=0.5$: dashed green line.

$$v_{xt}+\frac{b+c}{4}v_{xx}w_z+bw_{xx}v_z+cv_yv_{xz}+\frac{b+c}{4}v_xv_{yz}+ev_{xxyz}=0,\quad v_y=w_x, \tag{3}$$

by using $v=u_x, w=u_y$ [6]. Based on the enlightenment properties of the Bäcklund transformation, we proposed that the solutions of

$$v=\frac{v_0}{\phi}+v_1,\quad w=\frac{w_0}{\phi}+w_1, \tag{4}$$

where $\{v_1,w_1\}$ is an arbitrary known soliton solution of Eq. (3), while v_0, w_0, and ϕ could only be determined by substituting Eq. (4) into Eq. (3). To get significant solutions, the seed solution was fixed as $v_1=0$, $w_1=w_1(y,z,t)$ being arbitrary function of y,z,t.

By means of a computerized algebra (i.e., Maple), we obtained

$$v=\frac{8ek_1}{(b+c)[1+\exp(-k_1x-f)]}, \tag{5a}$$

$$w=\frac{8e}{(b+c)}\left(\frac{f_y}{1+\exp(-k_1x-f)}+g^{-1}g_y\right)+w_1, \tag{5b}$$

where b,c,e and k_1 are arbitrary constants, $f\equiv f(y,z,t)$ is an arbitrary function of $\{y,z,t\}$, and g,w_1 satisfy

$$2g_t-ek_1g_{yz}+ek_1(2g^{-1}g_yg_z+f_yg_z+f_zg_y)=0,$$

$$\frac{b+c}{4}k_1w_{1z}+f_t+ek_1(f_{yz}+f_yf_z)+[f_t+ek_1(f_yg_z+f_zg_y+g_{yz})]g^{-1}=0, \tag{6}$$

where $g\equiv g(y,z,t)$ and $w_1\equiv w_1(y,z,t)$.

After introducing some minor modifications, the following formula [the general form of Eq. (5a)] could be produced:

$$U\equiv\frac{\alpha\beta}{\gamma[1+\exp(-\beta x-f)]}. \tag{7}$$

This equation was valid for some suitable fields or potential quantities of a diversity of (3+1)-dimensional physically models, including the Burgers system, the NNV equation, the Jimbo-Miwa system, the potential-Yu-Toda-Sasa-Fukuyama equation, and the Korteweg-de Vries-type equation. In Eq. (7), $f\equiv f(y,z,t)$ is an arbitrary function of the indicated variable, while α,β and γ are taken as constants. In the universal formula (7), the appearance of the arbitrary function f seems to be closely related to the arbitrary boundary conditions in some types of quantities for the related models. In some well-regarded publications, the effects of the arbitrary boundary conditions were studied using the (2+1)-dimensional integrable systems [10–14]. We trust that one of the interesting results obtained by Fokas and Santini is worthy to be singled out here, in which the localized traveling solutions (e.g., dromions) did not preserve their form upon interactions and hence exchange their energy, but only some specified spectral parameters kept the solutions preserve their forms. Similar results were also found in other (2+1)-integrable models.

In this paper, the appropriate selection conditions of the arbitrary functions were presented in the universal formula (7) in order to obtain the complete elastic interactions.

Based on the arguments introduced by Fokas and Santini [10], one could, in principle, investigate the stability properties of the solutions and their relevance as asymptotic states for suitable initial boundary-value problems. However, in this paper, we only studied the interactive behavior among the localized solutions by analyzing the asymptotic properties of the universal formula (7), which were valid for more than one system.

This paper is organized as follows. In Sec. II, some (3+1)-dimensional solitonic solutions, such as the embedded soliton, the taperlike soliton, the plateau-type soliton, and the rectangle soliton, are revealed by introducing the appropriate boundary conditions and/or initial qualifications. In Sec. III, the interactive properties of the solitonic solutions are ana-

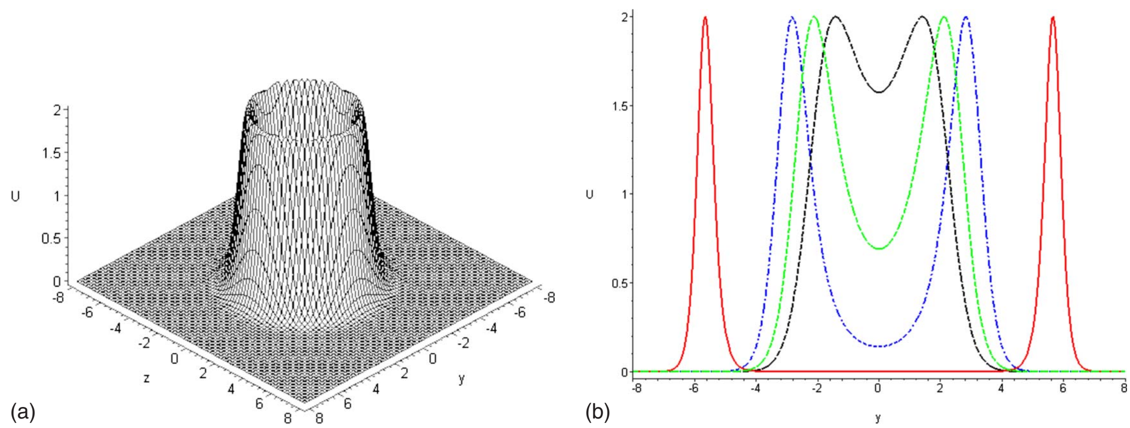

FIG. 2. (Color online) (a) Profile of the embedded solitons for the field v given by Eq. (5a) with the condition (9) and $x=0$, $b=c=e=k_1=1$. (b) A sectional view related to (a) at $z=0$ and $t=-4$: solid red line, $t=-2$: dotted and dashed blue line, $t=-1$: dashed black line, and $t=1.5$: dashed green line.

lyzed both analytically and graphically. A brief discussion and summary is given in the last section.

II. SOLITONIC SOLUTIONS OF THE (3+1)-dimensional breaking soliton equation

Because of the arbitrariness introduced by $f(y,z,t)$, solution (5a) may exhibit a number of abundant structures. Dromions, camber-type, ring-shape, and bubblelike solitons had been reported in the literature [6]. In this paper, we listed and plotted only some solitons, including the embedded soliton, the taperlike soliton, the plateau-type soliton, and the rectangle soliton, from the exact solution shown by Eq. (5a).

Given that some interesting lower dimensions embedded solitons were reported in Ref. [15], a natural and important question may remain on if we can find some types of (3+1)-dimensional embedded solitons from the "universal" formula. Fortunately, a positive response can be obtained based on the arbitrariness of the function f in the universal formula. For instance, when choosing f to be

$$f(y,z,t) = -y^2 - z^2 + \ln[\sin(y^2 + z^2 - t^2)], \tag{8a}$$

then we could derive an embedded soliton for the physical field v expressed by Eq. (5a). The corresponding profile is presented in Figs. 1(a) and 1(b), where Fig. 1(a) is a special structure of the embedded-soliton type of solution with $x=0$ and the parameter selections

$$b = c = e = k_1 = 1, \tag{8b}$$

at time $t=-4$. Figure 1(b) showed the sectional structure of embedded soliton expressed by Eq. (5a) with Eq. (8a) and the parameter selections are the same as those in Eq. (8b), except for $z=0$, where the embedded soliton suggested a breathe property.

