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Patterned and disordered continuous Abelian sandpile model

N. Azimi-Tafreshi* and S. Moghimi-Araghi†
Department of Physics, Sharif University of Technology, P.O. Box 11365-9161, Tehran, 11369 Iran
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We study critical properties of the continuous Abelian sandpile model with anisotropies in toppling rules that produce ordered patterns on it. Also, we consider the continuous directed sandpile model perturbed by a weak quenched randomness, study critical behavior of the model using perturbative conformal field theory, and show that the model has a random fixed point.

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I. INTRODUCTION

The idea of self-organized criticality, introduced by Bak, Tang, and Wiesenfeld (BTW) [1], provides a useful framework for the study of nonequilibrium systems which dynamically evolve into a critical state without the tuning of a control parameter. At critical state, these systems show scaling behaviors characterized by critical exponents [2].

The BTW sandpile model, renamed Abelian sandpile model (ASM) after Dhar's work [3], is the simplest lattice model that displays self-organized critical behavior. The Abelian structure of the model allows the theoretical determination of many of its properties [4,5]. This model is usually defined on a square lattice. At each site of the lattice, an integer height variable between 1 and 4 is assigned which represents the number of sand grains at that site. The evolution of the model at each time step is simple: a grain of sand is added to a random site. If the height of that site becomes greater than the critical height $h_c=4$, the site will be unstable; it topples and four grains leave the site and each of the four neighbors gets one of the grains. As a result, some of the neighbors may become unstable and toppling continues. The process continues until no unstable site remains and the avalanche ends. To achieve this, one should let some grains of sand leave the system and this happens at the boundary sites. Every avalanche can be represented as a sequence of toppling waves such that each site at a wave topples only once [6]. While the scaling behavior of avalanches is complex and usually not governed by simple scaling laws, it has been shown that the probability distributions for waves display clear power-law asymptotic behavior [7]. Deviations of pure power laws for avalanche distributions had been seen in original simulations but were usually interpreted as finite size effects due to avalanches which touch the boundary of the lattice. However, it turns out these deviations exist even for avalanches which do not reach the boundary of lattice [8]. Hence, waves as objects that show simple scaling picture are more useful for understanding of self-organized criticality dynamics.

The scaling exponents of the system show little dependence on parameters such as the number of neighbors. However, if we make the toppling rule anisotropic, then new universality classes may emerge: by selecting a particular transport direction in BTW model, Hwa and Kardar defined an anisotropic sandpile model such that the grains are allowed to leave the system only at one edge of the system [9]. They determined the critical exponents with a dynamical renormalization group (RG) method. Dhar and Ramaswamy defined a directed version of the BTW model and determined the critical exponents and the two-point correlation functions exactly in any dimension [10]. In [11], the effect of anisotropy in a continuous version of sandpile model (Zhang model) is investigated. In this paper, a d-dimensional lattice is considered. This d-dimensional space is divided into two a-dimensional and $(d-a)$-dimensional subspaces. It is assumed that the energy (sand) is propagated differently for the two subspaces, but inside the subspaces the propagation of energy is isotropic. It is then shown that the peaked energy distribution and critical exponents of the distribution of avalanche sizes are affected by the anisotropy. In [12] two variations of continuous Abelian sandpile model are introduced, the directed model and the elliptical model. It is shown that the elliptical anisotropy does not change the universality class of the isotropic model whereas the critical exponents are sensitive to the directed anisotropy. Karmakar *et al.* showed that in a quenched disorder sandpile model, the symmetric or asymmetric flows of sands in each bond of the lattice determine the universality class of the undirected model [13]. Also, a quenched disorder directed sandpile model has the same critical exponents as the BTW model when the local flow balance exists between inflow and outflow of sands at a site. Otherwise the model falls in the universality class of the Manna sandpile model [14].

The original isotropic model could be represented with a conformal field theory known as $c=-2$ theory [15]. When we insert anisotropy in the toppling rules, the rotational symmetry of the lattice is broken and the field theory associated with the model cannot be a conformal field theory. However, in an anisotropic sandpile model, it may be possible to restore the rotational symmetry at the large scales or statistically. To do this, one can introduce models in which the toppling rules have some patterns on the lattice in a way that at larger scales there will be no preferred directions; that is, locally you have preferred directions which differ from site to site in a regular pattern such that on larger scales the system looks isotropic. Another possibility is to assume a quenched randomness for anisotropy in the toppling of lattice sites; that is, we add anisotropy to the toppling rule for each site such that the amount of anisotropy and the pre-

*azimi@physics.sharif.ir
†samanimi@sharif.edu

ferred direction of anisotropy differ from site to site randomly. In this way there may be no preferred direction statistically.

