Interest rates in quantum finance: The Wilson expansion and Hamiltonian

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Interest rate instruments form a major component of the capital markets. The Libor market model (LMM) is the finance industry standard interest rate model for both Libor and Euribor, which are the most important interest rates. The quantum finance formulation of the Libor market model is given in this paper and leads to a key generalization: all the Libors, for different future times, are *imperfectly* correlated. A key difference between a forward interest rate model and the LMM lies in the fact that the LMM is calibrated directly from the observed market interest rates. The short distance Wilson expansion [Phys. Rev. **179**, 1499 (1969)] of a Gaussian quantum field is shown to provide the generalization of Ito calculus; in particular, the Wilson expansion of the Gaussian quantum field $\mathcal{A}(t,x)$ driving the Libors yields a derivation of the Libor drift term that incorporates imperfect correlations of the different Libors. The logarithm of Libor $\phi(t,x)$ is defined and provides an efficient and compact representation of the quantum field theory of the Libor market model. The Lagrangian and Feynman path integrals of the Libor market model of interest rates are obtained, as well as a derivation given by its Hamiltonian. The Hamiltonian formulation of the martingale condition provides an exact solution for the nonlinear drift of the Libor market model. The quantum finance formulation of the LMM is shown to reduce to the industry standard Bruce-Gatarek-Musiela-Jamshidian model when the forward interest rates are taken to be exactly correlated.

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I. INTRODUCTION

Interest rates are the return earned on cash deposits. The two main international currencies are the United States dollar and the European Union euro. Cash fixed deposits in these currencies account for almost 90% of the simple interest rates that are traded in the capital markets. Cash deposits in U.S. dollar as well as the British pound earn interest at the rate fixed by Libor and deposits in euro earn interest rates fixed by Euribor. A brief discussion of these instruments is given below for motivating the study of interest rates [1].

A. Libor

The interest rates offered by commercial banks for cash time deposits are often based on Libor, the *London interbank offered rate* [2]. Libor is one of the main instruments for interest rates in the debt market and is widely used for multifarious purposes.

Libor was launched on 1 January 1986 by British Bankers' Association. Libor is a daily quoted rate based on the interest rates at which *commercial banks* are willing to lend funds to other banks in the London interbank money market. The minimum deposit for a Libor has a par value of $1 000 000. Libor is a simple interest rate for fixed bank deposits and the British Bankers' Association has daily Libor quotes for loans in the money market of the following duration: overnight (24 h); one and two weeks; and 1, 3, 4, 5, 6, 9, and 12 months. Libors of longer duration are obtained from the interest rate swap market and are quoted for future loans of duration from 2 to 30 years. A Libor zero coupon yield curve is constructed from the swap market and is quoted by vendors of financial data. The Libor market is active in maturities ranging from a few days to 30 years, with the greatest depth in the 90 and 180 day time deposits.

Libor is defined for duration of cash deposits that are integral multiples of ℓ, where ℓ is called the tenor of the deposit; in particular, $L(t,T_n)$ is the forward simple interest rate, fixed at time t, for a future cash deposit from time T_n to $T_n+\ell$. Libor time is defined by $T_n=n\ell$; both calendar time and future time are defined on a time lattice as shown in Fig. 1.

The three-month Libor is the benchmark rate that forms the basis of the Libor derivative market. All Libor swaps, futures, caps, floors, swaptions, and so on are based on the three-month deposit. The term Libor will be taken to synonymous with the three-month Libor.

In 1999 the open positions on Eurodollar futures had a par value of about U.S. $750 billion and had grown tremendously since then. The Chicago Mercantile Exchange (CME) Libor futures represent one-month Libor rates on a $3 million deposit. In 2008, CME had Eurodollar futures and options on Libor with open interest of over 40 million Libor contracts and an average daily volume of 3.0 million. Libor is among the world's most liquid short term interest rate

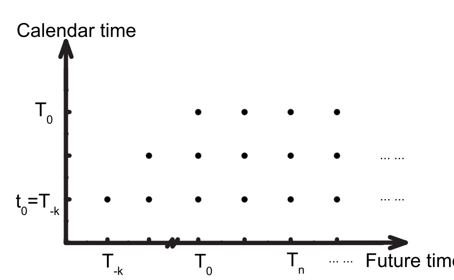

FIG. 1. Libor calendar and future time lattice; the tenor (future time lattice spacing) is given by $\ell=90$ days.

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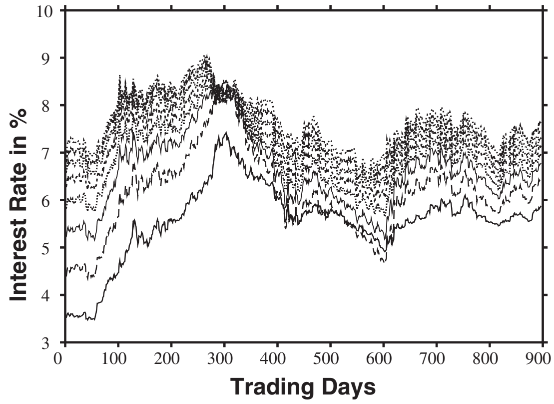

FIG. 2. Daily Libor forward interest rates $L(t,t+7\ \text{years})$, $L(t,t+6\ \text{years})$, ... $L(t,t+1\ \text{year})$, and $L(t,t+0.25\ \text{years})$ with calendar time being $t \in [1996,1999]$.

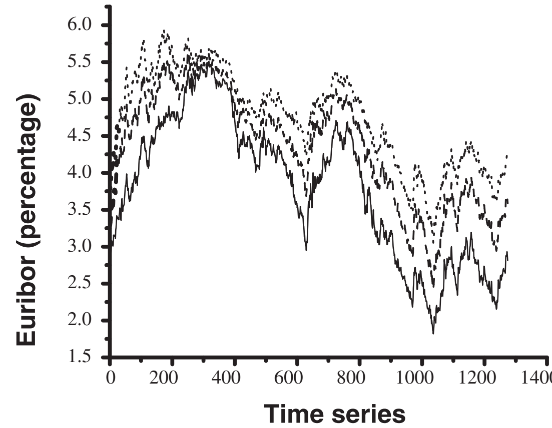

FIG. 3. Euribor maturing one, two, and three years in the future from 26 May 1999 to 17 May 2004.

future contracts. Interest rate swaps, with Libor taken as the floating rate, currently trade on the interbank market for maturities of up to 50 years.

Market data on Libor futures are given for daily time t in the form of $L(t,T_i-t)$, with fixed dates of maturity T_i (March, June, September, and December) and are shown in Fig. 2.

B. Euribor

Euribor (euro interbank offered rate) is the benchmark rate of the euro money market, which has emerged since 1999. Euribor is simple interest on fixed deposits in the euro currency; the duration of the deposits can vary from overnight, weekly, monthly, and three monthly out to long duration deposits of 10 years and longer. Euribor is sponsored by the Financial Markets Association and by the European Banking Federation, which represents 4500 banks in the 24 member states of the European Union and in Iceland, Norway, and Switzerland. Euribor is the rate at which euro interbank term deposits are offered by one prime bank to another.

The choice of banks quoting for Euribor is based on market criteria. These banks are of first class credit standing and are selected to ensure that the diversity of the euro money market is adequately reflected, thereby making Euribor an efficient and representative benchmark. All the features discussed for Libor can also be applied to Euribor.

Euribor was first announced on 30 December 1998 for deposits starting on 4 January 1999. Figure 3 shows daily values for Euribor forward interest rate on 90 day deposits for deposits one, two, and three years in the future. Since its launch, Euribor has been actively traded on the option markets and is the underlying rate of many derivative transactions both over the counter and exchange traded. Euribor is one of the most liquid global interest rate instruments, second only to Libor. The Euribor zero coupon yield curve, based on the rates, is contracted in the Euribor swap market and extends out to 50 years in the future.

The term Libor is used for generic interest rate instruments and includes both Libor and Euribor.

II. LIBOR MARKET MODEL

From Figs. 2 and 3, it is clear that interest rates $L(t,T)$, which at every instant t form a curve extending in future time out to almost 50 years, evolve in a random manner. It is hence natural to take the interest rates to be a *random* function; in other words, $L(t,T)$ is taken to be a random two-dimensional stochastic field. The observed market behavior of interest rates is obtained by averaging over all possible values of the stochastic field. Since the functional averaging is identical to the functional averaging over all possible configurations of a quantum field, interest rates $L(t,T)$ are modeled as a two-dimensional Euclidean quantum field [3].

Interest rate instruments can be modeled using either the zero coupon bonds $B(t,T)$ or the simple interest Libor $L(t,T)$. Both these approaches are, in principle, equivalent but are quite different from an empirical, computational, and analytical point of view.

$B(t,T)$ is the price of a zero coupon bond—at time t—that pays a prefixed amount, say 1 euro, at future time T. Recall $L(t,T)$ is the simple interest, agreed at time t, for a fixed deposit from future time T to $T+\ell$. The forward interest rate, denoted by $f(t,x)$, is the interest rate, fixed at time t, for an instantaneous loan at some future time $x>t$; note that x and t refer to future and calendar time, respectively. One can take the view that there exists one set of underlying forward interest rates $f(t,x)$ that can be used for modeling both $L(t,T)$ and zero coupon bonds $B(t,T)$. Forward interest rates are strictly positive, that is, $f(t,x) \geq 0$. The positivity of $f(t,x)$ is intuitively obvious and also required by absence of arbitrage. The Heath-Jarrow-Morton (HJM) [4] model of $f(t,x)$—and its quantum finance generalization [2]—goes a long way in accurately modeling interest rate instruments. However, the HJM model and its quantum finance generalization have one serious shortcoming: both allow $f(t,x)$ to be *negative* with a finite probability, which, in turn, implies that simple interest rate $L(t,T)$ has a finite probability of being negative.

Giving up $f(t,x) \geq 0$ does not pose a very serious problem for the bond sector since $B(t,T)$ is strictly positive even for those configurations for which $f(t,x) \leq 0$. However, for the interest rate sector of the debt market, a model that allows Libor to be negative can yield results that allow for arbitrage and hence are not permissible as a consistent model for interest rate instruments. One needs to go beyond modeling $f(t,x)$ and instead develop a model based directly on Libor $L(t,T)$.

The Libor market model (LMM) aims at modeling interest rates in terms of debt instruments that are *directly* traded in the financial markets. In particular, forward interest rates $f(t,x)$ are not directly traded, but instead what are traded are (a) Libor and Euribor for fixed time deposits and (b) zero coupon bonds $B(t,T)$ as well as coupon bonds. LMM takes the traded values of Libor $L(t,T)$ to be the main ingredient in modeling interest rates—instead of deriving Libor from an underlying Libor forward interest rate model. In the LMM all Libors are *strictly positive*: $L(t,T)>0$.

Zero coupon bonds and the Libor forward interest rates are both derived from Libor instead from $f(t,x)$. Strictly positive Libor has the added advantage that all zero coupon bonds and hence coupon bonds as well are all strictly positive.

The LMM approach was pioneered by Bruce-Gatarek-Musiela (BGM) [5] and Jamshidian [6], with many of its subsequent developments discussed by Rebonata [7,8]. One of the biggest achievements of the LMM is a derivation of Black's formula for pricing interest rate caplets from an arbitrage free model—something that many experts thought was not possible. Various extensions of the LMM have been made; Anderson and Andresean [9] and Joshi and Rebonata [10] incorporated stochastic volatility into the LMM whereas Labordere [11] combined LMM with the stochastic alpha, beta, rho (SABR) model [12]. The calibration and applications of BGM-Jamshidian model have been extensively studied [7,13].

In the BGM-Jamshidian approach, similar to the HJM modeling of the forward interest rates, all Libors for different future times are exactly correlated. In contrast, in the quantum finance formulation, Libors are driven not by white noise but rather by the two-dimensional stochastic field $\mathcal{A}(t,x)$. The values of all Libor instruments are given by averaging $\mathcal{A}(t,x)$ over all its possible values. Hence, $\mathcal{A}(t,x)$ is mathematically equivalent to a two-dimensional quantum field.

The equal time Wilson expansion [14,15] of the bilinear product of the quantum field $\mathcal{A}(t,x)$ is discussed in Sec. V and provides a generalization of Ito calculus. The key link in deriving the quantum finance version of the LMM and in, particular, of the Libor drift is the singular property of the bilinear product of Gaussian quantum field $\mathcal{A}(t,x)$. The quantum finance generalization of the Libor market model contains crucial correlation terms reflecting the imperfect correlation of the different Libors and avoids systematic errors that arise from the assumption of perfectly correlated Libors.

A derivation of Libor drift independent of the Wilson expansion and based on the Hamiltonian formulation of the Libor market model is given in Sec. XV.

The LMM is driven by $f(t,x)$, the *Libor forward interest rates*, which is distinct from both the empirical forward interest rates and the bond forward interest rates. It is shown that $f(t,x)$ has a nonlinear evolution equation with both its drift and volatility being stochastic. Libor forward interest rates are strictly positive and nonsingular, being finite for all calendar and future time. An efficient description of Libor instruments is obtained by doing a nonlinear change of independent variables from $f(t,x)$ to $L(t,T_n)$ and then to logarithmic Libor field, namely, $\phi(t,x)$.

It is shown that, when the limit of perfectly correlated Libor is taken, the quantum finance LMM reduces to the BGM-Jamshidian model and, in turn, yields—in the limit of zero Libor tenor ($\ell\to 0$)—the HJM model for the bond forward interest rates.

FIG. 4. Libor future time.

III. LIBOR AND ZERO COUPON BONDS

In terms of Libor forward interest rates $f(t,x)$ Libor zero coupon bond $B(t,T)$ is defined as

$$B(t,T) = \exp\left\{-\int_t^T dx f(t,x)\right\} \tag{1}$$

and Libor $L(t,T_n)$ is given by

$$L(t,T_n) = \frac{1}{\ell}\left[\exp\left\{\int_{T_n}^{T_n+\ell} dx f(t,x)\right\} - 1\right]. \tag{2}$$

Libor zero coupon bonds $B(t,T)$ are not actual instruments traded in the market but rather a way of encoding the discounting of future cash flows consistent with all the Libors. The price of a traded zero coupon treasury bond $B(t,T)$ is not equal to a Libor bond $B(t,T)$, but these differences are small and will be ignored.

From Eq. (2), Libor is given by

$$L(t,T_n) = \frac{B(t,T_n) - B(t,T_{n+1})}{\ell B(t,T_{n+1})}$$

and which yields

$$B(t,T_n+\ell) = \frac{B(t,T_n)}{1+\ell L(t,T_n)}. \tag{3}$$

Equation (3) provides a recursion equation that allows one to express $B(t,T_n)$ solely in terms of $L(t,T_n)$. Recall that Libors are only defined for discrete future time given by *Libor future time* $T=T_n=n\ell$, $n=0,\pm 1,\pm 2,\ldots,\pm\infty$. Libor future time lattice is shown in Fig. 4. Hence, from Eq. (3)

$$B(t,T_{k+1}) = \frac{B(t,T_k)}{1+\ell L_k(t)} = B(t,T_0)\prod_{n=0}^{k}\frac{1}{1+\ell L_n(t)},$$

where $L_n(t)\equiv L(t,T_n)$.

Bonds $B(t,T_0)$ that have time t not at a Libor time ℓk cannot be expressed solely in terms of Libor rates. Consider only zero coupon bonds that are issued at Libor time, say T_0, and mature at another Libor time T_{k+1}; since $B(T_0,T_0)=1$, $B(T_0,T_{k+1})$ can be expressed entirely in terms of Libor as follows:

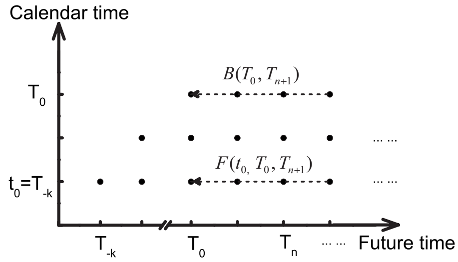

FIG. 5. The zero coupon bond $B(T_0,T_{n+1})$ is issued at T_0 and expires at $T_n+\ell$. Its forward Libor bond price $F(t_0,T_0,T_{n+1})$ is given at present (Libor) time $t_0=T_{-k}$.

$$B(T_0,T_{k+1}) = \prod_{n=0}^{k} \frac{1}{1+\ell L_n(T_0)}. \quad (4)$$

Forward bond price and Libor

Let us present time be $t_0=T_{-k}$. Suppose a zero coupon bond $B(T_0,T_n+\ell)$ is going to be issued at some future time $T_0>t_0=T_{-k}$, with expiry at time $T_n+\ell$; the zero coupon bond and its forward price are defined for Libor time and shown in Fig. 5. The forward bond price is the price—at present time t_0—of a zero coupon bond that is to be issued at some future time $T_0>t_0$. From Eq. (4), the forward bond price is given by

$$F(t_0,T_0,T_n+\ell) \equiv \frac{B(t_0,T_{n+1})}{B(t_0,T_0)} = \left\{\prod_{i=-k}^{n} \frac{1}{1+\ell L(t_0,T_i)}\right\} \Big/ \left\{\prod_{i=-k}^{-1} \frac{1}{1+\ell L(t_0,T_i)}\right\} = \prod_{i=0}^{n} \frac{1}{1+\ell L_i(t_0)}. \quad (5)$$

IV. LIBOR MARKET MODEL AND QUANTUM FINANCE

The Libor market model is defined in the framework of quantum finance by defining the time evolution of the Libor rates $L(t,T)$.