Similarly, if taking f as

$$f(y,z,t) = \ln[\text{sech}(0.5y^2 + 0.5z^2 - t^2)], \tag{9}$$

then we could derived another kind of embedded soliton for the physical field v in Eq. (5a), as shown in Figs. 2(a) and 2(b).

When considering f to be

$$f(y,z,t) = -\sqrt{y^2 + z^2}, \tag{10}$$

then a taperlike soliton from the physical field v of Eq. (5a) could be obtained as shown in Fig. 3.

Furthermore, when choosing f to be

$$f(y,z,t) = \ln\{2.715 - \exp[\tanh(y^2 + z^2 - t^2)]\}, \tag{11}$$

then we could obtain a plateau-type ring soliton (named by Lou [16]). From the physical field v in Eq. (5a), as shown in Fig. 4.

Similarly, when choosing f as

$$f(y,z,t) = \ln\{2.715 - \exp[\tanh(y^4 + z^4 - t^2)]\}, \tag{12}$$

then we could obtain a rectangle soliton from the physical field v of Eq. (5a), as shown in Fig. 5.

III. INTERACTIVE PROPERTIES OF THE (3+1)-dimensional solitonic solutions

In a previous study from our laboratory, we plotted some interaction figures for two special types of solitonic solutions (i.e., the semifolded solitary waves and semifoldons) with or without complete interactive properties [17]. Subsequently, the complete interactive properties of the plateau-type, basin-

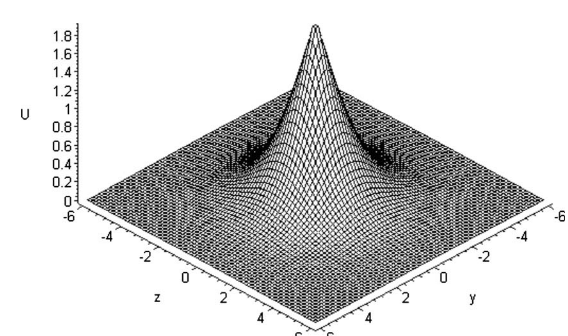

FIG. 3. Plot of the taperlike solitons for the field v given by Eq. (5a) with the condition (10) and $x=0$, $b=c=e=k_1=1$.

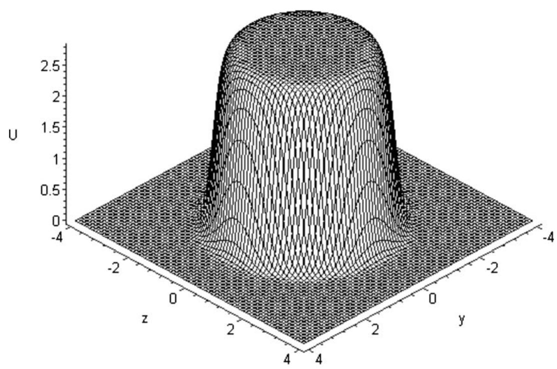

FIG. 4. Profile of the plateau-type ring solitons for the field v given by Eq. (5a) with the condition (11) and $x=0$, $b=c=e=k_1=1$.

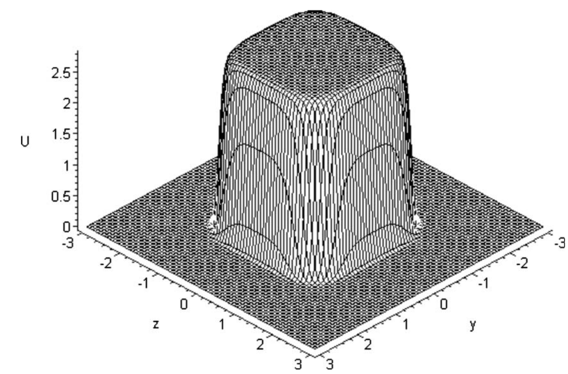

FIG. 5. Plot of the rectangle solitons for the field v given by Eq. (5a) with the condition (12) and $x=0$, $b=c=e=k_1=1$.

type, and bowl-type ring solitons of the (2+1)-dimensional modified Nizhnik-Novikov-Vesselov system were further discussed [18]. In this section, as an extension, we discussed the interactive properties of the solitonic solutions related to the (3+1)-dimensional system.

A. Asymptotic behaviors of the localized solitonic excitations produced from Eq. (7)

In general, if the function f is selected as localized solitonic excitations with