Modeling in finance widely uses the concept of stochastic differential equations. The *bond* forward interest rates $f_B(t,x)$ and Libor $L_k(t)$, as defined by the HJM and BGM-Jamshidian models, respectively, are both expressed as functions of white noise and given by

$$\frac{\partial f_B(t,x)}{\partial t} = \alpha(t,x) + \sigma(t,x)R(t) \quad \text{(HJM model)}, \quad (6)$$

$$\frac{1}{L_k(t)}\frac{\partial L_k(t)}{\partial t} = \zeta_k(t) + \gamma_k(t)R(t) \quad \text{(BGM-Jamshidian model)}, \quad (7)$$

where $R(t)$ is Gaussian white noise,

$$E[R(t)] = 0, \quad E[R(t)R(t')] = \delta(t-t').$$

The volatility functions $\sigma(t,x), \gamma_k(t)$ are deterministic. The drift $\alpha(t,x)$ is deterministic in the HJM model, whereas Libor drift $\zeta_k(t)$ depends on Libors $L_k(t)$ for the BGM-Jamshidian model.

Future time x and T_k have been introduced in both the HJM and BGM-Jamshidian models *only* in the drift and volatility of the interest rate term structure. A *single* white noise $R(t)$ drives the entire forward interest rate curve and leads, as follows, to perfectly correlated rates:

$$E\left[\frac{\partial f_B(t,x)/\partial t - \alpha(t,x)}{\sigma(t,x)} \frac{\partial f_B(t',x')/\partial t' - \alpha(t',x')}{\sigma(t',x')}\right] = \delta(t-t'),$$

$$E\left[\frac{L_k^{-1}(t)\,\partial L_k(t)/\partial t - \zeta_k(t)}{\gamma_k(t)} \frac{L_{k'}^{-1}(t')\,\partial L_{k'}(t')/\partial t' - \zeta_{k'}(t')}{\gamma_{k'}(t')}\right] = \delta(t-t'). \quad (8)$$

Note that the right-hand side of above equations is *independent* of x,x' and $T_k,T_{k'}$, respectively, showing perfect correlation in future time.

The quantum finance model of the bond forward interest rates $f_B(t,x)$ given in [2] and its HJM limit have a major unavoidable side effect: there is a finite probability that $f_B(t,x)$ can take negative values. Since, empirical forward interest rates can never be negative, the Libor market model takes the view that for the debt market one should *replace* the bond forward interest rates $f_B(t,x)$ by *strictly positive* Libor forward interest rates $f(t,x)$. These rates are used for modeling all interest rate instruments and, in particular, yield all $L(t,T)$ as always being positive.

In the Libor market model, market interest rates $L(T_0,T_n)$ and coupon and zero coupon bonds $\mathcal{B}(T_N)$ and $B(T_n,T_N)$—given at Libor times T_n,T_N, respectively—are expressed solely in terms of Libor $L(T_0,T_n)$, as in Eq. (4), without any direct reference to the underlying Libor forward interest rates $f(t,x)$. Moreover, positive Libor rates automatically yield coupon and zero coupon bonds that are strictly positive, as seen in Eq. (4).

Only the quantum finance *differential formulation* of the Libor forward interest rates is the main focus of this paper; a similar generalization of the bond forward interest rates and of the HJM model has been extensively discussed in [2]. The BGM-Jamshidian model of the LMM is generalized by "promoting" white noise $R(t)$ to a two-dimensional quantum field $\mathcal{A}(t,x)$ and yields the following:

$$\frac{\partial f(t,x)}{\partial t} = \mu(t,x) + v(t,x)\mathcal{A}(t,x), \quad (9)$$

$$f(t,x) = f(t_0,x) + \int_{t_0}^{t} dt\mu(t,x) + \int_{t_0}^{t} dtv(t,x)\mathcal{A}(t,x). \quad (10)$$

In terms of the Libors, the quantum LMM is given by

$$\frac{1}{L(t,T_k)}\frac{\partial L(t,T_k)}{\partial t} = \zeta_k(t) + \int_{T_k}^{T_{k+1}} dx\gamma(t,x)\mathcal{A}(t,x). \quad (11)$$

In the LMM formulation of Libor forward interest rates $f(t,x)$ both the drift $\mu(t,x)$ and volatility $v(t,x)$ are stochastic and depend on $f(t,x)$; in particular, Libor volatility $v(t,x) \propto [1-\exp\{-\ell f(t,x)\}]$.

The dynamics of the quantum field $\mathcal{A}(t,x)$ is given by the "stiff Lagrangian" [16],

$$\mathcal{L}[\mathcal{A}] = -\frac{1}{2}\left\{\mathcal{A}^2(t,z) + \frac{1}{\mu^2}\left(\frac{\partial \mathcal{A}(t,z)}{\partial z}\right)^2 + \frac{1}{\lambda^4}\left(\frac{\partial^2 \mathcal{A}(t,z)}{\partial^2 z}\right)^2\right\},$$

$$z = (x-t)^\nu; \quad z \in [0,\infty]. \tag{12}$$

The quantum field $\mathcal{A}(t,z)$ satisfies the Neumann boundary conditions

$$\left.\frac{\partial \mathcal{A}(t,z)}{\partial t}\right|_{z=0} = 0.$$

The provenance of the stiff quantum field $\mathcal{A}(t,z)$ is from the market behavior of interest rates [16]; to explain the empirical behavior of both Euribor and Libor, one needs to introduce remaining market time $z=(x-t)^\nu$ as well as strongly correlate the field $\mathcal{A}(t,z)$ using fourth-order derivatives [17].

The Libor market model is quantized in Appendix A using the Feynman path integral. The correlation function of the two-dimensional quantum field $\mathcal{A}(t,x)$ is given in Eq. (A6) by

$$E[\mathcal{A}(t,x)] = 0, \tag{13}$$

$$E[\mathcal{A}(t,x)\mathcal{A}(t',x')] = \delta(t-t')\mathcal{D}(x,x';t), \tag{14}$$

$$M_v(x,x';t) = v(t,x)\mathcal{D}(x,x';t)v(t,x'). \tag{15}$$

As expected, the Libor forward interest rates are imperfectly correlated,

$$E\left[\frac{\partial f(t,x)/\partial t - \mu(t,x)}{v(t,x)}\frac{\partial f(t',x')/\partial t' - \mu(t',x')}{v(t',x')}\right] = \delta(t - t')\mathcal{D}(x,x';t) \quad \text{(imperfectly correlated)}.$$

V. WILSON EXPANSION OF QUANTUM FIELD $\mathcal{A}(t,x)$

Modeling in finance widely uses the concept of stochastic differential equations. The time derivative of various quantities such as a security $S(t)$ is generically expressed as follows:

$$\frac{dS(t)}{dt} = \mu(t) + \sigma(t)R(t).$$

The HJM and BGM-Jamshidian interest rate models are examples of stochastic differential equations, as can be seen from Eqs. (6) and (7).

Ito's stochastic calculus, for discrete time $t=n\epsilon$, is a result of the following identity [2]:

$$E[R(t)R(t)] = \delta(t-t') \Rightarrow R^2(t) = \frac{1}{\epsilon} + O(1). \tag{16}$$

The singular piece of $R^2(t)$ is *deterministic* and to leading order is equal to $1/\epsilon$; all the random terms that occur for $R^2(t)$ are finite as $\epsilon \to 0$.

For Gaussian quantum fields such as $\mathcal{A}(t,x)$, which have a quadratic action, one can give differential formulation of the theory of forward interest rates, as given in Eq. (9), and that is similar to the HJM and BGM formulations. This is possible because the full content of a Gaussian (free) quantum field, as discussed in Appendix A, is encoded in its *propagator*.

Similar to white noise, the correlation function $E[\mathcal{A}(t,x)\mathcal{A}(t',x')]$ is infinite for $t=t'$ (equal calendar time). The product of nonlinear (non-Gaussian) quantum fields is the subject matter of the short "distance" Wilson expansion [14]. The singular product of two Gaussian quantum fields is the simplest case of the Wilson expansion and the singularity, similar to Eq. (16), is expressed as follows:

$$\mathcal{A}(t,x)\mathcal{A}(t,x') = \frac{1}{\epsilon}\mathcal{D}(x,x';t) + O(1). \tag{17}$$

The correlation of $\mathcal{A}(t,x)\mathcal{A}(t,x')$ is singular for $t=t'$—very much like the singularity of white noise $R(t)$. All the fluctuating components, which are contained in $\mathcal{A}(t,x)\mathcal{A}(t,x')$, are regular and finite as $\epsilon \to 0$.

Since, for each x and each t, $\mathcal{A}(t,x)$ is an integration variable one may question as to how can one assign it a deterministic numerical value as in Eq. (17). What Eq. (17) means is that in any correlation function, wherever a product of fields is at the same time, namely, $\mathcal{A}(t,x)\mathcal{A}(t,x')$, then—to leading order in ϵ—the product can be replaced by the deterministic quantity $\mathcal{D}(x,x';t)/\epsilon$. In terms of symbols, Eq. (17) states the following:

$$E[\mathcal{A}(t_1,x_1)\mathcal{A}(t_2,x_2)\ldots \overbrace{\mathcal{A}(t,x_n)\mathcal{A}(t,x_{n+1})}^{\text{equal time}}\ldots\mathcal{A}(t_N,x_N)] = \frac{1}{\epsilon}E[\mathcal{A}(t_1,x_1)\mathcal{A}(t_2,x_2)\ldots\mathcal{A}(t_{n-1},x_{n-1})\mathcal{D}(x_n,x_{n+1};t) \times \mathcal{A}(t_{n+2},x_{n+2})\ldots\mathcal{A}(t_N,x_N)] + O(1).$$

As discussed in [17], one can choose the normalization of $\sigma(t,x)$ so that $\mathcal{D}(x,x;t)=1/\epsilon$ and which yields from Eq. (17)

$$\mathcal{A}^2(t,x) = \frac{1}{\epsilon^2} + O(1), \tag{18}$$

showing even more clearly the similarity of Eqs. (16) and (18).

The HJM model is a special case of quantum finance, given by taking the limit

$$\mathcal{A}(t,x) \to R(t), \quad \mathcal{D}(x,x';t) \to 1 \Rightarrow E[\mathcal{A}(t,x)\mathcal{A}(t',x')] \to E[R(t)R(t)] = \delta(t-t'). \tag{19}$$

Since Eqs. (16) and (17) have a similar singularity structure, one expects that there should be a natural generalization of Ito calculus for Gaussian quantum fields. The singularity of the equal time quadratic product of the quantum field, in

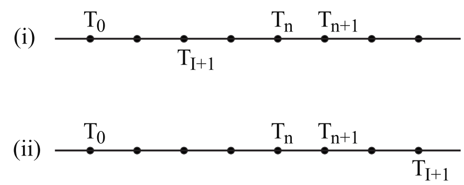

FIG. 6. Libor time lattice for the forward bond numeraire $B(t,T_{I+1})$ with (i) $T_{I+1}<T_n$ and (ii) $T_{I+1}>T_n$.

particular, leads to a differential formulation of the martingale condition for discounted zero coupon bonds and is discussed in Appendix B.

VI. LIBOR MARTINGALES AND FORWARD BOND NUMERAIRE

The factor for discounting of the future cash flows of traded instruments is of fundamental importance in finance and is called a numeraire [2]. A wide class of numeraires can be used to render all traded assets into martingales. Choose the zero coupon bond $B(t,T_{I+1})$, with fixed index I, as the forward bond numeraire. The combination $L(t,T_n)B(t,T_{n+1})$, from Eq. (2), is equivalent to a portfolio of zero coupon bonds and hence is a traded asset. By a suitable choice of the drift, all traded assets can be made into martingales. In particular, all instruments $\mathcal{X}_n(t)$, defined below, for $n=0,\pm1,\pm2,\ldots,\pm\infty$ are martingales [18]; in other words, for all n

$$\mathcal{X}_n(t) \equiv \frac{L(t,T_n)B(t,T_{n+1})}{B(t,T_{I+1})} \quad \text{(martingales)}. \tag{20}$$

Note that, for $n=I$, the portfolio $\mathcal{X}_I(t)$ is equal to $L_I(t) \equiv L(t,T_I)$; hence, for the forward bond numeraire given by zero coupon bond $B(t,T_{I+1})$, the Libor rate $L(t,T_I)$ is a martingale. As shown in (i) and (ii) in Fig. 6, time T_n can be either less than, equal to, or greater than T_I.

In terms of the Libor forward interest rates, from Eqs. (1) and (2), the martingale is

$$\ell\mathcal{X}_n(t) = \exp\left\{-\int_{T_{I+1}}^{T_n} dx f(t,x)\right\} - \exp\left\{-\int_{T_{I+1}}^{T_{n+1}} dx f(t,x)\right\}.$$

Differentiating portfolio $\mathcal{X}_n(t)$ using Eqs. (9) and (14) and the rules derived in Appendix B yields the following:

$$\begin{aligned}\ell\frac{\partial\mathcal{X}_n(t)}{\partial t} = &\left[-\int_{T_{I+1}}^{T_n} dx\mu(t,x) + \frac{1}{2}\int_{T_{I+1}}^{T_n} dxdx' M_v(x,x';t)\right.\\ &\left.- \int_{T_{I+1}}^{T_n} dx v(t,x)\mathcal{A}(t,x)\right]\exp\left[-\int_{T_{I+1}}^{T_n} dx f(t,x)\right]\\ &+\left[\int_{T_{I+1}}^{T_{n+1}} dx\mu(t,x) - \frac{1}{2}\int_{T_{I+1}}^{T_{n+1}} dxdx' M_v(x,x';t)\right.\\ &\left.+\int_{T_{I+1}}^{T_{n+1}} dx v(t,x)\mathcal{A}(t,x)\right]\exp\left[-\int_{T_{I+1}}^{T_{n+1}} dx f(t,x)\right].\end{aligned}$$

Note that in obtaining $\partial\mathcal{X}_n(t)/\partial t$ *no condition* has been placed on either the drift $\mu(t,x)$ and/or the volatility $v(t,x)$, both of which can be arbitrary nonlinear functions of $f(t,x)$.

The bond portfolio $\mathcal{X}_n$ is a martingale, as discussed in Eq. (C6) of Appendix C, if and only if

$$E\left[\frac{\partial\mathcal{X}_n(t)}{\partial t}\right] = 0. \tag{21}$$

The random terms in Eq. (21) are proportional to $\mathcal{A}(t,x)$. Since, from Eq. (13), $E[\mathcal{A}(t,x)]=0$, the martingale condition given in Eq. (21) requires that the drift—namely, terms independent of $\mathcal{A}(t,x)$—must be zero and yields

$$\int_{T_{I+1}}^{T_n} dx\mu_I(t,x) = \frac{1}{2}\int_{T_{I+1}}^{T_n} dxdx' M_v(x,x';t).$$

The martingale condition given above is satisfied by choosing the following value for drift:

$$\mu_I(t,x) = \int_{T_{I+1}}^{x} dx' M_v(x,x';t). \tag{22}$$

VII. TIME EVOLUTION OF LIBOR

The drift term, as given in Eq. (22), is expressed in terms of the Libor forward interest rate volatility function $v(t,x)$. The main theoretical objective of the Libor market model is to completely remove $v(t,x)$ from the Libor evolution equation. More specifically, the objective is to express the drift of the Libor rates in terms of deterministic Libor volatility $\gamma(t,x)$ [defined later in Eq. (26)].

Consider the definition of Libor given in Eq. (2) and choose the drift μ to be equal to the μ_I given in Eq. (22). Equation (9) and the Wilson expansion for $\mathcal{A}(t,x)$ given in Eq. (17) yield

$$\begin{aligned}\ell\frac{\partial L(t,T_n)}{\partial t} = \exp\left\{\int_{T_n}^{T_{n+1}} dx f(t,x)\right\}&\left[\int_{T_n}^{T_{n+1}} dx\mu_I(t,x)\right.\\ &+\frac{1}{2}\int_{T_n}^{T_{n+1}} dx M_v(x,x';t)\\ &\left.+\int_{T_n}^{T_{n+1}} dx v(t,x)\mathcal{A}(t,x)\right].\end{aligned} \tag{23}$$

The drift for $\partial L(t,T_n)/\partial t$, from Eq. (22), has the following simplification,

$$\begin{aligned}&\int_{T_n}^{T_{n+1}} dx\mu_I(t,x) + \frac{1}{2}\int_{T_n}^{T_{n+1}} dxdx' M_v(x,x';t)\\ &= \int_{T_n}^{T_{n+1}} dx\left[\int_{T_{I+1}}^{T_n} dx' + \int_{T_n}^{x} dx'\right] M_v(x,x';t)\\ &\quad + \int_{T_n}^{T_{n+1}} dx\int_{T_n}^{x} dx' M_v(x,x';t)\\ &= \int_{T_{I+1}}^{T_n} dx\int_{T_n}^{T_{n+1}} dx' M_v(x,x';t)\end{aligned}$$

$$+\int_{T_n}^{T_{n+1}} dx \int_{T_n}^{T_{n+1}} dx' M_v(x,x';t)$$
$$= \int_{T_{I+1}}^{T_{n+1}} dx \int_{T_n}^{T_{n+1}} dx' M_v(x,x';t), \quad (24)$$

and yields, from Eqs. (2), (23), and (24), the following,

$$\frac{\partial L(t,T_n)}{\partial t} = \left[\int_{T_{I+1}}^{T_{n+1}} dx \int_{T_n}^{T_{n+1}} dx' M_v(x,x';t) + \int_{T_n}^{T_{n+1}} dx v(t,x)\mathcal{A}(t,x) \right] \frac{[1+\ell L(t,T_n)]}{\ell}. \quad (25)$$

Note that, as expected, the drift is zero for $n=I$, making $\mathcal{X}_I(t)=L(t,T_I)$ a martingale. Libor drift $\zeta(t,T_n)$ and volatility $\gamma(t,x)$ are *defined* as follows:

$$\frac{1}{L(t,T_n)} \frac{\partial L(t,T_n)}{\partial t} = \zeta(t,T_n) + \int_{T_n}^{T_{n+1}} dx \gamma(t,x)\mathcal{A}(t,x). \quad (26)$$

Volatility $\gamma(t,x)$ is a deterministic function—independent of $L(t,T_n)$. The drift $\zeta(t,T_n)$ is a nonlinear function of $L(t,T_n)$ that is determined by the martingale condition. Volatility $\gamma(t,x)$ and drift $\zeta(t,T_n)$ are discussed in Secs. VIII and IX, respectively.

VIII. VOLATILITY $\gamma(t,x)$ FOR POSITIVE LIBOR

A key *assumption* of the Libor market model is that the Libor volatility function $\gamma(t,x)$ is a deterministic function that is independent of the Libor rates. The main results of this section is to give an explicit derivation for Eq. (26) and verify that $\gamma(t,x)$ is, in fact, a deterministic function. The market value for $\gamma(t,x)$—for Libor and Euribor—has been obtained in [17].

As it stands, Eq. (25) for $\partial L(t,T_n)/\partial t$ does not imply that the Libor interest rates $L(t,T_n)$ are strictly positive. Libors are strictly positive only if Eq. (26) holds; namely, if there exists a Libor volatility function $\gamma(t,x)$ such that, from Eqs. (25) and (26),

$$\int_{T_n}^{T_{n+1}} dx v(t,x)\mathcal{A}(t,x) = \frac{\ell L(t,T_n)}{1+\ell L(t,T_n)} \int_{T_n}^{T_{n+1}} dx \gamma(t,x)\mathcal{A}(t,x) \quad (27)$$

$$\Rightarrow v(t,x) = \frac{\ell L(t,T_n)}{1+\ell L(t,T_n)} \gamma(t,x), \quad x \in [T_n,T_{n+1}). \quad (28)$$

In the Libor market model, $v(t,x)$ yields a model of the Libor forward interest rates with stochastic volatility. Equation (27) can be viewed as fixing the volatility function $v(t,x)$ of the forward interest rates $f(t,x)$ so as to ensure that $\gamma(t,x)$ is deterministic and leads to all Libor $L(t,T_n)$ being strictly positive.

To have a better understanding of $v(t,x)$ consider the limit of $\ell \to 0$, which yields $\int_{T_n}^{T_{n+1}} dx f(t,x) \simeq \ell f(t,x)$. From Eqs. (2) and (28)

$$v(t,x) \simeq [1 - e^{-\ell f(t,x)}]\gamma(t,x).$$

The following are the two limiting cases:

$$v(t,x) = \begin{cases} \ell \gamma(t,x) f(t,x), & \ell f(t,x) \ll 1 \\ \gamma(t,x), & \ell f(t,x) \gg 1. \end{cases} \quad (29)$$

For small values of $f(t,x)$, the volatility $v(t,x)$ is proportional to $f(t,x)$. It is known [19] that Libor forward interest rates $f(t,x)$ with volatility $v(t,x) \simeq f(t,x)$ are unstable and diverge after a finite time. However, in Libor market model, when the Libor forward rates become large, that is $\ell f(t,x) \gg 1$, the volatility $v(t,x)$ becomes deterministic and equals $\gamma(t,x)$. It is shown in Appendix E that Libor forward interest rates $f(t,x)$ are never divergent and Libor dynamics yields finite $f(t,x)$ for all future and calendar times.

IX. LIBOR DRIFT FOR MARTINGALES $\chi_n(t)$

The main motivation for introducing the Libor market model is to have manifestly positive interest rates and bonds. To ensure that the Libor rates $L(t,T_n)$ are always positive, it is sufficient to show that they are the exponential of real variables. To obtain positive Libor rates requires a nontrivial drift; a quantum finance derivation of the drift term is given in this section and generalizes earlier results of the BGM-Jamshidian approach.

The drift $\zeta(t,T_n)$ in Eq. (26) is chosen to make $\chi_n(t)$—given in Eq. (20)—a martingales for all n. One needs to express the Libor drift $\zeta(t,T_n)$ solely in terms of Libor volatility function $\gamma(t,x)$. The Libor drift term $\zeta(t,T_n)$ is defined, from Eqs. (25) and (26), as follows:

$$\zeta(t,T_n) = \frac{[1+\ell L(t,T_n)]}{\ell L(t,T_n)} \times \int_{T_{I+1}}^{T_{n+1}} dx v(t,x) \int_{T_n}^{T_{n+1}} dx' \mathcal{D}(x,x';t) v(t,x'). \quad (30)$$

The Libor forward interest rate volatility function $v(t,x)$ needs to be expressed in terms of the Libor volatility function $\gamma(t,x)$. To do so, a recursion equation is obtained from Eq. (27) in the following manner. Multiply both sides of Eq. (27) by $\mathcal{A}(t,x')v(t,x')$ and use Eq. (17), namely,

$$\mathcal{A}(t,x)\mathcal{A}(t,x') = \frac{1}{\epsilon}\mathcal{D}(x,x';t).$$

This removes the quantum field from Eq. (27) and, by equating the $1/\epsilon$ term from both sides of the resulting equation, one obtains

$$\int_{T_n}^{T_{n+1}} dx v(t,x)\mathcal{D}(x,x';t)v(t,x') = \frac{\ell L(t,T_n)}{1+\ell L(t,T_n)} \int_{T_n}^{T_{n+1}} dx \gamma(t,x)\mathcal{D}(x,x';t)v(t,x'). \quad (31)$$

Since the dynamics of $L(t,T_n)$ is being analyzed, we integrate variable x' from T_n to T_{n+1} and obtain

$$\int_{T_n}^{T_{n+1}} dx \int_{T_n}^{T_{n+1}} dx' v(t,x)\mathcal{D}(x,x';t)v(t,x')$$
$$= \frac{\ell L(t,T_n)}{1+\ell L(t,T_n)} \int_{T_n}^{T_{n+1}} dx v(t,x)\omega_n(t,x), \quad (32)$$

where $\omega_n(t,x)$ is defined by

$$\omega_n(t,x) = \int_{T_n}^{T_{n+1}} dx' \mathcal{D}(x,x';t)\gamma(t,x'). \quad (33)$$

Hence, from Eqs. (32) and (30)

$$\zeta(t,T_n) = \int_{T_{I+1}}^{T_{n+1}} dx v(t,x)\omega_n(t,x). \quad (34)$$

The drift obtained in Eq. (34) still depends on the volatility function $v(t,x)$. To express this integral solely in terms of the volatility function $\gamma(t,x)$ one has to carry out a calculation similar to the one used in obtaining Eq. (34).

Multiplying both sides of Eq. (27), this time by $\mathcal{A}(t,x')\gamma(t,x')$, and using Eq. (17) yield the following:

$$\int_{T_n}^{T_{n+1}} dx v(t,x)\mathcal{D}(x,x';t)\gamma(t,x')$$
$$= \frac{\ell L(t,T_n)}{1+\ell L(t,T_n)} \int_{T_n}^{T_{n+1}} dx \gamma(t,x)\mathcal{D}(x,x';t)\gamma(t,x').$$

Integrating x' from T_n to T_{n+1} yields

$$\int_{T_n}^{T_{n+1}} dx v(t,x)\omega_n(t,x) = \frac{\ell L(t,T_n)}{1+\ell L(t,T_n)} \int_{T_n}^{T_{n+1}} dx dx' M_\gamma(x,x';t), \quad (35)$$

where

$$M_\gamma(x,x';t) \equiv \gamma(t,x)\mathcal{D}(x,x';t)\gamma(t,x'). \quad (36)$$

The recursion equation

$$\int_t^{T_{n+1}} dx v(t,x)\omega_n(t,x) = \int_t^{T_n} dx v(t,x)\omega_n(t,x) + \int_{T_n}^{T_{n+1}} dx v(t,x)\omega_n(t,x) \quad (37)$$

yields, from Eqs. (35), the following:

$$\int_t^{T_{n+1}} dx v(t,x)\omega_n(t,x) = \int_t^{T_n} dx v(t,x)\omega_n(t,x) + \frac{\ell L(t,T_n)}{1+\ell L(t,T_n)} \int_{T_n}^{T_{n+1}} dx \gamma(t,x)\omega_n(t,x).$$

For simplicity, let time $t=T_0$; recursing above equation yields, using Eq. (33), the following:

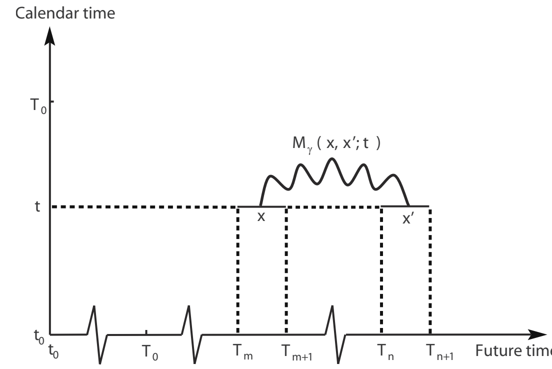

FIG. 7. Libor propagator $\Lambda_{mn}(t)$ yields nontrivial and imperfect correlation between the different Libors.

$$\int_{T_0}^{T_{n+1}} dx v(t,x)\omega_n(t,x)$$
$$= \sum_{m=0}^{n} \frac{\ell L(t,T_m)}{1+\ell L(t,T_m)} \int_{T_m}^{T_{m+1}} dx \gamma(t,x)\omega_n(t,x)$$
$$= \sum_{m=0}^{n} \frac{\ell L(t,T_m)}{1+\ell L(t,T_m)} \Lambda_{mn}(t), \quad (38)$$

where, as shown in Fig. 7, the Libor propagator is given by

$$\Lambda_{mn}(t) \equiv \int_{T_m}^{T_{m+1}} dx \gamma(t,x)\omega_n(t,x)$$
$$= \int_{T_m}^{T_{m+1}} dx \int_{T_n}^{T_{n+1}} dx' \gamma(t,x)\mathcal{D}(x,x';t)\gamma(t,x')$$
$$= \int_{T_m}^{T_{m+1}} dx \int_{T_n}^{T_{n+1}} dx' M_\gamma(x,x';t). \quad (39)$$

There are three cases for $\zeta(t,T_n)$ corresponding to $T_n=T_I$, $T_n>T_I$, and $T_n<T_I$, as shown in Fig. 6.

Case (i) $T_n=T_I$. From Eq. (34)

$$\zeta(t,T_I) = 0.$$

Case (ii) $T_n>T_I$. Equation (38) yields the following:

$$\zeta(t,T_n) = \int_{T_{I+1}}^{T_{n+1}} dx v(t,x)\omega_n(t,x)$$
$$= \int_{T_0}^{T_{n+1}} dx v(t,x)\omega_n(t,x) - \int_{T_0}^{T_{I+1}} dx v(t,x)\omega_n(t,x)$$
$$= \sum_{m=I+1}^{n} \frac{\ell L(t,T_m)}{1+\ell L(t,T_m)} \Lambda_{mn}(t).$$

Case (iii) $T_n<T_I$. From Eq. (38), one has the following:

$$\zeta(t,T_n) = \int_{T_{I+1}}^{T_{n+1}} dx v(t,x)\omega_n(t,x)$$
$$= -\left[\int_{T_0}^{T_{I+1}} dx v(t,x)\omega_n(t,x) - \int_{T_0}^{T_{n+1}} dx v(t,x)\omega_n(t,x)\right]$$
$$= -\sum_{m=n+1}^{I} \frac{\ell L(t,T_m)}{1+\ell L(t,T_m)}\Lambda_{mn}(t).$$

Collecting the results from above yields [18,20]

$$\zeta(t,T_n) = \begin{cases} \sum_{m=I+1}^{n} \frac{\ell L(t,T_m)}{1+\ell L(t,T_m)}\Lambda_{mn}(t), & T_n > T_I \\ 0, & T_n = T_I \\ -\sum_{m=n+1}^{I} \frac{\ell L(t,T_m)}{1+\ell L(t,T_m)}\Lambda_{mn}(t), & T_n < T_I, \end{cases} \tag{40}$$

where $\Lambda_{mn}(t)$ is given in Eq. (39).

Equation (40) is the main result of the Libor market model and the equation incorporates imperfect correlation of Libor thus generalizing the earlier BGM-Jamshidian result.

X. LIBOR DYNAMICS

As stated in Eq. (26), Libor dynamics is given by

$$\frac{1}{L(t,T_n)}\frac{\partial L(t,T_n)}{\partial t} = \zeta(t,T_n) + \int_{T_n}^{T_{n+1}} dx\gamma(t,x)\mathcal{A}(t,x). \tag{41}$$

In particular, since $\zeta(t,T_I)=0$, Libor $L(t,T_I)$ has a martingale evolution given by

$$\frac{\partial L(t,T_I)}{\partial t} = L(t,T_I)\int_{T_I}^{T_{I+1}} dx\gamma(t,x)\mathcal{A}(t,x). \tag{42}$$

The results obtained express the time evolution of Libor completely in terms of volatility $\gamma(t,x)$, which is a function that is empirically studied in [17]. Libor drift $\zeta(t,T_n)$ is fixed by Eq. (40) and is a nonlinear and nonlocal function of all Libors.

In the Libor evolution equations given in Eq. (41), all reference to the volatility function $v(t,x)$ of the Libor forward interest rates $f(t,x)$ has been removed—as indeed was the whole purpose of the derivations of Sec. IX—with the drift being completely expressed in terms of Libor $L(t,T_n)$ and its volatility $\gamma(t,x)$.

Equation (41) needs to be integrated to confirm that Libor dynamics yields positive valued Libors. Let $T_0 > t_0$ be the two points on the Libor time lattice. From Eqs. (17) and (41), the differential of logarithmic Libor is given by

$$\frac{\partial \ln L(t,T_n)}{\partial t} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[\ln L(t+\epsilon,T_n) - \ln L(t,T_n)]$$
$$= \frac{1}{L(t,T_n)}\frac{\partial L(t,T_n)}{\partial t} - \frac{\epsilon}{2}\left[\frac{1}{L(t,T_n)}\frac{\partial L(t,T_n)}{\partial t}\right]^2$$
$$+ O(\epsilon) \Rightarrow \frac{\partial \ln L(t,T_n)}{\partial t} = \zeta(t,T_n)$$

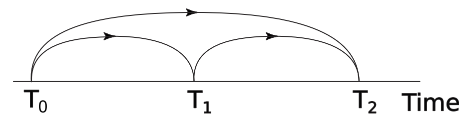

FIG. 8. Simple interest earned over Libor time interval T_0 to T_2. Simple interest earned over the two subintervals T_0 to T_1 and from T_1 to T_2 must be equal to the interest earned from T_0 to T_2.

$$+ \int_{T_n}^{T_{n+1}} dx\gamma(t,x)\mathcal{A}(t,x) - \frac{1}{2}\Lambda_{nn}(t). \tag{43}$$

Integrating above equation over time yields

$$L(T_0,T_n) = L(t_0,T_n)e^{\beta(t_0,T_0,T_n)+W_n}, \tag{44}$$

where

$$\beta(t_0,T_0,T_n) = \int_{t_0}^{T_0} dt\zeta(t,T_n), \quad q_n^2 = \int_{t_0}^{T_0} dt\Lambda_{nn}(t),$$

$$W_n = -\frac{1}{2}q_n^2 + \int_{t_0}^{T_0} dt\int_{T_n}^{T_{n+1}} dx\gamma(t,x)\mathcal{A}(t,x).$$

Libor is proportional to the exponential of real quantities, namely, a real drift $\zeta(t,T_n)-q_n^2/2$ and a real valued (Gaussian) quantum field $\mathcal{A}(t,x)$. Hence, Libor dynamics leads to positive Libor—as given in Eq. (44).