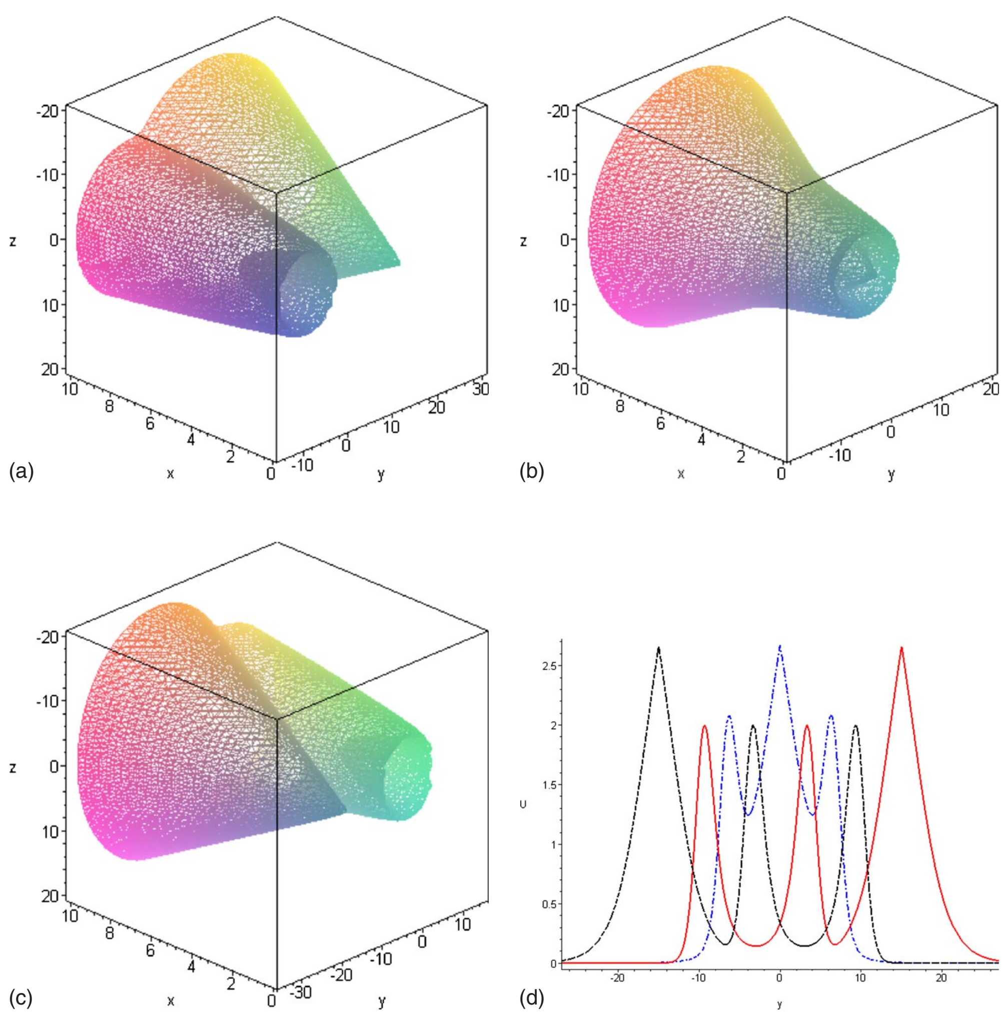

FIG. 6. (Color online) Time evolutional profiles of the interactions between an embedded soliton and a taperlike soliton for the field v given by Eq. (5a) with the condition (18) and $b=c=e=k_1=1$ at various time points. (a) $t=-3$, (b) $t=0$, (c) $t=3$, and (d) a sectional view related to (a), (b), and (c) at $x=z=0$: solid red line (a), dotted and dashed blue line (b), and dashed black line (c).

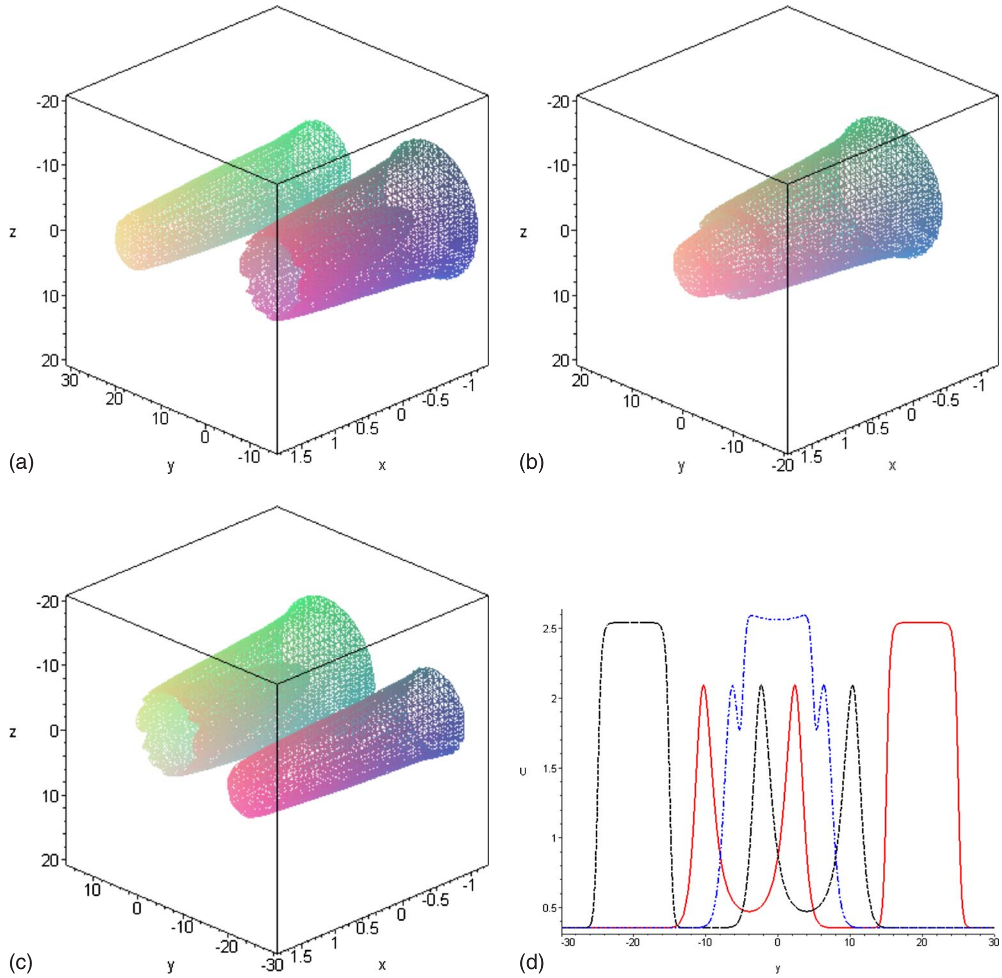

FIG. 7. (Color online) Time evolutional profiles of the interactions between an embedded soliton and a plateau-type soliton for the field v given by Eq. (5a) with the condition (19) and $b=c=e=k_1=1$ at various times points. (a) $t=-4$, (b) $t=0$, (c), $t=4$, and (d) a sectional view related to (a), (b), and (c) at $x=z=0$: solid red line (a), dotted and dashed blue line (b) and dashed black line (c).

$$f|_{t\to\mp\infty} = \sum_{i=1}^{M} f_i^{\mp}, \quad f_i^{\mp} \equiv f_i(z, y - c_i t + \delta_i^{\mp}), \tag{13}$$

where $\{f_i\}\forall i$ are localized functions, then the physical quantity U expressed by Eq. (7) can deliver M $(3+1)$-dimensional localized solitonic excitations with the asymptotic behavior

$$U|_{t\to\mp\infty} \to \sum_{i=1}^{M} \left\{ \frac{\alpha\beta}{\gamma\{1+\exp[-\beta x - (f_i^{\mp} + F_i^{\mp})]\}} \right\} \equiv \sum_{i=1}^{M} U_i^{\mp}(x, z, y - c_i t + \delta_i^{\mp}) \equiv \sum_{i=1}^{M} U_i^{\mp}, \tag{14}$$

where

$$F_i^{\mp} = \sum_{j<i} f_j(\mp\infty) + \sum_{j>i} f_j(\pm\infty), \tag{15}$$

assuming there was no loss of generality, $c_i > c_j$ if $i > j$.

Deduced from expression of Eq. (7), the ith localized excitation U_i preserves its shape during the interaction if

$$F_i^{+} = F_i^{-}. \tag{16}$$

Meanwhile, the phase shift of the ith localized excitation U_i reads as

$$\delta_i^{+} - \delta_i^{-} \tag{17}$$

in the y direction.

The above discussions demonstrated that the localized solitonic excitations for the universal quantity U could be constructed without difficulties via the $(2+1)$-dimensional localized excitations with the properties in Eqs. (13) and (16). As a matter of fact, all localized solutions (or their derivatives) with completely elastic, not completely elastic, or completely inelastic interactive behaviors of any known $(2+1)$-dimensional integrable models could be utilized to construct the $(3+1)$-dimensional localized solitonic solutions with complete elastic ($F_i^{+}=F_i^{-}$, for all i), not complete elastic or complete inelastic ($F_i^{+}\neq F_i^{-}$, at least for one of i) interactive properties, respectively. In order to observe the interactive behaviors directly and visually, we investigated some representing examples by fixing the arbitrary functions f in

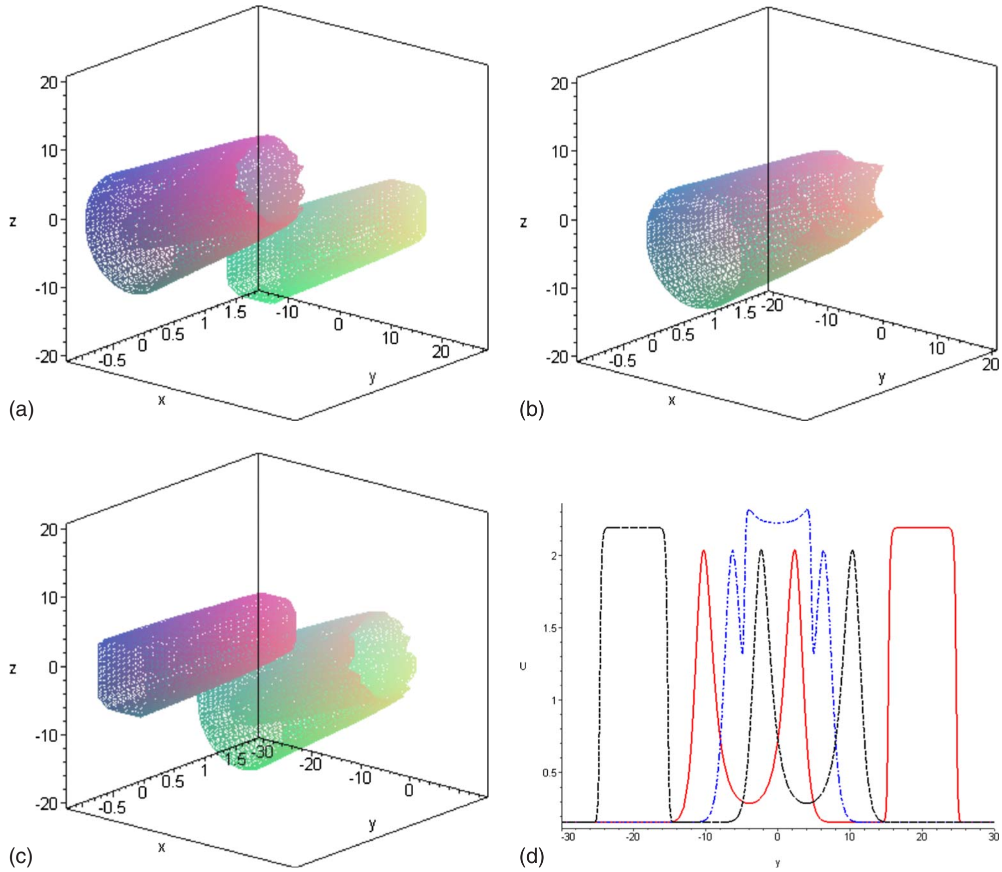

FIG. 8. (Color online) Time evolutional profiles of the interactions between an embedded soliton and a plateau-type soliton for the field v given by Eq. (5a) with the condition (20) and $b=c=e=k_1=1$ at various times points: (a) $t=-4$, (b) $t=0$, (c) $t=4$, and (d) a sectional view related to (a), (b), and (c) at $x=z=0$: solid red line (a), dotted and dashed blue line (b), and dashed black line (c).

Eq. (5a). For convenience, we set $b=c=e=k_1=1$ in Eq. (5a) in the following discussions.

B. Completely elastic interactions among the embedded solitons and other solitons on a flat background

In this section, some representing examples were studied on the interactions of the embedded solitons and other solitons. First, we considered interactions between an embedded soliton and a taperlike soliton. Therefore, when f was shown to be

$$f(y,z,t) = \ln\{\mathrm{sech}[0.1(y-t)^2 + 0.1z^2 - 4] + 2\exp[-0.5\sqrt{(y+5t)^2+z^2}]\}, \tag{18}$$

an embedded soliton and a taperlike soliton could be derived for the physical field v in Eq. (5a), while Fig. 6 showed the interactive property of Eq. (5a) under the proposed conditions (18) with different speeds. In Fig. 6, the interactions between the embedded soliton and the taperlike soliton were observed showing complete elastic. After collision, the result seems to be identical to the complete elastic collisions between the two classical particles, in terms of their amplitudes, velocities, and wave shapes.

Taking the similar approach above in the second case, the interactions between an embedded soliton and a plateau-type soliton were observed. When f was selected to be

$$f(y,z,t) = \ln(2 + \mathrm{sech}[0.1(y-t)^2 + 0.1z^2 - 4] - 0.7\exp\{\tanh[0.2(y+5t)^2 + 0.2z^2 - 4]\}), \tag{19}$$

then an embedded soliton and a plateau-type soliton could be derived from the physical field v in Eq. (5a). As shown in Fig. 7, the interactions between the embedded soliton and the plateau-type soliton were revealed being completely elastic.

In the third case below, the interactions between an embedded soliton and a rectangle soliton were discussed. If f was chosen to be

$$f(y,z,t) = \ln(1.4 + \mathrm{sech}[0.1(y-t)^2 + 0.1z^2 - 4] - 0.5\exp\{\tanh[0.01(y+5t)^4 + 0.01z^4 - 4]\}), \tag{20}$$

then an embedded soliton and a rectangle soliton could be derived for the physical field v in Eq. (5a), as shown in Fig. 8. Again, the interactions between an embedded soliton and a