XI. LOGARITHMIC LIBOR RATES $\phi(t,x)$

Since $\gamma(t,x)$, the volatility of $L(t,T_n)$, is deterministic it is convenient to change variables from $f(t,x)$ to $L(t,T_n)$. Equation (44) shows that Libor $L(t,T_n)$ is a positive random variable. A change of variables to logarithmic coordinates shows the structure of the Libor market model more clearly. Let $\phi(t,x)$ be a two-dimensional quantum field; define a change of variables by

$$\ell L(t,T_n) = \exp\left\{\int_{T_n}^{T_{n+1}} dx\phi(t,x)\right\} \equiv e^{\phi_n(t)}. \tag{45}$$

From its definition, $\phi(t,x)$ has dimensions of 1/time and can be thought of as the effective *logarithmic Libor* interest rates.

Consider, at some time t, a contract for a deposit to be made from future time T_0 to T_2; the principal plus simple interest earned, at time T_2, is given by $1+(T_2-T_0)L(t,T_0,T_2)$. This amount must be equal to that earned by first depositing the principal at time T_0, then rolling over, at $T_0+\ell=T_1$, the deposit and interest earned, and collecting the principal and interest at time $T_2=T_1+\ell$ (see Fig. 8). For there to be no arbitrage opportunities the two procedures must be equal, namely [21],

$$1 + (T_2-T_0)L(t,T_0,T_2) = [1+\ell L(t,T_0)][1+\ell L(t,T_1)] \Rightarrow (T_2-T_0)L(t,T_0,T_2) = e^{\phi_0(t)+\phi_1(t)} + e^{\phi_0(t)} + e^{\phi_1(t)} \Rightarrow e^{\phi_0(t)+\phi_1(t)}$$
$$= \exp\left[\int_{T_0}^{T_2} dx\phi(t,x)\right]$$

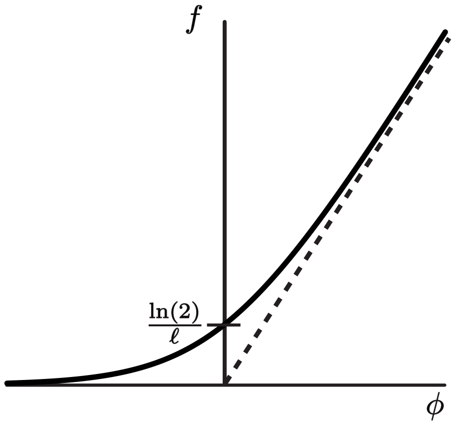

FIG. 9. The dependence of $\phi(t,x)$ on $f(t,x)$.

$$= \ell L(t,T_0)\ell L(t,T_1). \tag{46}$$

Here, $\exp\{\int_{T_n}^{T_m+\ell} dx\phi(t,x)\}$, similar to Eq. (46), is related to the future Libor rate $L(t,T_n,T_{m+1})$. The integral of $\phi(t,x)$ over many Libor future time intervals yields the following:

$$\exp\left\{\int_{T_n}^{T_m+\ell} dx\phi(t,x)\right\} = \prod_{i=n}^{m}[\ell L(t,T_i)].$$

The Libor forward interest rates $f(t,x)$ are related to logarithmic Libor by Eq. (2),

$$\exp\left\{\int_{T_n}^{T_{n+1}} dx f(t,x)\right\} = 1 + \ell L(t,T_n) \Rightarrow \exp\left\{\int_{T_n}^{T_n+\ell} dx f(t,x)\right\} = 1 + \exp\left\{\int_{T_n}^{T_n+\ell} dx\phi(t,x)\right\}.$$

The definition of $\phi(t,x)$ depends on the tenor, and for the benchmark case is taken to be $\ell = 90$ days (three months). For Libor, the tenor is always finite, being a minimum of overnight (24 h). The logarithmic Libor $\phi(t,x)$ is well defined for any nonzero tenor ℓ. For the limit of zero tenor, let $\ell = \epsilon \to 0$; from defining Eq. (2), it follows that, since $f(t,x)$ is always finite,

$$1 + \epsilon f(t,x) = 2 + \epsilon\phi(t,x) \Rightarrow \phi(t,x) = \frac{1}{\epsilon}[f(t,x) - 1] \to -\infty.$$

In other words, the zero tenor limit is singular for $\phi(t,x)$; however, for finite tenor $\ell \neq 0$, the field $\phi(t,x)$ is always well defined.

Since the interest derivative market is based on the three-month Libor, let $\ell = 1/4$ year; one can approximately evaluate the integral and obtain the following:

$$\exp\{\ell f(t,x)\} \simeq 1 + \exp\{\ell\phi(t,x)\}. \tag{47}$$

Equation (47) is plotted in Fig. 9. For $f(t,x) \ll \ln(2)/\ell \sim 400\%/\text{year}$, the value of $\phi(t,x) \simeq 0$; furthermore, for $\phi(t,x) \ll 1$ the value of $f(t,x) \simeq 0$. Only when both the rates $f(t,x)$ and $\phi(t,x)$ are large they are approximately equal.

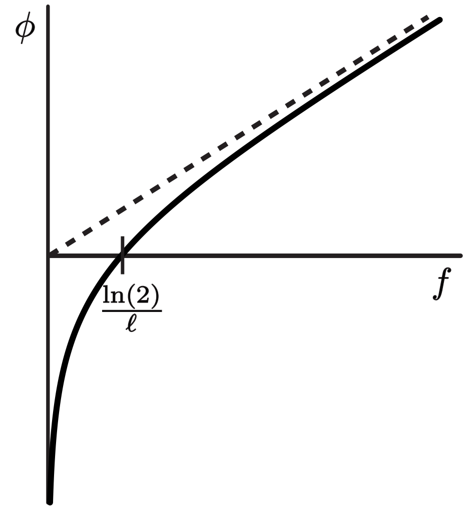

FIG. 10. The dependence of $f(t,x)$ on $\phi(t,x)$.

Hence, there is no domain where *both* the quantum fields $f(t,x)$ and $\phi(t,x)$ take *small* values and consequently there is no consistent scheme for *simultaneously* defining a perturbation expansion, in powers of $\phi(t,x)$ and $f(t,x)$, for both the quantum fields. In summary, one can define a perturbation expansion for either $f(t,x)$ or $\phi(t,x)$ but not simultaneously for both the fields.

Furthermore, as can be seen from Fig. 10, for Eq. (47) to have real values for both $f(t,x)$ and $\phi(t,x)$, the following is required:

$$0 \leq f(t,x) \leq +\infty, \quad -\infty \leq \phi(t,x) \leq +\infty.$$

Note that $\phi(t,x)$ is the natural quantum field for developing a Feynman perturbation expansion for Libor instruments as it is a flat degree of freedom taking values on the real line.

The dynamics of Libor $L(t,T_n)$ is specified in Eq. (41) and yields, from Eq. (43), the following *defining equation* for $\phi(t,x)$:

$$\frac{\partial}{\partial t}\int_{T_n}^{T_{n+1}} dx\phi(t,x) = \frac{\partial \ln[\ell L(t,T_n)]}{\partial t} = \zeta(t,T_n) + \int_{T_n}^{T_{n+1}} dx\gamma(t,x)\mathcal{A}(t,x) - \frac{1}{2}\Lambda_{nn}. \tag{48}$$

The drift $\zeta(t,T_n)$ for forward bond numeraire $B(t,T_{I+1})$ is given by Eqs. (39) and (40). Integrating Eq. (48) from calendar Libor time t_0 to T_0 yields

$$\int_{T_n}^{T_{n+1}} dx\phi(T_0,x) = \int_{T_n}^{T_{n+1}} dx\phi(t_0,x) + \int_{t_0}^{T_0} dt\left[\zeta(t,T_n) + \int_{T_n}^{T_{n+1}} dx\gamma(t,x)\mathcal{A}(t,x) - \frac{1}{2}\Lambda_{nn}\right]. \tag{49}$$

Exponentiating Eq. (49) yields Eq. (44) as expected.

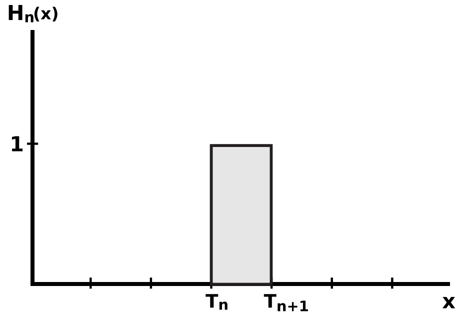

FIG. 11. The characteristic function $H_n(x)$ for the Libor interval $[T_n,T_{n+1})$.

Dropping the $\int_{T_n}^{T_{n+1}}dx$ integration from both sides of Eq. (48) yields, for $x\in[T_n,T_{n+1})$, the following time evolution for logarithmic Libor:

$$\frac{\partial\phi(t,x)}{\partial t}=-\frac{1}{2}\Lambda_n(t,x)+\rho_n(t,x)+\gamma(t,x)\mathcal{A}(t,x), \quad (50)$$

$$\Lambda_n(t,x)=\int_{T_n}^{T_{n+1}}dx'M(x,x';t). \quad (51)$$

The function $\rho_n(t,x)$ is defined as follows:

$$\zeta(t,T_n)=\int_{T_n}^{T_{n+1}}dx\rho_n(t,x). \quad (52)$$

Hence, from Eqs. (40) and (52), for $x\in[T_n,T_{n+1})$,

$$\rho_n(t,x)=\begin{cases}\sum_{m=I+1}^{n}\frac{e^{\phi_m(t)}}{1+e^{\phi_m(t)}}\Lambda_m(t,x), & T_n>T_I\\ 0, & T_n=T_I\\ -\sum_{m=n+1}^{I}\frac{e^{\phi_m(t)}}{1+e^{\phi_m(t)}}\Lambda_m(t,x), & T_n<T_I.\end{cases} \quad (53)$$

To write Eq. (50) in a more compact form, define the characteristic function $H_n(x)$ for the Libor time interval $[T_n,T_{n+1})$ given by

$$H_n(x)=\begin{cases}1, & T_n\le x<T_{n+1}\\ 0, & x\notin[T_n,T_{n+1})\end{cases} \quad (54)$$

$$\Rightarrow\int_{T_n}^{T_{n+1}}dxH_m(x)=\delta_{m-n} \quad (55)$$

and it is shown in Fig. 11. The characteristic function has the following important properties:

$$f(x)=\sum_{n=0}^{\infty}H_n(x)f_n(x),$$

$$f(x)=f_n(x),\quad x\in[T_n,T_{n+1}).$$

Hence, from Eqs. (50) and (29), for arbitrary future time x,

$$\frac{\partial\phi(t,x)}{\partial t}=\rho(t,x)-\frac{1}{2}\Lambda(t,x)+\gamma(t,x)\mathcal{A}(t,x), \quad (56)$$

$$\Lambda(t,x)=\sum_{n=0}^{\infty}H_n(x)\Lambda_n(t,x),$$

$$\rho(t,x)=\sum_{n=0}^{\infty}H_n(x)\rho_n(t,x). \quad (57)$$

It is convenient to separate out a "kinetic" drift $-\frac{1}{2}\Lambda(t,x)$ that does not depend on the Libors, with the remaining drift $\rho(t,x)$ being a nonlinear and nonlocal function of the Libors.

XII. LAGRANGIAN OF LOGARITHMIC LIBOR $\phi(t,x)$

Equation (56) encodes a change of variables relating two quantum fields $\phi(t,x)$ and $\mathcal{A}(t,x)$ and is given by

$$\mathcal{A}(t,x)=\frac{\partial\phi(t,x)/\partial t-\tilde{\rho}(t,x)}{\gamma(t,x)}, \quad (58)$$

$$\tilde{\rho}(t,x)\equiv\rho(t,x)-\frac{1}{2}\Lambda(t,x). \quad (59)$$

The Lagrangian and action for the Gaussian "stiff" quantum field $\mathcal{A}(t,x)$, after doing an integration by parts in Eq. (12), are given by

$$\mathcal{L}[\mathcal{A}]=-\frac{1}{2}\mathcal{A}(t,x)\mathcal{D}^{-1}(t,x,x')\mathcal{A}(t,x'),$$

$$S[\mathcal{A}]=\int_{\mathcal{T}}\mathcal{L}[\mathcal{A}].$$

The semi-infinite trapezoidal domain $\mathcal{T}$ is given in Fig. 16. The partition function is given by the Feynman path integral

$$Z=\int D\mathcal{A}e^{S[\mathcal{A}]}.$$

The Lagrangian and action for logarithmic Libor quantum field $\phi(t,x)$ are given by

$$\mathcal{L}[\phi]=-\frac{1}{2}\left[\frac{\partial\phi(t,x)/\partial t-\tilde{\rho}(t,x)}{\gamma(t,x)}\right]\times\mathcal{D}^{-1}(t,x,x')\times\left[\frac{\partial\phi(t,x')/\partial t-\tilde{\rho}(t,x')}{\gamma(t,x')}\right],$$

$$S[\phi]=\int_{t_0}^{\infty}dt\int_{t}^{\infty}dxdx'\mathcal{L}[\phi]. \quad (60)$$

The Neumann boundary conditions $\mathcal{A}(t,x)$ yields the following boundary conditions on $\phi(t,x)$ [2]:

$$\frac{\partial}{\partial x}\left[\frac{\partial\phi(t,x)/\partial t-\tilde{\rho}(t,x)}{\gamma(t,x)}\right]\Bigg|_{x=t}=0. \quad (61)$$

It is shown in Appendix F that the Jacobian of the functional change of variables given in Eq. (58) is a constant, independent of $\phi(t,x)$ even though the transformation in Eq. (58) in nonlinear due to the nonlinearity of Libor drift $\rho(t,x)$. A constant Jacobian leads to $\phi(t,x)$ being *flat variables*, with no measure term in the path integral. Flat variables have a well-defined leading order Gaussian path integral that generates a Feynman perturbation expansion for all financial instruments, thus greatly simplifying all calculations that are based on $\phi(t,x)$.

Hence, up to an irrelevant constant, the logarithmic Libor path integral measure is given by

$$\int D\mathcal{A} = \int D\phi = \prod_{t=t_0}^{\infty}\prod_{x=t}^{\infty}\int_{-\infty}^{+\infty} d\phi(t,x).$$

The partition function for ϕ is

$$Z = \int D\phi e^{S[\phi]} = \int D\mathcal{A} e^{S[\mathcal{A}]}$$

and the expectation value of a financial instrument $\mathcal{O}$ is given by

$$E[\mathcal{O}] = \frac{1}{Z}\int D\mathcal{A}\mathcal{O}[\mathcal{A}]e^{S[\mathcal{A}]} = \frac{1}{Z}\int D\phi\mathcal{O}[\phi]e^{S[\phi]}. \quad (62)$$

Path integral derivation of Libor martingale

Consider the $n=I$ special case of the portfolio; from Eq. (40), drift $\zeta(t,T_I)=0$ and hence the instrument $\mathcal{X}_I(t)=L(t,T_I)$ is a martingale. The integral formulation of the martingale condition states that the present value of a martingale instrument is the conditional expectation value of its future value; in other words, the martingale condition is given by

$$L(t_0,T_I) = E[L(T_0,T_I)], \quad T_0 > t_0.$$

The path integral for the right-hand side, from Eq. (62), is given by the expectation value of a financial instrument $\mathcal{O}=L(T_0,T_I)$; hence

$$E[L(T_0,T_I)] = \frac{1}{Z}\int D\phi e^{S[\phi]}L(T_0,T_I). \quad (63)$$

For Libor $L(t,T_I)$, the drift is zero $\rho_I=0$ and hence, from Eq. (52), $\zeta(t,T_I)=0$; Eqs. (45) and (49) yield

$$\ln \ell L(T_0,T_I) = \int_{T_I}^{T_{I+1}} dx\phi(T_0,x) = \int_{T_I}^{T_{I+1}} dx\phi(t_0,x) + \int_{t_0}^{T_0} dt\left\{\int_{T_I}^{T_{I+1}} dx\gamma(t,x)\mathcal{A}(t,x) - \frac{1}{2}\Lambda_{\text{II}}(t)\right\}.$$

Changing path integration variables from $\phi(t,x)$ to $\mathcal{A}(t,x)$ and using the generating functional given in Eq. (A5) yield

$$E[\ell L(T_0,T_I)] = \ell L(t_0,T_I) \times \frac{1}{Z}\int D\mathcal{A}\exp\left\{\int_{t_0}^{T_0} dt\left[\int_{T_I}^{T_{I+1}} dx\gamma(t,x)\mathcal{A}(t,x) - \frac{1}{2}\Lambda_{\text{II}}(t)\right]\right\}e^{S[\mathcal{A}]} = \ell L(t_0,T_I) \quad \text{(martingale)}.$$

The calculation confirms that $\mathcal{X}_I(t)=L(t,T_I)$ is in fact a martingale.

The path integral derivation of the martingale condition for the Libor $\chi_I(T_0)=L(T_0,T_I)$ cannot be applied for the general case of $\chi_n(t)$. The reason being that, in general, drift ρ_n is a nonlinear function of the quantum field $\mathcal{A}(t,x)$; evaluating the drift requires that a nonlinear path integral be evaluated exactly—something that is computationally intractable.

XIII. DYNAMICS OF LIBOR FORWARD INTEREST RATES $f(t,x)$

The dynamics of logarithmic Libor given in Eq. (56) also defines the dynamics of the quantum field $f(t,x)$. Differentiating Eq. (2) and using Eqs. (45) and (56) yield the following:

$$\int_{T_n}^{T_{n+1}} dx\frac{\partial f(t,x)}{\partial t} = \frac{e^{\phi_n(t)}}{1+e^{\phi_n(t)}}\int_{T_n}^{T_{n+1}} dx\frac{\partial\phi(t,x)}{\partial t} = \frac{e^{\phi_n(t)}}{1+e^{\phi_n(t)}} \quad (64)$$

$$\times\int_{T_n}^{T_{n+1}} dx\left[-\frac{1}{2}\Lambda(t,x) + \rho(t,x) + \gamma(t,x)\mathcal{A}(t,x)\right]. \quad (65)$$

From Eqs. (9) and (65)

$$\frac{\partial f(t,x)}{\partial t} = \mu(t,x) + v(t,x)\mathcal{A}(t,x) \Rightarrow \int_{T_n}^{T_{n+1}} dx\mu(t,x) = \frac{e^{\phi_n(t)}}{1+e^{\phi_n(t)}} \times\int_{T_n}^{T_{n+1}} dx\left[-\frac{1}{2}\Lambda(t,x) + \rho(t,x)\right], \quad (66)$$

$$\int_{T_n}^{T_{n+1}} dxv(t,x) = \frac{e^{\phi_n(t)}}{1+e^{\phi_n(t)}}\int_{T_n}^{T_{n+1}} \gamma(t,x). \quad (67)$$

The result for $v(t,x)$ has been obtained earlier in Eq. (28); the value of μ is a new result. Writing the drift and volatility in terms of $f(t,x)$ yields, from Eqs. (2) and (57), the following:

$$f_n(t) \equiv \int_{T_n}^{T_{n+1}} dxf(t,x), \quad \frac{e^{\phi_n(t)}}{1+e^{\phi_n(t)}} = 1 - e^{-f_n(t)}, \quad (68)$$

$$\mu(t,x) = u(t,x)\left[-\frac{1}{2}\Lambda(t,x) + \rho(t,x)\right], \quad (69)$$

$$v(t,x) = u(t,x)\gamma(t,x), \quad (70)$$

where $u(t,x)=\Sigma_{n=0}^{\infty} H_n(x)[1-e^{-f_n(t)}]$.

The drift $\mu(t,x)$ and volatility $v(t,x)$ are both functions of only $\exp\{-f_n(t)\}$, which, in turn, is the forward price of a zero coupon bond.

XIV. HAMILTONIAN FORMULATION OF MARTINGALE CONDITION

The Hamiltonian is a differential operator that acts on an underlying state space [22]. Financial instruments are elements of the state space and the evolution is determined by a Hamiltonian.

As discussed in Sec. VI, choose the zero coupon bond $B(t,T_{I+1})$ to be the forward bond numeraire; then, from Eq. (20), for every n

$$\mathcal{X}_n(t) \equiv \frac{L(t,T_n)B(t,T_{n+1})}{B(t,T_{I+1})} \quad \text{(martingale)}.$$

Recall Libor time T_n can be less than, equal to, or greater than T_I—as shown in Fig. 6.

The drift $\rho(t,x)$ is fixed in the Hamiltonian framework by imposing the martingale condition on $\mathcal{X}_n(t)$, namely, that [2]

$$\mathcal{H}(t)[\mathcal{X}_n(t)] = \mathcal{H}(t)\left[\frac{L(t,T_n)B(t,T_{n+1})}{B(t,T_{I+1})}\right] = 0. \tag{71}$$

The instrument $\mathcal{X}_n(t)$ is an element of the Libor interest rate state space, which is discussed in Appendix G, and $\mathcal{H}(t)$ is the Hamiltonian. The differential formulation of the martingale given in Eq. (71) states that for the Hamiltonian evolving interest rate instruments to be free from arbitrage opportunities, it must annihilate $\mathcal{X}_n(t)$.

For notational ease, write

$$\mathcal{X}_n(t) = \frac{L(t,T_n)B(t,T_{n+1})}{B(t,T_{I+1})} \equiv L_n\frac{B(t,T_{n+1})}{B(t,T_{I+1})}.$$

Let t be a Libor time; from Eq. (4), the zero coupon bond is given by

$$B(t,T_{n+1}) = \prod_{k=0}^{n}\left(\frac{1}{1+\ell L_k}\right). \tag{72}$$

The following are the three cases for $\mathcal{X}_n(t)$:

(i) For $n=I$,

$$\mathcal{X}_I(t) = L_I = L(t,T_I). \tag{73}$$

(ii) For $n>I$,

$$\mathcal{X}_n(t) = L_n\prod_{k=I+1}^{n}\left(\frac{1}{1+\ell L_k}\right) = L_n \exp\left\{-\sum_{k=I+1}^{n}\ln(1+\ell L_k)\right\}. \tag{74}$$

(iii) For $n<I$,

$$\mathcal{X}_n(t) = L_n\prod_{k=n+1}^{I}(1+\ell L_k) = L_n \exp\left\{\sum_{k=n+1}^{I}\ln(1+\ell L_k)\right\}. \tag{75}$$

XV. HAMILTONIAN DERIVATION OF LIBOR MARKET MODEL DRIFT

Libor market model drift $\rho(t,x)$ was derived in Sec. IX using the Wilson expansion. The drift is given an independent derivation in this section based directly on the Hamiltonian formulation of the martingale condition given in Eq. (71).

The derivation for Libor drift $\rho(t,x)$ for the Libor market model given in Sec. IX was quite circuitous; the Libor forward interest rates $f(t,x)$ were used as scaffolding and it was not clear why one could not evaluate Libor drift directly using only Libor $L(t,T_n)$. The result is also quite opaque, with the drift having summations and minus signs that do not have a clear explanation. In contrast, the Hamiltonian framework yields a clear and transparent derivation of Libor drift directly using Libor variables $L(t,T_n)$ and the result is intuitively clear.

The logarithmic Libor Hamiltonian is derived in Appendix H and is given by Eq. (H8); for notational convenience, a kinetic piece is subtracted out from the drift $\rho(t,x)$. Hence [23]

$$\mathcal{H}(t) = -\frac{1}{2}\int_{x,x'} M(x,x';t)\frac{\delta^2}{\delta\phi(x)\,\delta\phi(x')} + \frac{1}{2}\int_x [\Lambda(t,x) - \rho(t,x)]\frac{\delta}{\delta\phi(x)},$$

$$\int_x \equiv \int_t^{+\infty} dx. \tag{76}$$

The martingale condition given in Eq. (71) requires the calculation of $\delta/\delta\phi$ acting on $\mathcal{X}_n(t)$, which in turn needs the following computation. The definition of logarithm Libor given in Eq. (45), namely,

$$\ell L(t,T_n) \equiv \ell L_n = \exp\left\{\int_{T_n}^{T_{n+1}} dx\,\phi_t(x)\right\} \equiv e^{\phi_n},$$

yields

$$\frac{\delta}{\delta\phi(x)}L_k = H_k(x)L_k, \tag{77}$$

where the characteristic function is given in Eq. (54) and is shown in Fig. 11.

For the case of $n=I$, from Eq. (73), $\mathcal{X}_I(t)=L_I=L(t,T_I)$; by inspection, it can be easily seen that $\rho_I(t,x)=0$.

A. Case (i) $n>I$

For the case of $\mathcal{X}_n(t)$, $n>I$, Eqs. (74) and (77) yield

$$\frac{1}{\mathcal{X}_n(t)}\frac{\delta\mathcal{X}_n(t)}{\delta\phi(x)} = H_n(x) - \sum_{k=I+1}^{n}\frac{e^{\phi_k}H_k(x)}{1+e^{\phi_k}}. \tag{78}$$

Note that the summation term above is due to the discounting by the forward numeraire $B(t,T_{I+1})$. The second functional derivative yields

$$\frac{1}{\mathcal{X}_n(t)}\frac{\delta^2\mathcal{X}_n(t)}{\delta\phi(x)\delta\phi(x')} = \left[H_n(x) - \sum_{j=I+1}^{n}\frac{e^{\phi_j}H_j(x)}{1+e^{\phi_j}}\right]\left[H_n(x') - \sum_{k=I+1}^{n}\frac{e^{\phi_k}H_k(x')}{1+e^{\phi_k}}\right] - \sum_{j=I+1}^{n}\frac{e^{\phi_j}H_j(x)H_j(x')}{1+e^{\phi_j}} + \sum_{j=I+1}^{n}\left[\frac{e^{\phi_j}}{1+e^{\phi_j}}\right]^2 H_j(x)H_j(x'). \quad (79)$$

On applying logarithmic Libor Hamiltonian on $\mathcal{X}_n(t)$, $n>I$, Eqs. (77)–(79) yield, after a few obvious cancellations,

$$\frac{\mathcal{H}[\mathcal{X}_n(t)]}{\mathcal{X}_n(t)} = \frac{1}{2}\sum_{j=I+1}^{n}\left[\frac{e^{\phi_j}}{1+e^{\phi_j}}\right]^2\Lambda_{jj} - \sum_{j=I+1}^{n}\frac{e^{\phi_j}}{1+e^{\phi_j}}\Lambda_{jn} + \frac{1}{2}\sum_{j,k=I+1}^{n}\frac{e^{\phi_j+\phi_k}}{[1+e^{\phi_j}][1+e^{\phi_k}]}\Lambda_{jk} + \zeta_n - \sum_{j=I+1}^{n}\frac{e^{\phi_j}}{1+e^{\phi_j}}\zeta_j, \quad (80)$$

where

$$\Lambda_{mn} = \int_{T_m}^{T_{m+1}} dx \int_{T_n}^{T_{n+1}} dx' M(x,x';t)$$

and, as in Eq. (52),

$$\zeta_n \equiv \zeta(t,T_n) = \int_{T_n}^{T_{n+1}} dx \rho_n(t,x). \quad (81)$$

Inspecting the result in Eq. (80) leads to the following ansatz:

$$\zeta_n = \sum_{j=I+1}^{n}\frac{e^{\phi_j}}{1+e^{\phi_j}}\Lambda_{jn}. \quad (82)$$

Hence

$$\sum_{j=I+1}^{n}\frac{e^{\phi_j}}{1+e^{\phi_j}}\zeta_j = \sum_{j=I+1}^{n}\frac{e^{\phi_j}}{1+e^{\phi_j}}\sum_{k=I+1}^{j}\frac{e^{\phi_k}}{1+e^{\phi_k}}\Lambda_{jk}. \quad (83)$$

The remarkable identity

$$\sum_{j=I+1}^{n}\sum_{k=I+1}^{j} A_{jk} = \frac{1}{2}\sum_{j,k=I+1}^{n} A_{jk} + \frac{1}{2}\sum_{j=I+1}^{n} A_{jj} \quad (84)$$

applied to Eq. (83) leads to the cancellation of all the terms on the right-hand side of Eq. (80) and yields the final result,

$$\mathcal{H}[\mathcal{X}_n(t)] = 0 \quad \text{(martingale)}. \quad (85)$$

Hence, from Eqs. (81) and (82), Libor drift is given by

$$\int_{T_n}^{T_{n+1}} dx \rho_n(t,x) = \sum_{j=I+1}^{n}\frac{e^{\phi_j(t)}}{1+e^{\phi_j(t)}} \times \int_{T_n}^{T_{n+1}} dx \int_{T_j}^{T_{j+1}} dx' M(x,x';t). \quad (86)$$

Therefore, for $T_n \le x < T_{n+1}$, the drift is given by

$$\rho_n(t,x) = \sum_{j=I+1}^{n}\frac{e^{\phi_j(t)}}{1+e^{\phi_j(t)}}\int_{T_j}^{T_{j+1}} dx' M(x,x';t) = \sum_{j=I+1}^{n}\frac{e^{\phi_j(t)}}{1+e^{\phi_j(t)}}\Lambda_j(t,x). \quad (87)$$

B. Case (ii) $n<I$

A derivation similar to case (i) yields the result for $\mathcal{X}_n(t)$, $n<I$. One needs to keep track of the relative negative sign in χ_n, given in Eqs. (74) and (75) arising from the difference in the discounting factor. The following is the final result:

$$\rho_n(t,x) = -\sum_{j=n+1}^{I}\frac{e^{\phi_j(t)}}{1+e^{\phi_j(t)}}\int_{T_j}^{T_{j+1}} dx' M(x,x';t). \quad (88)$$

The exact results given in Eqs. (82), (87), and (88) yield Libor drift as follows:

$$\rho(t,x) = \sum_{n=0}^{\infty} H_n(x)\rho_n(t,x),$$

where

$$\rho_n(t,x) = \begin{cases} \sum_{m=I+1}^{n}\frac{e^{\phi_m(t)}}{1+e^{\phi_m(t)}}\Lambda_m(t,x), & n>I \\ 0, & n=I \\ -\sum_{m=n+1}^{I}\frac{e^{\phi_m(t)}}{1+e^{\phi_m(t)}}\Lambda_m(t,x), & n<I. \end{cases} \quad (89)$$

The result for Libor drift obtained in Eq. (89) agrees exactly with the result obtained in Eq. (40) using the Wilson short distance expansion.

XVI. LIBOR MARKET MODEL: HAMILTONIAN, LAGRANGIAN, AND $\mathcal{A}(t,x)$

The Hamiltonian formulation of the martingale condition, given in Eq. (71), yields the Libor drift in a fairly transparent and direct manner compared to the rather roundabout approach adopted in Sec. IX. The summation that appears in the drift term is due to expressing the ratio $B(t,T_{n+1})/B(t,T_{I+1})$ as a product of Libor variables $L(t,T_k)$. The relative minus sign in the summation term of the drift for $n<I$ and $n>I$ arises from the ratio $B(t,T_{n+1})/B(t,T_{I+1})$ being the product of $1+\ell L(t,T_k)$ or of its inverse.

The derivation of Libor drift given in Sec. IX follows the general spirit of the BGM-Jamshidian approach—generalized to the case of quantum finance by the use of the

Wilson expansion. The martingale condition was first expressed in terms of the Libor forward interest rates $f(t,x)$; one then did a change of variables and re-expressed the drift in terms of the Libor variables. To carry out this change of variables for the quantum finance case, the Wilson expansion for the velocity quantum field $\mathcal{A}(t,x)$ was crucial in capturing the nontrivial correlation terms.

In contrast, in the Hamiltonian derivation of Libor drift, there is no need to employ the $\mathcal{A}(t,x)$ field and all the correlation effects are produced by the Hamiltonian. The Libor Hamiltonian is expressed directly in terms of log Libor variables $\phi(t,x)$, making no reference to $f(t,x)$. The martingale condition is expressed directly in terms of the Hamiltonian $\mathcal{H}$ and leads to an exact derivation of Libor drift. The fact that the Jacobian of the transformation from $\mathcal{A}(t,x)$ to $\phi(t,x)$ is a constant, as shown in Appendix F, is essential for obtaining the logarithmic Libor Hamiltonian; a nontrivial Jacobian would give rise to new terms.

The Hamiltonian derivation of Libor drift leads to some general conclusions in the context of the Libor market model. The extension of the martingale condition from the case for equity to that of interest rate instruments is nontrivial due to (a) the need to treat the discounting factor as being stochastic and (b) the time-dependent state space [2]. The derivation of the nontrivial and nonlinear drift of the Libor market model verifies the correctness of the martingale condition stated in Eq. (71).