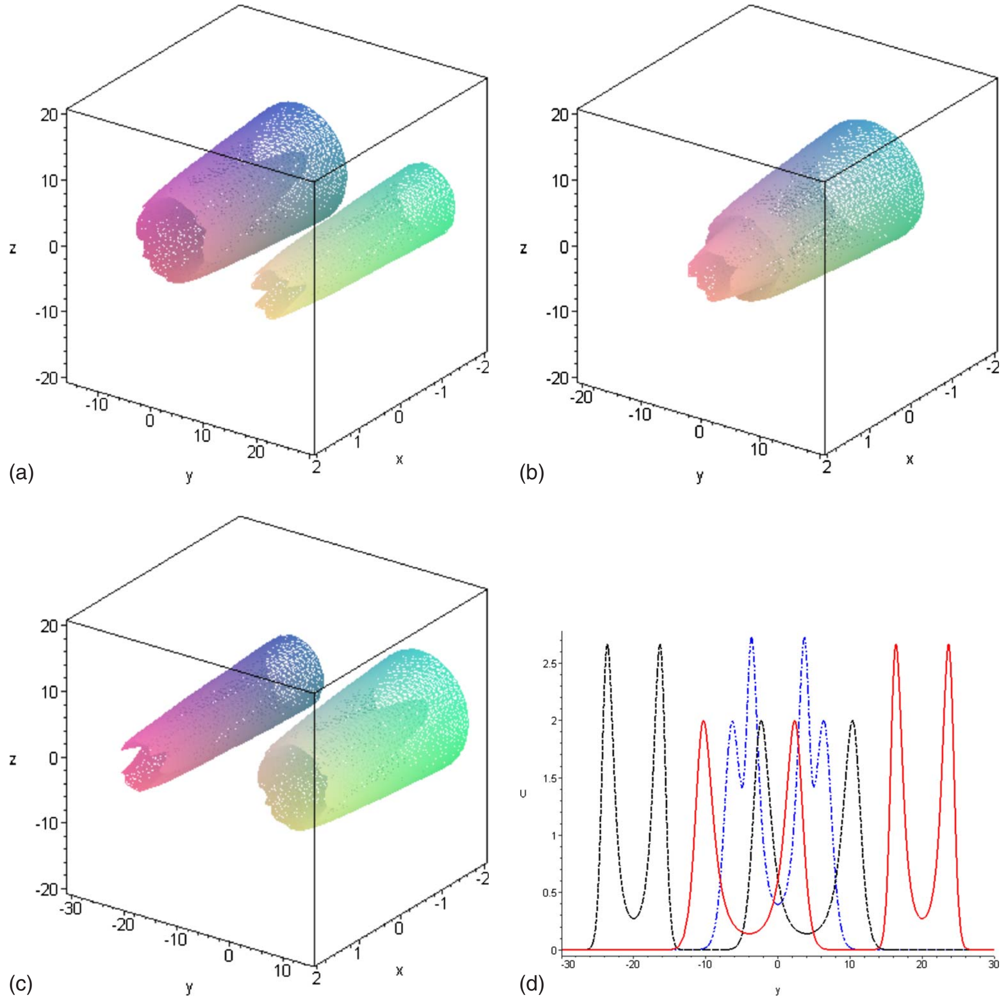

FIG. 9. (Color online) Time evolutional profiles of the interactions between two embedded solitons for the field v given by Eq. (5a) with the condition (21) and $b=c=e=k_1=1$ at various times points: (a) $t=-4$, (b) $t=0$, (c) $t=4$, and (d) a sectional view related to (a), (b), and (c) at $x=z=0$: solid red line (a), dotted and dashed blue line (b), and dashed black line (c).

rectangle soliton were completely elastic, indicating no changes in their amplitudes, velocities, and wave shapes after collision.

In this paper, the fourth case focused on the interactions between two embedded solitons. When f was taken to be

$$f(y,z,t) = \ln\{\text{sech}[0.1(y-t)^2 + 0.1z^2 - 4] + 2\text{sech}[0.3(y+5t)^2 + 0.3z^2 - 4]\}, \quad (21)$$

then two embedded solitons were derived for the physical field v in Eq. (5a) presented in Fig. 9, in which the interactions of the two embedded solitons showed completely elastic properties.

Remark 1. Along the same line of our arguments and the performance of this study, we derived interactions between (1) a taperlike soliton and a plateau-type soliton, (2) a taperlike soliton and a rectangle soliton, (3) a plateau-type soliton and a rectangle soliton, (4) two taperlike solitons, (5) two plateau-type solitons, and (6) two rectangle solitons for the physical field v in Eq. (5a). Those interactions showed complete elastic properties. For the sake of simplicity, we omitted the evolutional profiles of the corresponding times in this paper.

Finally, when f was chosen to be

$$f(y,z,t) = \ln(1.4 + \text{sech}[0.1(y-t)^2 + 0.1z^2 - 4] - 0.5\exp\{\tanh[0.01(y+8t)^4 + 0.01z^4 - 4]\} - 0.7\exp\{\tanh[0.2(y+4t)^2 + 0.2z^2 - 4]\}), \quad (22)$$

then the three-soliton solution, including an embedded solitons, a plateau-type soliton, and a rectangle soliton, were derived for the physical field v in Eq. (5a) presented in Fig. 10. The interactive phenomena are similar to those of the two solitons described in the above section, showing elastically interacting with each other.

Remark 2. When we continued the above process, the others three-soliton solutions, four-soliton solutions, and even the N-soliton solutions were derived. For the sake of

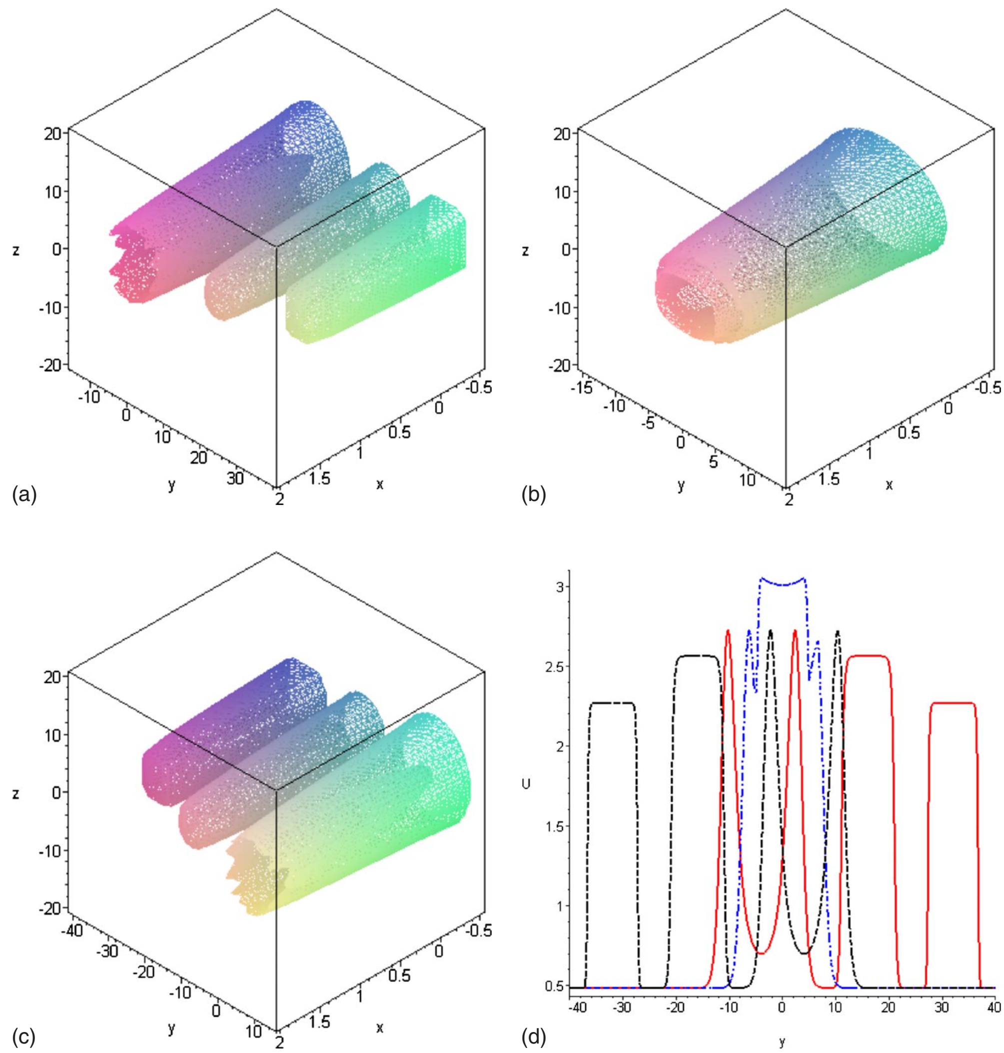

FIG. 10. (Color online) Time evolutional profiles of the interactions among an embedded soliton, a plateau-type soliton, and a rectangle soliton for the field v given by Eq. (5a) with the condition (22) and $b=c=e=k_1=1$ at various times points: (a) $t=-4$, (b) $t=0$, (c) $t=4$, and (d) a sectional view related to (a), (b), and (c) at $x=z=0$: solid red line (a), dotted and dashed blue line (b), and dashed black line (c).

simplicity, we would not discuss such evolution and the interactive behaviors in details in this paper.