The Hamiltonian derivation of Libor drift provides independent proof of the correctness of the earlier derivation of Libor drift in Sec. IX, which crucially hinged on the Wilson expansion. The Hamiltonian result shows that the Wilson short distance expansion for a Gaussian quantum field is the correct generalization of Ito's calculus and opens the way to applications of $\mathcal{A}(t,x)$ in theoretical finance.

XVII. SUMMARY

Quantum finance provides a natural and mathematically tractable formulation of the Libor market model. Gaussian white noise $R(t)$, in effect, is "promoted" by quantum finance to a quantum field $\mathcal{A}(t,x)$ that drives the time evolution of Libor. In particular, the Libor market model has been given three different and consistent formulations, namely, employing $\mathcal{A}(t,x)$, $\mathcal{L}[\phi]$, and $\mathcal{H}(t)$—thus displaying the versatility and flexibility of quantum finance.

In an economy where Libor rates are perfectly correlated across different maturities, a single volatility function is sufficient. In contrast, Libor term structure that is imperfectly correlated introduces many new features. In the quantum finance approach, due to the specific properties of Gaussian quantum fields, the entire Libor volatility function can be taken directly from the market and thus incorporates many key features of the market in a parameter free manner.

The Gaussian quantum field $\mathcal{A}(t,x)$ has a quadratic action and hence one can obtain a differential formulation of the Libor market model. The singular quadratic product $\mathcal{A}(t,x)$ has a "short distance" Wilson expansion that generalizes the results of Ito calculus and yields a derivation of Libor drift. It is seen that the underlying Libor forward interest rates $f(t,x)$ of the Libor market model are nonlinear and nonsingular.

The Libor forward interest rates $f(t,x)$ are related to logarithmic Libor by Eq. (2),

$$\exp\left\{\int_{T_n}^{T_n+\ell} dx f(t,x)\right\} = 1 + \exp\left\{\int_{T_n}^{T_n+\ell} dx \phi(t,x)\right\}. \tag{90}$$

The transformation from $\phi(t,x)$ to $f(t,x)$ is nonlinear and nonlocal and is well defined only for strictly positive $f(t,x)$, that is, for $f(t,x) \geq 0$. In particular, this means that $f(t,x)$ *cannot* be a Gaussian quantum field. The linear approximation that treats $f(t,x)$ as a Gaussian quantum field needs to be carefully examined to ascertain whether it can be applied to the interest rate (Libor and Euribor) market.

Note the transformation given in Eq. (90) is completely general and only requires that $f(t,x) \geq 0$; for example, one could take $f(t,x) \sim e^{\xi(t,x)}$, where $\xi(t,x)$ is any real quantum field and this yields strictly positive Libors. The Libor market model makes a very *specific* choice for $f(t,x)$, namely, one for which the logarithmic Libor quantum field $\phi(t,x)$ has deterministic volatility given $\gamma(t,x)$. This choice for $f(t,x)$ leads to the field $\phi(t,x)$ being a "flat" quantum field that takes values on the entire real line. The leading order effects of $\phi(t,x)$ are described by a Gaussian quantum field; the nonlinear terms are all contained in the drift and can be treated perturbatively using Feynman diagrams. Logarithmic Libor variables $\phi(t,x)$ yield strictly positive coupon bonds and Libors. A nonlinear quantum field theory for flat quantum field $\phi(t,x)$ is the most appropriate formalism for simultaneously analyzing coupon bond and Libor instruments.

The Hamiltonian realization of the martingale evolution of an instrument entails that the instrument be annihilated by the Hamiltonian. The Hamiltonian is the appropriate framework for imposing the martingale condition for nonlinear interest rates and, in particular, yields the exact expression for the Libor market model's nonlinear drift.

A new feature of the interest rate dynamics is that, unlike the case for equity, the interest rates' state space and Hamiltonian are time dependent; this leads to a number of new features for the Hamiltonian and, in particular, that the evolution operator is defined by a time ordered product. Choosing the forward bond numeraire leads to a major simplification: the interest rates' state space becomes equivalent to a fixed state space and one can dispense with time ordering, leading to a time independent evolution operator.

ACKNOWLEDGMENTS

I thank Cui Liang for many earlier discussions and for emphasizing the importance of the Libor market model. I thank Tang Pan and Cao Yang for useful discussions.

APPENDIX A: $\mathcal{A}(t,x)$'S GENERATING FUNCTIONAL

One is free to choose the dynamics of the quantum field $\mathcal{A}(t,x)$. Following Baaquie and Bouchaud [16], the Lagrangian that describes the evolution of instantaneous forward rates is defined by three parameters μ, λ, η and is given by

$$\mathcal{L}(A) = -\frac{1}{2}\left\{\mathcal{A}^2(t,z) + \frac{1}{\mu^2}\left(\frac{\partial\mathcal{A}(t,z)}{\partial z}\right)^2 + \frac{1}{\lambda^4}\left(\frac{\partial^2\mathcal{A}(t,z)}{\partial^2 z}\right)^2\right\} \times \left|\frac{\partial\mathcal{A}(t,z)}{\partial z}\right|_{z=0} = 0, \tag{A1}$$

where market (psychological) future time is defined by $z = (x-t)^{\nu}$.

The action $S[\mathcal{A}]$ of the Lagrangian is defined as

$$S[\mathcal{A}] = \int_{t_0}^{\infty} dt \int_0^{\infty} dz dz' \mathcal{L}(\mathcal{A}). \tag{A2}$$

The market value of all interest rates' financial instruments is obtained by performing a path integral over the (fluctuating) two-dimensional quantum field $\mathcal{A}(t,z)$. The expectation value of an instrument $F[\mathcal{A}]$, denoted by $E\{F[\mathcal{A}]\}$, is defined by the functional average over all values of $\mathcal{A}(t,z)$, weighted by the probability measure e^S/Z. Hence

$$E\{F[\mathcal{A}]\} \equiv \frac{1}{Z}\int D\mathcal{A}F[\mathcal{A}]e^{S[\mathcal{A}]}, \quad Z = \int D\mathcal{A}e^{S[\mathcal{A}]}. \tag{A3}$$

The quantum field theory of the forward interest rates is defined by the generating function given by

$$Z[h] = E\left[\exp\left\{\int_{t_0}^{\infty} dt \int_0^{\infty} dz h(t,z)\mathcal{A}(t,z)\right\}\right] \equiv \frac{1}{Z}\int D\mathcal{A}\exp\left\{S[\mathcal{A}] + \int_{t_0}^{\infty} dt \int_0^{\infty} dz h(t,z)\mathcal{A}(t,z)\right\} \tag{A4}$$

$$= \exp\left\{\frac{1}{2}\int_{t_0}^{\infty} dt \int_0^{\infty} dz dz' h(t,z)D(z,z';t)h(t,z')\right\}. \tag{A5}$$

Hence the correlator of the $\mathcal{A}(t,x)$ quantum field is given by

$$E[\mathcal{A}(t,z)] = 0,$$

$$E[\mathcal{A}(t,z)\mathcal{A}(t',z')] = \delta(t-t')D(z,z';t). \tag{A6}$$

For notational simplicity only the case of $\nu=1$ will be considered; all integrations over z are replaced with those over future time x. For $\nu=1$ from Eq. (A2) the dimension of the quantum field $\mathcal{A}(t,x)$ is 1/time and the volatility $\sigma(t,x)$ of the forward interest rates also has dimension of 1/time.

The expression for $D(x,x';t)$ has been studied in [16,17] and provides a very accurate description of the correlation of the forward interest rates. In the present paper the explicit value of the propagator $D(x,x';t)$ is not required.

APPENDIX B: TIME EVOLUTION OF A BOND

To illustrate the content of the singularity in the equal time quadratic product of the quantum field, namely, $\mathcal{A}(t,x)\mathcal{A}(t,x')$ given in Eq. (17), a concrete analysis is carried out of the evolution of zero coupon bond—with and without discounting.

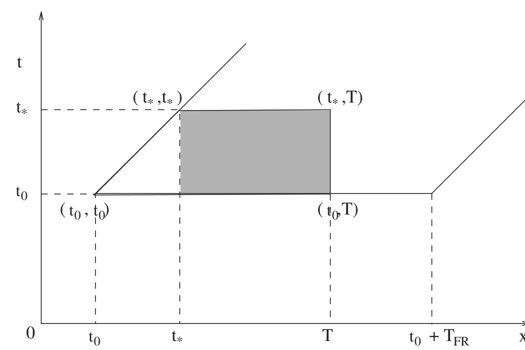

FIG. 12. The domain of the forward interest rates $f(t,x)$, in calendar and future time, required for evaluating the time evolution of a zero coupon bond.

Consider for simplicity the bond forward interest rates given by

$$\frac{\partial f_B(t,x)}{\partial t} = \alpha(t,x) + \sigma(t,x)\mathcal{A}(t,x), \tag{B1}$$

$$f_B(t,x) = f_B(t_0,x) + \int_{t_0}^{t} dt\alpha(t,x) + \int_{t_0}^{t} dt\sigma(t,x)\mathcal{A}(t,x), \tag{B2}$$

where the drift $\alpha(t,x)$ and volatility $\sigma(t,x)$ are both deterministic functions.

From Eq. (1), a zero coupon and its forward bond are given by

$$B(t_*,T) = \exp\left\{-\int_{t_*}^{T} dx f_B(t_*,x)\right\},$$

$$F(t_0,t_*,T) = \exp\left\{-\int_{t_*}^{T} dx f_B(t_0,x)\right\},$$

$$B(t_*,T) = F(t_0,t_*,T)\exp\left\{-\int_{t_0}^{t_*} dt \int_{t_*}^{T} dx\alpha(t,x)\right\} \times \exp\left\{-\int_{t_0}^{t_*} dt \int_{t_*}^{T} dx\sigma(t,x)\mathcal{A}(t,x)\right\}. \tag{B3}$$

The domain of integration is a rectangle $\mathcal{R}$, equal to $[t_0,t_*] \times [t_*,T]$ and shown in Fig. 12.

To calculate the time evolution of the bond one needs to compute the time derivative of $\exp\{-\int_{t_0}^{t_*}dt\int_{t_*}^{T}dx\sigma(t,x)\mathcal{A}(t,x)\}$; for doing this computation one needs to take account of the fact that the quadratic power of $\mathcal{A}(t,x)$ is singular. Consider [24]

$$\exp\left\{+\int_{t_0}^{t_*}dt\int_{t_*}^{T}dx\sigma(t,x)\mathcal{A}(t,x)\right\}$$
$$\times\frac{\partial}{\partial t_*}\exp\left\{-\int_{t_0}^{t_*}dt\int_{t_*}^{T}dx\sigma(t,x)\mathcal{A}(t,x)\right\}$$
$$=\frac{1}{\epsilon}\left[\exp\left\{-\epsilon\int_{t_*}^{T}dx\sigma(t_*,x)\mathcal{A}(t_*,x)\right\}\right.$$
$$\left.\times\exp\left\{+\epsilon\int_{t_0}^{t_*}dt\sigma(t,t_*)\mathcal{A}(t,t_*)\right\}-1\right].$$

Expanding the exponential to second order yields all the nontrivial terms as follows [25]:

$$\frac{1}{\epsilon}\left[\exp\left\{-\epsilon\int_{t_*}^{T}dx\sigma(t_*,x)\mathcal{A}(t_*,x)\right\}\right.$$
$$\left.\times\exp\left\{+\epsilon\int_{t_0}^{t_*}dt\sigma(t,t_*)\mathcal{A}(t,t_*)\right\}-1\right]$$
$$=-\int_{t_*}^{T}dx\sigma(t_*,x)\mathcal{A}(t_*,x)+\int_{t_0}^{t_*}dt\sigma(t,t_*)\mathcal{A}(t,t_*)$$
$$+\frac{\epsilon}{2}\left(\int_{t_*}^{T}dx\sigma(t_*,x)\mathcal{A}(t_*,x)\right)^2+O(\epsilon)$$
$$=-\int_{t_*}^{T}dx\sigma(t_*,x)\mathcal{A}(t_*,x)+\int_{t_0}^{t_*}dt\sigma(t,t_*)\mathcal{A}(t,t_*)$$
$$+\frac{1}{2}\int_{t_*}^{T}dx\int_{t_*}^{T}dx'\mathcal{M}_{\sigma}(x,x';t_*),$$

where, from Eq. (17),

$$\left(\int_{t_*}^{T}dx\sigma(t_*,x)\mathcal{A}(t_*,x)\right)^2$$
$$=\frac{1}{\epsilon}\int_{t_*}^{T}dx\int_{t_*}^{T}dx'\mathcal{M}_{\sigma}(x,x';t_*)\mathcal{M}_{\sigma}(x,x';t_*)$$
$$\equiv\sigma(t_*,x)\sigma(t_*,x')\mathcal{D}(x,x';t_*).$$

Collecting all the results yields

$$\exp\left\{+\int_{t_0}^{t_*}dt\int_{t_*}^{T}dx\sigma(t,x)\mathcal{A}(t,x)\right\}$$
$$\times\frac{\partial}{\partial t_*}\exp\left\{-\int_{t_0}^{t_*}dt\int_{t_*}^{T}dx\sigma(t,x)\mathcal{A}(t,x)\right\}$$
$$=-\int_{t_*}^{T}dx\sigma(t_*,x)\mathcal{A}(t_*,x)+\int_{t_0}^{t_*}dt\sigma(t,t_*)\mathcal{A}(t,t_*)$$
$$+\frac{1}{2}\int_{t_*}^{T}dx\int_{t_*}^{T}dx'\mathcal{M}_{\sigma}(x,x';t_*).$$

The only term in the zero coupon bond that needs to be examined carefully is the one involving $\mathcal{A}(t,x)$; the other terms all obey the usual rules of calculus. Hence, since $\partial F(t_0,t_*,T)/\partial t_*=+f(t_0,t_*)F(t_0,t_*,T)$ and using Eq. (B1), one obtains from Eq. (B3)

$$\frac{1}{B(t_*,T)}\frac{\partial B(t_*,T)}{\partial t_*}=f(t_*,t_*)-\int_{t_*}^{T}dx\alpha(t_*,x)$$
$$-\int_{t_*}^{T}dx\sigma(t_*,x)\mathcal{A}(t_*,x)$$
$$+\frac{1}{2}\int_{t_*}^{T}dx\int_{t_*}^{T}dx'\mathcal{M}(x,x';t_*).\quad\text{(B4)}$$

The martingale condition on the drift $\alpha(t,x)$ for a numeraire using discounting by the money market account, namely, by $\exp\{-\int_{t_0}^{t_*}dtr(t)\}$, where $r(t)=f(t,t)$ is the spot interest rate, is given by

$$\alpha(t,x)=\sigma(t,x)\int_{t}^{x}dx'\mathcal{D}(x,x';t)\sigma(t,x').$$

This yields

$$\frac{\partial B(t_*,T)}{\partial t_*}=\left[f(t_*,t_*)-\int_{t_*}^{T}dx\sigma(t_*,x)\mathcal{A}(t_*,x)\right]B(t_*,T).$$

Similarly, the time derivative of a discounted bond obeys a martingale evolution. The *discounted* zero coupon bond is given by

$$D(t_*,T)\equiv\exp\left\{-\int_{t_0}^{t_*}dtf(t,t)\right\}B(t_*,T)$$

and a derivation similar to the result obtained in Eq. (B4) yields

$$\frac{\partial D(t_*,T)}{\partial t_*}=-\left[\int_{t_*}^{T}dx\sigma(t_*,x)\mathcal{A}(t_*,x)\right]D(t_*,T).$$

From Eq. (C9), it is seen that the discounted bond follows a martingale process.

APPENDIX C: MARTINGALE

A martingale refers to a special category of stochastic processes. An arbitrary discrete stochastic process is a collection of random variables X_i, $i=1,2,\ldots,N$ that is described by a joint probability distribution function $p(x_1,x_2,\ldots,x_N)$. The stochastic process is a martingale if it satisfies

$$E[X_{n+1}|x_1,x_2,\ldots,x_n]=x_n\quad\text{(martingale)}.\quad\text{(C1)}$$

In other words, the expected value of the random variable X_{n+1}—conditioned on the occurrence of x_i for random variables X_i, $i=1,2,\ldots,n$—is simply x_n itself.

Consider a general stochastic functional differential equation for a two-dimensional function $\chi(t,x)$,

$$\frac{\partial\chi(t,x)}{\partial t}=d(t,x)+\int dx'G(x,x';t)v(t,x')\mathcal{A}(t,x'),\quad\text{(C2)}$$

where $\mathcal{A}(t,x)$ is the Gaussian two-dimensional quantum field defined by the action given in Eq. (12). $G(x,x';t)$ is a deterministic function and the quantities $d(t,x)$ and $v(t,x)$ can, in general, depend on $\chi(t,x)$. Equation (C2) is a stochastic differential equation that is encountered in the study of Libor.