C. Interactions among embedded solitons and other solitons on a nonflat background

As far as we understood, localized structures derived from some solitary wave solutions or rational solutions in (1+1)- and (2+1)-dimensional system were usually considered to propagating on a constant background (or an ideal background), which does not exist actually in the real world since some background waves are always encountered. In fact, a lot of physical phenomena need to be described with certain background waves. Given that the real physical space time is in (3+1) dimensions, its localized excitations, especially for those related to propagation on a background wave, have attracted attention of many mathematicians and physicists for years, although little progress has been made in this direction. As a different approach, our focus in this section was on some evolutional properties of the interactions between the embedded solitons and other type of solitons that occur in the nonflat background in this study. First, we considered interactions between an embedded soliton and a taperlike soliton on a periodic wave background of trigonometric function. Therefore, when f was shown to be

$$\begin{aligned} f(y,z,t) = \ln\{&-0.02\sin(0.05y^2+0.05z^2-t^2) \\ &+\operatorname{sech}[0.1(y-t)^2+0.1z^2-4] \\ &+2\exp[-0.4\sqrt{(y+5t)^2+z^2}]\}, \end{aligned} \tag{23}$$

an embedded soliton and a taperlike soliton in the periodic wave background could be derived for the physical field v in Eq. (5a). Figure 11 showed the corresponding profiles of the complex wave excitations presenting the propagation of an embedded soliton moving along the y axis in the positive direction and a taperlike soliton moving along the y axis in the negative direction in the determined periodic wave background [Eq. (23)] with different speeds. As shown in Fig. 11, during the process of propagation, the amplitude of the complex waves changed due to the superposition of the solitary

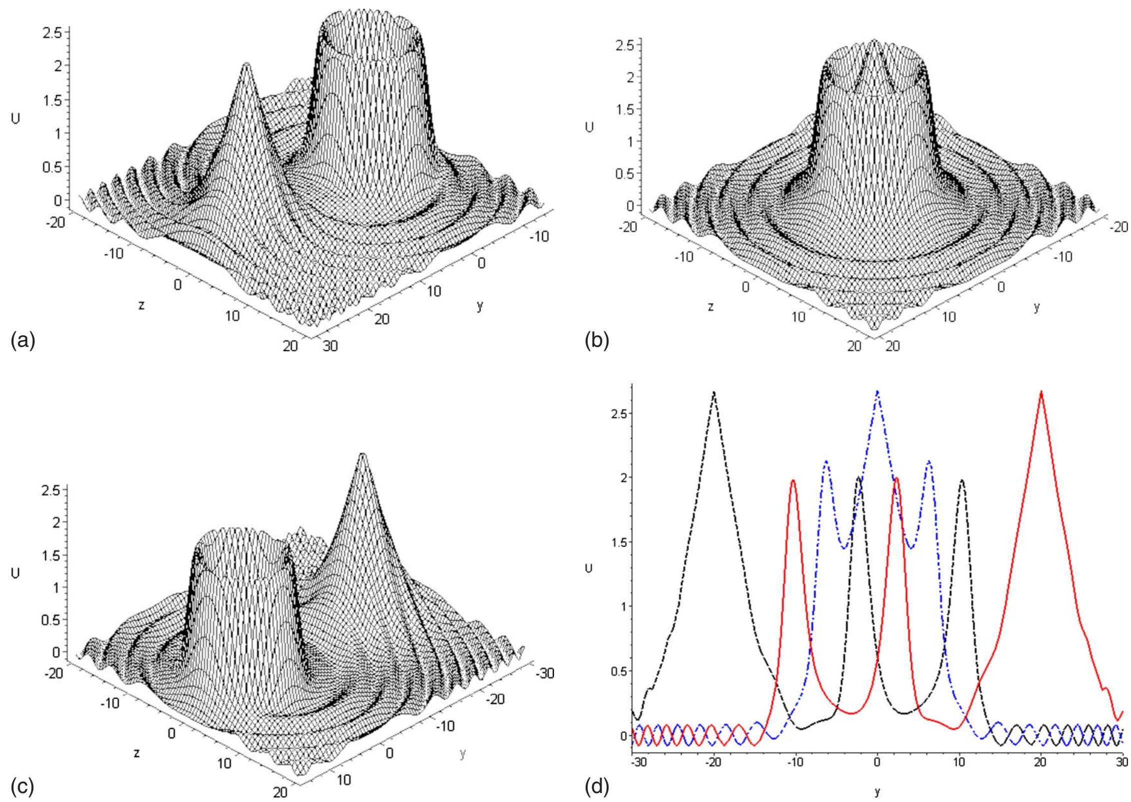

FIG. 11. (Color online) Time evolutional profiles of the interactions between an embedded soliton and a taperlike soliton in the periodic wave background for the field v given by Eq. (5a) with the condition [Eq. (23)] and $x=0$, $b=c=e=k_1=1$ at various time points. (a) $t=-3$, (b) $t=0$, (c) $t=3$, and (d) a sectional view related to (a), (b), and (c) at $z=0$: solid red line (a), dotted and dashed blue line (b), and dashed black line (c).

wave and underlying periodic wave. If the background wave amplitude and the soliton amplitude increased, we could find the increases in the complex wave amplitude based on wave superposition theorem. However, the shapes and velocities of the embedded solitons and the taperlike solitons did not suffer from any changes. The findings were very similar to many actual physical processes in the natural world, such as solitary waves in the study of water waves.

Remark 3. Further to the above findings, if we considered the interactions among the embedded solitons and the other solitons (e.g., the plateaulike soliton, rectangle soliton, and embedded soliton) under the periodic wave background, phenomena similar to the above findings in Sec. III C could also be observed. Moreover, supposing that the background of these interactions were Jacobi elliptic waves or Bessel function waves, we could then obtain the same conclusion. To avoid extensive repeating in this paper, we omitted the corresponding content in details here.

Interactions are an important part of the soliton theory; solitons are actually defined in a traditional way by referring to their elastic interactions [19]. As mentioned in Sec. III A, there are some kinds of interaction forms. We noticed that the background of these interactions was the flat plane or the periodic wave plane. An interesting question may be raised on how the interactions between solitons occur in the kink background? For studying such interactive form, we selected the f in Eq. (5a) as

$$f(y,z,t) = \ln(2 - 0.2\tanh(0.5y + z + 20) + \mathrm{sech}[0.1(y-4t)^2 + 0.1z^2 - 4] - 0.8\exp\{\tanh[0.1(y+3t)^2 + 0.1z^2 - 4]\}). \tag{24}$$

Figure 12 showed the interactions of the two solitons on the kink background. At the initial phase, the two solitons were distant in space. As time passed, the two solitons collided and then separated. As indicated, all these interactions were elastic. Further analysis showed that the interactions had rich behavior because of the kink background. One interesting aspect was that the two solitons also interacted with the background kink. To clearly display these interactions, we plotted the evolution of soliton interactions in Figs. 13–15 at section $z=0$.

At the start of the interactions between the embedded solitons and the background kink, the interactions were not elastic (Fig. 13). Gradually, the embedded solitons and the plateaulike solitons met at the lower branch of the background kink, and their interactions became elastic (Fig. 14). Following the separation of the embedded solitons and the plateaulike solitons, the plateaulike solitons moved to the upper branch of the kink. The interactions between the plateaulike solitons and the kink were not elastic (Fig. 15). When the embedded solitons and plateaulike solitons were in the lower and upper branches of the kink, they moved independently.

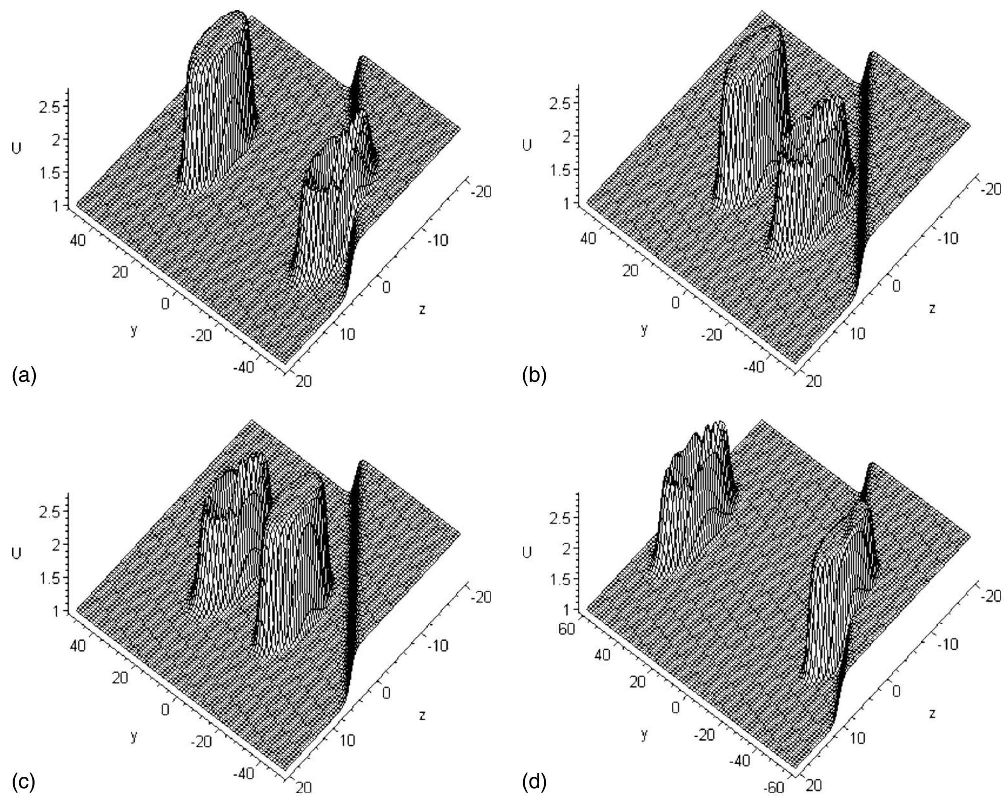

FIG. 12. Time evolutional profiles of the interactions between an embedded soliton and a plateaulike soliton on the kink background for the field v given by Eq. (5a) with the condition [Eq. (24)] and $x=0$, $b=c=e=k_1=1$ at various time points: (a) $t=-8$, (b) $t=-4$, (c) $t=4$, and (d) $t=12$.

By viewing the interactions on the kink background from another perspective, we believed that the background kink might work as a "reservoir." When the embedded soliton interacted with the reservoir kink, it would transfer some information to the reservoir. However, when the plateaulike soliton interacted with the reservoir, it would obtain the information from the embedded solitons based on the interactions with the kink. With a high possibility, such interactions might be useful in the communication based on the soliton. If the solitons were the carrier of the transferred information and the interactions between the solitons are elastic, the information transfer or exchange between the solitons would be difficult. As for interactions on the soliton background, the first soliton (the carrier) would initially transfer the information to the background. The second soliton would then get the information by the interactions with the background.

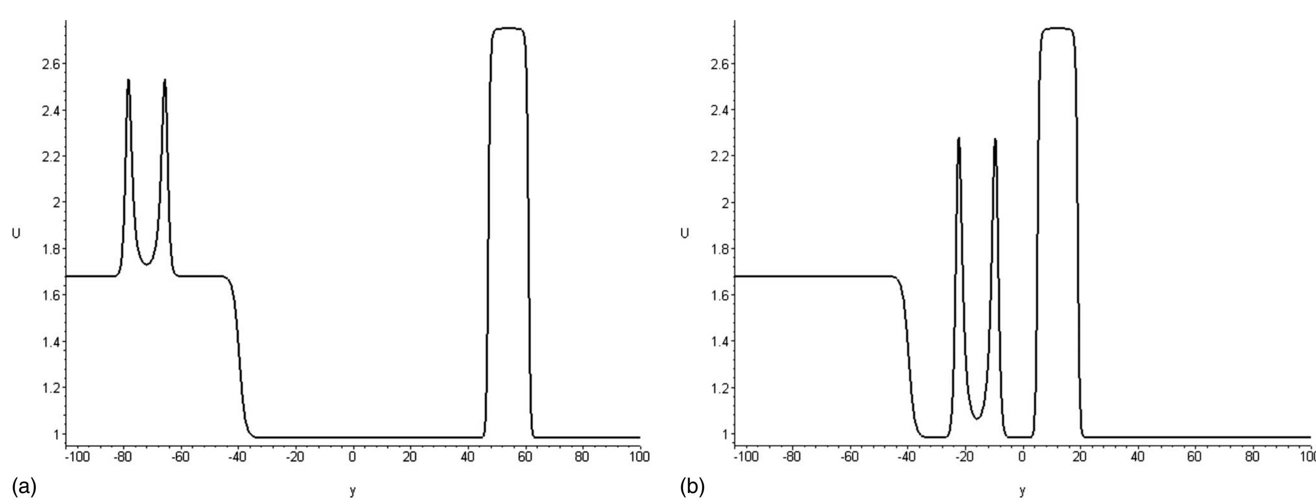

FIG. 13. The evolutional of the interactions between an embedded soliton and a plateaulike soliton on the kink background for the field v given by Eq. (5a) with the condition [Eq. (24)] and $x=0$, $b=c=e=k_1=1$ at various time points: (a) $t=-18$; (b) $t=-4$. The interactions between the embedded soliton and background kink are not elastic.