An initial (or final) condition needs to be specified to obtain a solution for Eq. (C2); for applications in finance, the initial condition is specified as follows:

$$\text{boundary condition:}\chi(t_0,x) = \text{fixed}.$$

Discretizing time in Eq. (C2) yields for infinitesimal ϵ

$$\chi(t+\epsilon,x) = \chi(t,x) + \epsilon d(t,x) + \epsilon\int dx' G(x,x';t)v(t,x')\mathcal{A}(t,x'). \quad \text{(C3)}$$

The martingale condition given in Eq. (C1) requires that the expectation value of $\chi(t+\epsilon,x)$ is taken conditioned on $\chi(t,x)$ having a *fixed* value. In taking the expectation value of Eq. (C3), the functions $d(t,x)$ and $v(t,x)$ are deterministic since they depend only on $\chi(t,x)$ and hence can be taken outside the expectation value. Since $E[\mathcal{A}(t,x')]=0$, taking the conditional expectation value of both sides of Eq. (C3) yields the following:

$$E[\chi(t+\epsilon,x)|\chi(t,x)] = \chi(t,x) + \epsilon d(t,x) + \epsilon\int dx' G(x,x';t)v(t,x')E[\mathcal{A}(t,x')] = \chi(t,x) + \epsilon d(t,x). \quad \text{(C4)}$$

The martingale condition given in Eq. (C1) requires $E[\chi(t+\epsilon,x)|\chi(t,x)]=\chi(t,x)$; hence, from Eqs. (C1) and (C4)

$$d(t,x) = 0 \quad \text{(martingale condition)}. \quad \text{(C5)}$$

Hence, the unconditional expectation value is given by

$$E[\chi(t+\epsilon,x)] = E[\chi(t,x)] \Rightarrow E\left[\frac{\partial\chi(t,x)}{\partial t}\right] = 0 \quad \text{(martingale)}. \quad \text{(C6)}$$

In summary, for $\chi(t,x)$ to be a martingale, it is sufficient that

$$\frac{\partial\chi(t,x)}{\partial t} = \int dx' G(x,x';t)v(t,x')\mathcal{A}(t,x'). \quad \text{(C7)}$$

The martingale condition can be further generalized. The stochastic evolution equation

$$\frac{1}{\xi(t,x)}\frac{\partial\xi(t,x)}{\partial t} = \int dx' G(x,x';t)v(t,x')\mathcal{A}(t,x') \quad \text{(C8)}$$

yields the following conditional expectation value:

$$E[\xi(t+\epsilon,x)] = \xi(t,x) \quad \text{(martingale)}, \quad \text{(C9)}$$

which is the martingale condition.

APPENDIX D: LIMITS OF THE LIBOR MARKET MODEL

The following three different limits of the Libor market model are taken:

(i) the limit of zero tenor, namely, $\ell \to 0$,

(ii) the BGM-Jamshidian limit, and

(iii) the HJM limit.

1. Tenor $\ell \to 0$

Consider the limit of the tenor $\ell \to 0$. Let $x=T_n$, $x'=T_m$, and $x,x' > T_I$. From Eq. (28), $v(t,x) \simeq \ell f(t,x)\gamma(t,x)$; hence, from Eq. (69)

$$x \in [T_n,T_{n+1}], \quad L(t,T_n) \sim f(t,x) + O(\ell),$$

$$e^{\phi(t)} \sim \ell f(t,x), \quad 1 - e^{-f_n(t)} \simeq \ell f(t,x),$$

$$\Lambda_n \simeq \ell M_\gamma(x,x;t) = \ell\,\gamma(t,x)\mathcal{D}(x,x;t)\gamma(t,x),$$

$$\mu(t,x) = \sum_{j=0}^{\infty} H_j(x)[1 - e^{-f_j(t)}]\left[-\frac{1}{2}\Lambda_j(t,x) + \rho_j(t,x)\right] \simeq [\ell f(t,x)]\left[-\frac{\ell}{2}\gamma(t,x)\mathcal{D}(x,x;t)\gamma(t,x) + \sum_{m=I+1}^{n}\frac{\ell f(t,T_m)}{1+\ell f(t,x)}\ell\,\gamma(t,x)\mathcal{D}(x,T_m;t)\gamma(t,T_m)\right].$$

The sum $\Sigma_j H_j(x)$ has collapsed to one term since $x \in [T_n,T_{n+1})$. The limit $\ell \to 0$ is taken holding $v(t,x) = \ell f(t,x)\gamma(t,x)$ fixed; hence, in this limit

$$\mu(t,x) \simeq -\frac{\ell}{2}[\ell f(t,x)\gamma(t,x)]\mathcal{D}(x,x;t)\gamma(t,x) + \ell f(t,x)\ell^2\sum_{m=I+1}^{n}[f(t,T_m)\gamma(t,x)\mathcal{D}(x,T_m;t)\gamma(t,T_m)] \simeq v(t,x)\int_{T_I}^{x} dx'\mathcal{D}(x,x';t)v(t,x') + O(\ell). \quad \text{(D1)}$$

Hence, from Eq. (9)

$$\frac{\partial f(t,x)}{\partial t} \simeq \ell^2 f(t,x)\int_{T_I}^{x} dx'\mathcal{D}(x,x';t)f(t,x') + \ell^2 f(^2t,x)\mathcal{A}(t,x). \quad \text{(D2)}$$

The limit of tenor $\ell \to 0$ yields an evolution equation for the Libor forward interest rates $f(t,x)$ that looks similar to the HJM model, except $v(t,x)=\ell f(t,x)\gamma(t,x)$ is stochastic. Recall the drift results from requiring a martingale evolution of the Libor forward interest rates $f(t,x)$ with the forward bond numeraire being the zero coupon bond $B(t,T_I)$ with T_I fixed.

2. BGM-Jamshidian limit

The BGM-Jamshidian limit of the quantum finance formulation of Libor market model can be obtained using the following prescription:

$$\mathcal{D}(x,x';t)|_{\text{BGM}} \to 1,$$

$$\mathcal{A}(t,x)|_{\text{BGM}} \to R(t),$$

$$E[R(t)R(t')] = \delta(t-t'), \tag{D3}$$

where $R(t)$ is Gaussian white noise.

The Libor evolution for $T_I < T_n$, given in Eq. (26), yields the BGM-Jamshidian limit. From Eq. (39)

$$\Lambda_{mn}(t) = \int_{T_m}^{T_{m+1}} dx\,\gamma(t,x)\int_{T_n}^{T_{n+1}} dx'\mathcal{D}(x,x';t)\gamma(t,x')$$

$$\to \gamma_n(t)\gamma_n(t), \quad \text{where} \quad \gamma_n(t) = \int_{T_n}^{T_{n+1}} dx\,\gamma(t,x).$$

Hence, the BGM-Jamshidian limit of the Libor market model evolution equation, from Eq. (26), (53), and (D3), is given by

$$\frac{1}{L(t,T_n)}\frac{\partial L(t,T_n)}{\partial t} = \zeta_n(t) + \gamma_n(t)R(t),$$

$$\zeta_n(t) = \gamma_n(t)\sum_{m=I+1}^{n}\frac{\ell L(t,T_m)}{1+\ell L(t,T_m)}\gamma_m(t). \tag{D4}$$

3. HJM limit

The HJM limit requires the following three conditions:

(i) the zero tenor limit $\ell \to 0$ is taken,

(ii) it assumed that $v(t,x) = \ell f(t,x)\gamma(t,x)$ is a deterministic function equal to the HJM-volatility function, that is, $v(t,x) \to \sigma(t,x)$, the HJM bond forward interest rate volatility is independent of $f(t,x)$, and

(iii) $\mathcal{D}(x,x';t)|_{\text{HJM}} \to 1$.

With these assumptions Eq. (D2) yields

$$\frac{\partial f(t,x)}{\partial t} = \sigma(t,x)\int_{T_I}^{x} dx'\sigma(t,x') + \sigma(t,x)R(t),$$

which is the expected HJM equation; note that the drift is the one appropriate for the forward bond numeraire $B(t,T_I)$.

APPENDIX E: NONSINGULAR LIBOR FORWARD INTEREST RATES $f(t,x)$

The underlying Libor forward interest rates driving all the Libors, from Eq. (9), are given by the following:

$$\frac{\partial f(t,x)}{\partial t} = \mu(t,x) + v(t,x)\mathcal{A}(t,x).$$

The theory is nonlinear due to the dependence of the volatility $v(t,x)$ and drift $\mu(t,x)$ on the underlying Libor forward interest rates $f(t,x)$. A nonlinear drift that renders the time evolution of Libor to be a martingale apparently implies that the underlying Libor forward interest rates are singular [19]. This aspect of the Libor forward interest rates is analyzed.

An approximation of Eq. (70) that is adequate for the analysis of this appendix is

$$v(t,x) \simeq [1-e^{-\ell f(t,x)}]\gamma(t,x).$$

Recall from Eq. (54) that volatility $v(t,x)$ has the following two limiting cases:

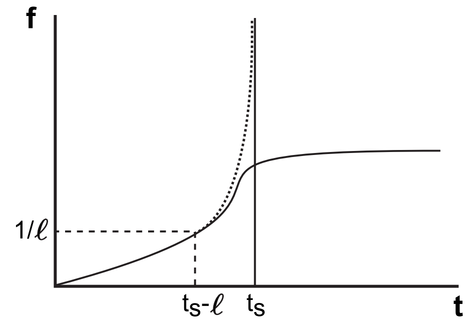

FIG. 13. Time evolution of $f(t,x)$. The singularity at time t_s is spurious since nonlinear effects take over at time $t_s - \ell$ leading to a finite $f(t,x)$ for all time.

$$v(t,x) = \begin{cases} \ell\gamma(t,x)f(t,x), & \ell f(t,x) \ll 1 \\ \gamma(t,x), & \ell f(t,x) \gg 1. \end{cases} \tag{E1}$$

Here $v(t,x) = \ell f(t,x)\gamma(t,x) \sim 0$, ignoring the integration that does not qualitatively change the results. Recall from Eq. (22) that the drift is

$$\mu(t,x) = v(t,x)\int_{T_{I+1}}^{x} dx'\mathcal{D}(x,x';t)v(t,x') + O(\ell).$$

The limiting cases for $\mu(t,x)$, from Eq. (E1), are the following [26]:

For case (i) $\ell f(t,x) \ll 1$,

$$\mu(t,x) \simeq [\ell f(t,x)\gamma(t,x)]^2 + O(\ell). \tag{E2}$$

For case (ii) $\ell f(t,x) \gg 1$,

$$\mu(t,x) \simeq -\frac{1}{2}\int_{T_n}^{T_{n+1}} dx' M_\gamma(x,x';t) + \int_{T_{I+1}}^{T_{n+1}} dx' M_\gamma(x,x';t).$$

In the limit $\ell f(t,x) \gg 1$, volatility $v(t,x) \to \gamma(t,x)$. The results obtained in Eqs. (E2) and (E1) for this limit are consistent with the earlier result given in Eq. (22); the two results agree only in the limit of $\ell \to 0$. For the case when $\ell f(t,x)\gamma(t,x)$ is independent of ℓ, the extra term $\int_{T_n}^{T_{n+1}} M_\gamma(x,x';t)/2$ in Eq. (E2) is of $O(\ell)$ and goes to zero.

For small values of $\ell f(t,x)$, $\mu(t,x) \simeq [\ell\gamma(t,x)f(t,x)]^2$ follows from Eq. (E2). The stochastic term can be ignored as these do not qualitatively change the impact of the quadratic term on the evolution of $f(t,x)$. The simplified dynamics for the Libor forward interest rates, from Eq. (9), is the following [27]:

$$\frac{\partial f(t,x)}{\partial t} \simeq \ell^2\gamma^2 f^2(t,x) + \text{random terms} \Rightarrow f(t,x)$$

$$\simeq \ell\gamma\frac{f(0,x)}{1-[\ell\gamma f(0,x)]t} + \text{random terms} \Rightarrow f(t,x)$$

$$\to \infty \quad \text{for} \quad t_s = \frac{1}{\ell\gamma f(0,x)}, \quad f(0,x) > 0. \tag{E3}$$

From Eq. (E3), all the Libor forward interest rates seem to become infinite as $t \to t_s = 1/[\ell\gamma f(0,x)] > 0$ and are shown as the dashed line in Fig. 13. If $f(t,x)$, in fact, becomes singular

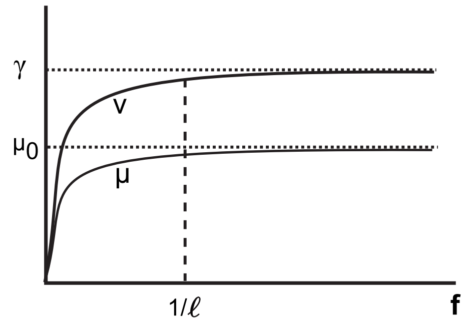

FIG. 14. The behavior of volatility $v(t,x)$ and drift $\mu(t,x)$ as a function of $f(t,x)$.

without the stochastic term, then, on including the stochastic term, the singularity is even more severe with the Libor forward interest rates becoming singular almost instantaneously [19]. However, the result that the forward interest rates become singular is not correct. When $\ell f(t,x) \sim 1$, the approximation leading to Eq. (E3) is no longer valid and the solution breaks down for t_f given by

$$f(t,x) \simeq \frac{1}{\ell} = \ell\gamma \frac{f(0,x)}{1-[\ell\gamma f(0,x)]t_f} \Rightarrow t_f = t_s - \ell < t_s. \tag{E4}$$

Instead, as follows from Eqs. (E2) and (E1), for $\ell f(t,x) \sim 1$ the volatility and drift both become deterministic, leading to a finite evolution of $f(t,x)$. Ignoring the stochastic term yields

$$f(t,x) \sim f(0,x)e^{\mu_0 t}, \quad t > t_f.$$

The different domains for $f(t,x)$ are shown in Fig. 13; $f(t,x)$ grows slowly for $t > t_s$ since the coefficient $\mu_0 \sim \gamma^2 \ll 1$.

Recall that the drift $\mu(t,x)$ and volatility $v(t,x)$ are both functions of only $\exp\{-f_n(t)\} \simeq \exp\{-\ell f(t,T_n)\}$. As $f(t,x)$ grows large, both $\mu(t,x)$ and $v(t,x)$ rapidly become deterministic and independent of $f(t,x)$, as shown in Fig. 14. This in turn means that $f(t,x)$ can be described by a linear Gaussian quantum field. Gaussian fields have configurations where $f(t,x)$ takes small values and can, hence, revert back to the regime for its nonlinear evolution. In this manner, the Libor forward interest rates are driven by its exact evolution equation between the nonlinear and linear domains.

APPENDIX F: JACOBIAN OF TRANSFORMATION $\mathcal{A}(t,x) \to \phi(t,x)$

The change of variables from quantum field $\mathcal{A}(t,x)$ to $\phi(t,x)$ is, from Eq. (56), given by

$$\frac{\partial \phi(t,x)}{\partial t} = \tilde{\rho}(t,x) + \gamma(t,x)\mathcal{A}(t,x), \quad t \in [t_0,t_*]. \tag{F1}$$

Equation (F1) is a nonlinear change of variables since drift $\tilde{\rho}(t,x)$ depends on $\phi(t,x)$ and, in principle, can have a nontrivial Jacobian. Taking the differential of Eq. (F1) yields

$$\int_{t_0}^{t_*} dt' \left[\delta(t'-t)\frac{\partial}{\partial t'} - \frac{\delta\tilde{\rho}(t,x)}{\delta\phi(t',x)} \right] d\phi(t',x) = \gamma(t,x) d\mathcal{A}(t,x)$$

$$\Rightarrow \det[\mathcal{J}]D\phi = \text{const} \times D\mathcal{A}.$$

The change of variables factorizes for the x variable; hence, for notational simplicity, the x coordinate is suppressed and only the time variable t is displayed. The Jacobian is equal to $\det[\mathcal{J}]$, where matrix of transformation $\mathcal{J}$ is given by

$$\mathcal{J}(t',t) = \delta(t'-t)\frac{\partial}{\partial t} - \frac{\delta\tilde{\rho}(t',x)}{\delta\phi(t,x)} \equiv \delta(t'-t)\frac{\partial}{\partial t} - J(t',t) \Rightarrow \mathcal{J} = \mathcal{U}\frac{\partial}{\partial t}\mathcal{U}^{-1}, \quad J(t',t) = \frac{\delta\tilde{\rho}(t',x)}{\delta\phi(t,x)}. \tag{F2}$$

In Eq. (F2) matrix multiplication is an integration over t given by $\int_{t_0}^{t_*} dt$. Time is discretized $t \to t_n = t' + (n-1)\epsilon$, $n = 1,2,3,\ldots,M = (t-t')/\epsilon$ to explicitly write out the matrix elements of $\mathcal{U}$ as follows:

$$\mathcal{U}(t',t) = \left[\prod_{n=1}^{M-1} \int_{t_0}^{t_*} dt_n\right] \prod_{n=0}^{M-1} \exp\{\epsilon J(t_n,t_{n+1})\},$$

$$\text{boundary conditions}: t_1 = t', \quad t_M = t.$$

From Eq. (F2) the Jacobian is given by

$$\det[\mathcal{J}] = \det\left[\mathcal{U}\frac{\partial}{\partial t}\mathcal{U}^{-1}\right] = \det\left[\delta(t'-t)\frac{\partial}{\partial t}\right] = \text{const}.$$

Hence, the Jacobian of the transformation in going from $\mathcal{A}(t,x) \to \phi(t,x)$ is a constant; all constants involved in going from $\mathcal{A}(t,x) \to \phi(t,x)$ cancel due to division by the partition function Z. Henceforth, the path integration measure will taken to be invariant, namely,

$$\int D\phi = \int D\mathcal{A}.$$

The Jacobian being a constant is essential for the derivation of the Libor Hamiltonian given in Sec. XV.