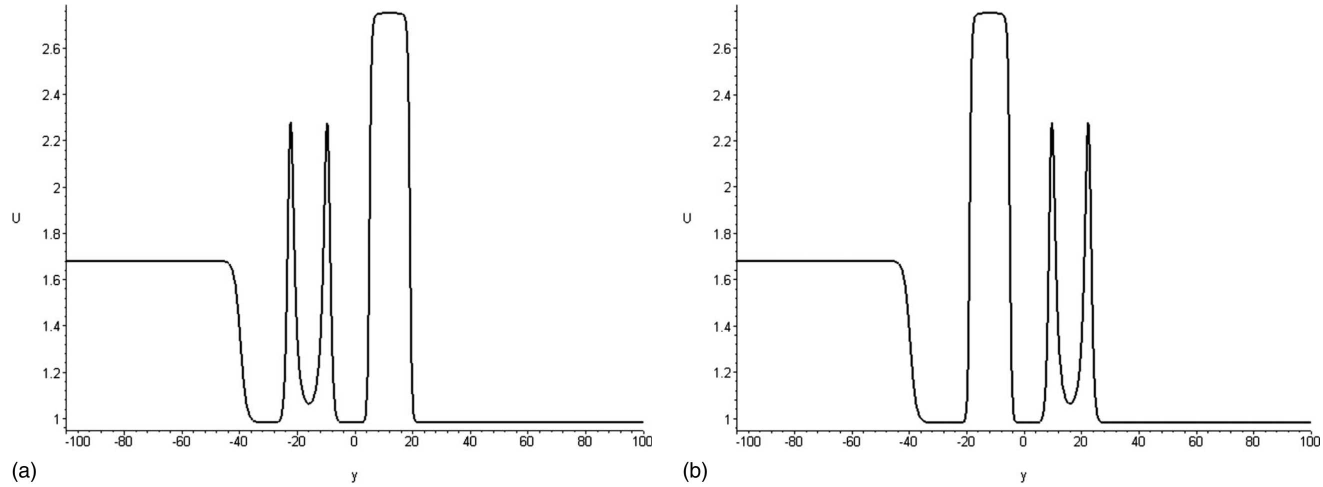

FIG. 14. The evolutional of the interactions between an embedded soliton and a plateaulike soliton on the kink background for the field v given by Eq. (5a) with the condition [Eq. (24)] and $x=0$, $b=c=e=k_1=1$ at various time points: (a) $t=-4$; (b) $t=4$. The interactions between the embedded soliton and the plateaulike soliton are elastic.

Therefore, the whole process could be completed using the background soliton for the information relay.

IV. SUMMARY AND DISCUSSION

Overall, it has been a convincing fact that the findings in the research of solitons provide very interesting prospects in many fields of the natural science. The interactive properties of solitons may play an important role in the future development of many scientific applications. Although the soliton structures and their properties of the $(1+1)$-dimensional integrable nonlinear evolution have been well studied, the understanding of the soliton structures and their interactions in the higher spatial dimensions continue to be limited because of the technical difficulties in finding suitable formulas in the higher dimensions nonlinear models.

By using a special variable separation approach for a $(3+1)$-dimensional model, a formula was established in this paper, in which its arbitrary variable-separated functions were able to be involved. Thanks to the existence of the arbitrary functions in the universal formula, various special types of the explicit multiple wave interactions of the solutions on a flat background, including the embedded solitons, the taperlike soliton, the plateau-type soliton, and the rectangle soliton, were explicitly given both analytically and graphically. Through our study, only some representing cases to those interaction wave solutions were analyzed using a suitable selection process for the arbitrary functions according to the asymptotic results of Eq. (14). The study demonstrated that the interactive behaviors among them were elastic. Moreover, the soliton interactions on the periodic wave background and the kink background were also obtained, showing the interactions between the two solitons as elastic. However, the interactions between the solitons and background kink were not elastic.

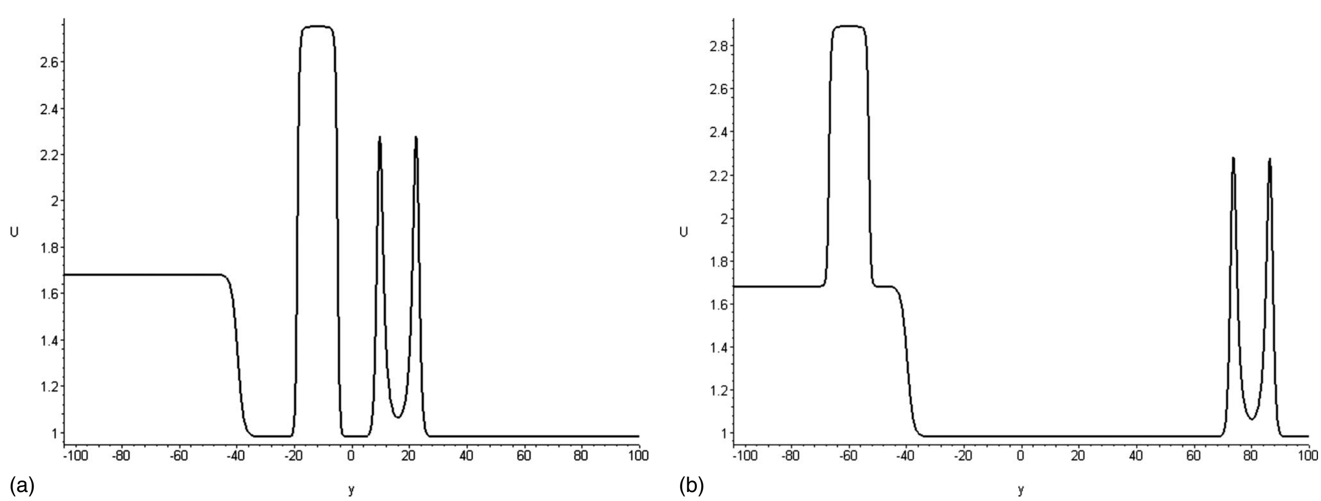

FIG. 15. The evolutional of the interactions between an embedded soliton and a plateaulike soliton on the kink background for the field v given by Eq. (5a) with the condition [Eq. (24)] and $x=0$, $b=c=e=k_1=1$ at various time points: (a) $t=4$; (b) $t=20$. The interactions between the plateaulike soliton and the background kink are not elastic.

ACKNOWLEDGMENTS

The author is very grateful to Professor Christopher for his enthusiastic help on English grammar. C.-L.B. also would like to express his sincere thanks to anonymous referees for their many helpful advice and suggestions. This work was supported by the National Natural Science Foundation of China and the Natural Science Foundation of Shandong Province in China.

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