APPENDIX G: INTEREST RATE STATE SPACE $\mathcal{V}_t$

The Hamiltonian and the state space are two independent ingredients of a quantum system; taken together they reproduce the Lagrangian and the path integral. The essential features of the interest rates' Hamiltonian and state space are reviewed; a detailed discussion is given in [2].

The state space of a quantum field theory at time t, similar to all quantum systems, is a linear vector space—denoted by $\mathcal{V}_t$. The dual space of $\mathcal{V}_t$—denoted by $\mathcal{V}_{t,\text{Dual}}$—consists of all linear mappings from $\mathcal{V}_t$ to the complex numbers and is also a linear vector space. The Hamiltonian $\mathcal{H}_t$ is an operator—the quantum generalization of energy—that is an element of the tensor product space $\mathcal{V}_t \otimes \mathcal{V}_{t,\text{dual}}$ and maps the state space to itself, that is, $\mathcal{H}_t: \mathcal{V}_t \to \mathcal{V}_t$.

For each time slice, the state space is defined for interest rates with $x > t$, as shown in Fig. 15. The state space has a nontrivial structure due to the underlying trapezoidal domain

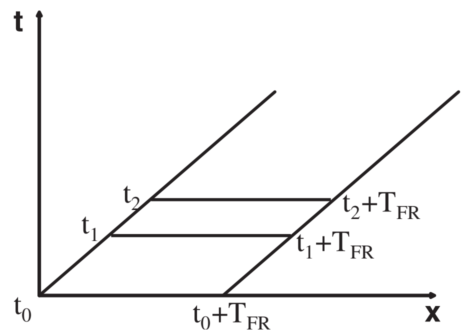

FIG. 15. The domain of the state space of the interest rates. In the figure the state space $\mathcal{V}_t$ is indicated for two distinct calendar times t_1 and t_2.

$\mathcal{T}$ of the xt space. On composing the state space for each time slice, the trapezoidal structure for finite time, as shown in Fig. 16, is seen to emerge from the state space defined for each time slice.

The Hamiltonian for log Libor $\phi(t,x)$ is nonlinear and defined on a nontrivial domain. Since the quantum field $\phi(t,x)$ exists only for future time, that is, for $x>t$ and hence $x\in[t,t+T_{FR}]$. In particular, the interest rates' quantum field has a *distinct* state space $\mathcal{V}_t$ for every instant t.

The state space at time t is labeled by $\mathcal{V}_t$, and the coordinate eigenvector of $\mathcal{V}_t$ is denoted by $|\phi_t\rangle$. For fixed time t, the state space $\mathcal{V}_t$ consists of all possible functions of the interest rates, with future time $x\in[t,t+T_{FR}]$. The elements of the state space of the forward interest rates $\mathcal{V}_t$ include *all* possible debt instruments that are traded in the market at time t. In continuum notation, the coordinate basis eigenstates of $\mathcal{V}_t$ are tensor products over the future time x,

$$|\phi_t\rangle=\prod_{t\le x\le t+T_{FR}}|\phi(t,x)\rangle,$$

and satisfy the following completeness equation:

$$\mathcal{I}_t=\prod_{t\le x\le t+T_{FR}}\int_{-\infty}^{+\infty}d\phi(t,x)|\phi_t\rangle\langle\phi_t|\equiv\int D\phi_t|\phi_t\rangle\langle\phi_t|. \tag{G1}$$

Figure 15 shows the domain of the state space as a function of time, shown for typical times t_1 and t_2.

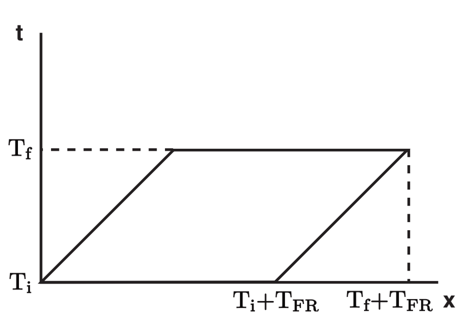

FIG. 16. The trapezoidal domain $\mathcal{T}$ of the forward interest rates required for computing the transition amplitude $\langle\phi_{\text{initial}}|\mathcal{T}\{\exp -\int_{T_i}^{T_f}\mathcal{H}(t)dt\}|\phi_{\text{final}}\rangle$.

The time-dependent interest rate Hamiltonian $\mathcal{H}(t)$ is the backward Fokker-Planck Hamiltonian and propagates the interest rates *backward* in time—taking the final state $|\phi_{\text{final}}\rangle$ given at future calendar time T_f backward to an initial state $\langle\phi_{\text{initial}}|$ at the earlier time T_i.

The transition amplitude Z for a time interval $[T_i,T_f]$ can be constructed from the Hamiltonian and state space by applying the time slicing method; since both the state space and Hamiltonian are time dependent one has to use the time-ordering operator $\mathcal{T}$ to keep track of the time dependence: $\mathcal{H}(t)$ for earlier time t is placed to the left of $\mathcal{H}(t)$ that refers to later time. The transition amplitude between a final (coordinate basis) state $|\phi_{\text{final}}\rangle$ at time T_f to an arbitrary initial (coordinate basis) state $\langle\phi_{\text{initial}}|$ at time T_i is given by the following [2]:

$$Z=\langle\phi_{\text{initial}}|\mathcal{T}\left\{\exp-\int_{T_i}^{T_f}\mathcal{H}(t)dt\right\}|\phi_{\text{final}}\rangle. \tag{G2}$$

Due to the time dependence of the state spaces $\mathcal{V}_t$ the forward interest rates that determine Z form a trapezoidal domain shown in Fig. 16.

APPENDIX H: INTEREST RATE HAMILTONIAN

Consider the Lagrangian density $\mathcal{L}(t,x)$ for logarithmic Libor field $\phi(t,x)$ given by Eq. (60),

$$\mathcal{L}(t,x)=-\frac{1}{2}\left[\frac{\partial\phi(t,x)/\partial t-\tilde{\rho}(t,x)}{\gamma(t,x)}\right]\times\mathcal{D}^{-1}(t,x,x')\times\left[\frac{\partial\phi(t,x')/\partial t-\tilde{\rho}(t,x')}{\gamma(t,x')}\right]-\infty\le\phi(t,x)\le+\infty.$$

The volatility $\gamma(t,x)$ is deterministic and $\tilde{\rho}(t,x)$ is a nonlinear drift term defined in Eq. (59). Neumann boundary condition, given in Eq. (61), has been incorporated into the expression for the Lagrangian. The derivation for the Hamiltonian is done for an arbitrary propagator $\mathcal{D}^{-1}(t,x,x')$, although for most applications a specific choice, such as the stiff propagator, is made.

Discretizing time into a lattice of spacing ϵ, with $t\to t_n=n\epsilon$, yields the Lagrangian $\mathcal{L}(t_n)$,

$$\mathcal{L}(t_n)\equiv\int_{t_n}^{t_n+T_{FR}}dx\mathcal{L}(t_n,x),\quad\int_x\equiv\int_{t_n}^{t_n+T_{FR}}dx=-\frac{1}{2\epsilon^2}\int_x\mathcal{A}(t_n,x)\mathcal{D}^{-1}(t,x,x')\mathcal{A}(t_n,x), \tag{H1}$$

$$\mathcal{A}(t_n,x)=\frac{(\phi_{t_n+\epsilon}-\phi_{t_n}-\epsilon\tilde{\rho}_{t_n})(x)}{\gamma(t_n,x)}. \tag{H2}$$

Note $\phi(t_n,x)\equiv\phi_{t_n}(x)$ has been written to emphasize that time t_n is a parameter for the interest rate Hamiltonian.

The Dirac-Feynman formula relates the Lagrangian $\mathcal{L}(t_n)$ to the Hamiltonian operator and yields

$$\langle\phi_{t_n}|e^{-\epsilon\mathcal{H}_f}|\phi_{t_n+\epsilon}\rangle=\mathcal{N}e^{\epsilon\mathcal{L}(t_n)}, \tag{H3}$$

where $\mathcal{N}$ is a normalization. Equation H(H3) is rewritten using Gaussian integration and (ignoring henceforth irrelevant constants), using notation $\Pi_x \int dp(x) \equiv \int Dp$ yields

$$e^{\epsilon\mathcal{L}(t_n)} = \int Dp \exp\left[-\frac{\epsilon}{2}\int_{x,x'} p(x)\mathcal{D}(t,x,x')p(x') + i\int_x p(x)A(x)\right]. \tag{H4}$$

The propagator $\mathcal{D}(t,x,x')$ is the inverse of $\mathcal{D}^{-1}(t,x,x')$.

Rescaling the variable $p(x) \to \gamma(t,x)p(x)$, Eqs. (H1) and (H2) yield (up to an irrelevant constant) [28]

$$\exp\{\epsilon\mathcal{L}(t)\} = \int Dp \exp\left\{ i\int_x p(x)(\phi_{t+\epsilon} - \phi_t - \epsilon\tilde{\rho}_t)(x)\right\} \times \exp\left\{ -\frac{\epsilon}{2}\int_{x,x'} \gamma(t,x)p(x)\mathcal{D}(x,x';t)\gamma(t,x')p(x')\right\}. \tag{H5}$$

Hence, the Dirac-Feynman formula given in Eq. (H3) yields the Hamiltonian as follows:

$$\mathcal{N}e^{\epsilon\mathcal{L}(t)} = \langle \phi_t | e^{-\epsilon\mathcal{H}} | \phi_{t+\epsilon}\rangle \tag{H6}$$

$$= e^{-\epsilon\mathcal{H}_t(\delta/\delta\phi_t)} \int Dp \exp\left[i\int_x p(\phi_t - \phi_{t+\epsilon})\right]. \tag{H7}$$

For each instant of time, there are infinitely many independent interest rates (degrees of freedom), represented by the collection of variables $\phi_t(x), x \in [t, t+T_{FR}]$. Hence, one needs to use functional derivatives to represent the Hamiltonian as a functional differential operator and the Hamiltonian is written in terms of *functional derivatives* in the coordinates of the *dual* state space variables ϕ_t.

Unlike the action $S[\phi]$ that spans all instants of time—from the initial to the final time—the Hamiltonian is an infinitesimal generator in time and refers to only the instant of time at which it acts on the state space. The degrees of freedom $\phi_t(x)$ refer to time t only through the domain on which the Hamiltonian is defined. This is the reason that in the Hamiltonian the time index t can be dropped for the variables $\phi_t(x)$ and replaced by $\phi(x)$ with $t \le x \le t+T_{FR}$.

The Hamiltonian for logarithmic Libor interest rates, from Eqs. (H5)–(H7), is given by

$$\mathcal{H}(t) = -\frac{1}{2}\int_t^{t+T_{FR}} dx dx' M_\gamma(x,x';t)\frac{\delta^2}{\delta\phi(x)\delta\phi(x')} - \int_t^{t+T_{FR}} dx \tilde{\rho}(t,x)\frac{\delta}{\delta\phi(x)},$$

$$M_\gamma(x,x';t) = \gamma(t,x)\mathcal{D}(x,x';t)\gamma(t,x'). \tag{H8}$$

The derivation only assumed that the volatility $\gamma(t,x)$ is deterministic, which is a key feature of the Libor market model. The drift term $\tilde{\rho}(t,x)$ in Hamiltonian is completely *general* and can be any (nonlinear) functions of the interest rates [29].

General considerations related to the existence of a martingale measure rule out any potential terms for the interest rate Hamiltonian [2,30]. Nontrivial dynamics is contained in the kinetic term with the function $M_\gamma(x,x';t)$ encoding the model chosen for the interest rates; a wide variety of such models has been discussed in [2]. The drift term is completely fixed by the martingale condition and, in particular, by $M_\gamma(x,x')$.

The quantum fields $\phi(t,x)$ is more fundamental that the velocity quantum field $\mathcal{A}(t,x)$; the Hamiltonian cannot be written in terms of the $\mathcal{A}(t,x)$ degrees of freedom. The reason being that the dynamics of the forward interest rates are contained in the time derivative terms in the Lagrangian, namely, terms containing $\partial\phi(t,x)/\partial t$; in going to the Hamiltonian representation, these time derivatives essentially become differential operators $\delta/\delta\phi(t,x)$ [31].

[1] B. E. Baaquie, *Interest Rates and Coupon Bonds in Quantum Finance* (Cambridge University Press, Cambridge, 2009).
[2] B. E. Baaquie, *Quantum Finance* (Cambridge University Press, Cambridge, 2004).
[3] The Euclidean quantum field, namely, $\mathcal{A}(t,x)$, driving $L(t,T)$ is discussed in Sec. IV.
[4] D. Heath, R. Jarrow, and A. Morton, Econometrica **60**, 1 (1992).
[5] A. Brace, D. Gatarek, and M. Musiela, Math. Finance **7**, 127 (1997).
[6] F. Jamshidian, Finance Stoch. **1**, 293 (1997).
[7] R. Rebonata, *Modern Pricing of Interest-Rate Derivatives* (Princeton University Press, Princeton, 2002).
[8] R. Rebonata, *Interest-Rate Option Models* (Wiley, New York, 1996).
[9] L. Andersen and J. Andreasen, Appl. Math. Finance **7**, 1 (2000).
[10] M. Joshi and R. Rebonata, Quant. Finance **3**, 458 (2003).
[11] Henry Labordére, *Analysis, Geometry, and Modeling in Finance* (Chapman and Hall, Boca Raton, Fl, 2008).
[12] P. S. Hagan, D. Kumar, A. S. Lesniewski, and D. E. Woodward, Wilmott Magazine **84**, 108 (2004).
[13] D. Brigo and F. Mercurio, *Interest Rate Models—Theory and Practice* (Springer, New York, 2007).
[14] K. G. Wilson, Phys. Rev. **179**, 1499 (1969).
[15] K. G. Wilson and W. Zimmermann, Commun. Math. Phys. **24**, 87 (1972).
[16] B. E. Baaquie and J. P. Bouchaud, Wilmott Magazine **64**, 67 (2004).
[17] B. E. Baaquie and Cao Yang, Physica A **388**, 2666 (2009).
[18] I. Grubsic, Ph.D. thesis, Leiden University, 2002.
[19] S. E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models* (Springer, New York, 2008).
[20] P. Hunt and J. Kennedy, *Financial Derivatives in Theory and Practice* (Wiley, New York, 2000).
[21] Recall that $L(t,T)$ is the simple interest rate from T to $T+\ell$.

[22] L. D. Landau and L. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, 3rd ed. (Elsevier Science, Holland, 2003), Vol. 3.

[23] Henceforth, for notational convenience, the limit of $T_{FR} \rightarrow +\infty$ is taken.

[24] Note, on the right-hand side of the equation, the positive sign of the second term in the exponent.

[25] The second term in the exponential, namely, $\int_{t_0}^{t_*} dt \sigma(t,t_*)\mathcal{A}(t,t_*)$, does not contribute to the singular piece; when this term is squared the singular term coming from the quadratic product of $\mathcal{A}(t,x)$ at equal time is canceled by the vanishing integration measure.

[26] Note that, for $\ell \rightarrow 0$, one has the limit $\ell \Sigma_{m=I+1}^{n} \rightarrow (T_n - T_I)$: const.

[27] The dependence of γ and μ on t,x is henceforth ignored since it simplifies the calculation and does not change the main conclusions.

[28] Since only two time slices are henceforth considered, the subscript n on t_n is dropped as it is unnecessary.

[29] Drift is fixed by the choice of the numeraire. The logarithmic Libor rate $\phi(t,x)$ has a nonlinear drift.

[30] A potential term is a function only of $\phi(t,x)$ or $f(t,x)$; the interest rate Hamiltonian can only depend on the $\delta / \delta\phi(t,x)$ or $\delta / \delta f(t,x)$.

[31] If one wants to use the velocity degrees of freedom $\mathcal{A}(t,x)$ in the state space representation, one needs to use the formalism of phase space quantization [32].

[32] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomenon* (Oxford University Press, New York, 1993).