The question we address in this paper is whether the universality class of these modified models is different from the original sandpile model or not. We show that in some patterned sandpile models the universality class is the same as the isotropic Abelian sandpile model's universality class. However it turns out that the presence of disorder in a sandpile model may change the universality class of the system [13]. Here, we investigate the effect of disorder exploiting the replica technique; we consider anisotropy as a perturbation to the original conformal field theory and use renormalization group to describe the perturbative behaviors of the system [16,17].

This paper is organized as follows: in Sec. II we insert some anisotropies in the redistribution of sands in order to create some ordered patterns. We obtain the free energy function for these models by using one-to-one correspondence between the recurrent configurations of ASM and the spanning tree configurations on the same lattice [18]. The effect of these types of anisotropies on the critical behaviors of the system is investigated both theoretically and numerically. Next we consider a position dependent randomness in the toppling rule. Our procedure is based on the perturbative renormalization group approach around the conformal field theory describing the isotropic model and we obtain the renormalization group equations for coupling constants.

II. PATTERNED CONTINUOUS SANDPILE MODELS

It is known that the universality class of directed sandpile model is different from ordinary ASMs [10,11]. In the directed model, the sand grains always drift toward preferred direction, say the up-right corner. We would like to see that if the directedness is introduced to the model only in small scales, does the universality class change or not? To this end we add the directedness locally in a way that on average there will be no preferred direction toward which the sand grains move.

Consider the continuous ASM on a square lattice composed of N lattice sites [19,20]. To each site, a continuous height variable in the $[0,4)$ interval is assigned. We divide the sites into two groups, A and B, such that neighbors of one site in group A belong to group B and vice versa. We impose different anisotropic toppling rules for the points belonging to these two sublattice: when a toppling occurs in an A-site $1+\epsilon$ amount of sand is transferred to each of the right and up neighbors and $1-\epsilon$ amount to the down and left neighbor sites. In the case that a B site topples, $1+\epsilon$ amount of sand is given to each of the left and down sites and $1-\epsilon$ amount of sand is transferred to the right and up neighbors. Here, ϵ is a positive real parameter less than 1 that controls the amount anisotropy. For $\epsilon=0$ we will have the isotropic model and $\epsilon=1$ characterizes the fully anisotropic model. This toppling rule means that the A sites try to direct the avalanche toward the up-left corner and the B sites try to direct the avalanche to the down-right corner. Thus on average the sands do not move in any specific direction. In Fig. 1 such a lattice is sketched. If a toppling occurs, the amount of sand transferred via thick lines is $1+\epsilon$ and the amount of sand transferred via thin lines is $1-\epsilon$. It is clear that for $\epsilon=1$ the sands are only allowed to move along one of the thick zigzag paths and therefore the system becomes essentially a set of one-dimensional sandpile models. The elements of the toppling matrix can be written in the following form:

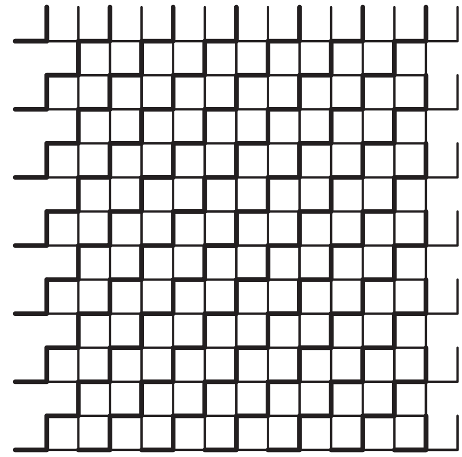

FIG. 1. Patterned ASM.

$$\Delta^{A}_{ij,i'j'}=\begin{cases}4 & \text{for } i=i',\ j=j'\\ -(1\pm\epsilon) & \text{for } i=i'\pm 1\\ -(1\mp\epsilon) & \text{for } j=j'\pm 1\\ 0 & \text{otherwise,}\end{cases} \tag{1}$$

$$\Delta^{B}_{ij,i'j'}=\begin{cases}4 & \text{for } i=i',\ j=j'\\ -(1\mp\epsilon) & \text{for } i=i'\pm 1\\ -(1\pm\epsilon) & \text{for } j=j'\pm 1\\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

A first step to deduce critical behaviors of the system could be finding the free energy function. The closed form of the free energy is obtained by enumerating the corresponding spanning trees on the lattice. The formulation for enumerating spanning trees for general lattices is given in [21]. We take the unit cells of two lattice sites as shown in Fig. 2. Following the standard procedure, we obtain the free energy,

$$f=\frac{1}{8\pi^2}\int_0^{2\pi}d\theta\int_0^{2\pi}d\phi\,\ln\det F(\theta,\phi), \tag{3}$$

where

$$F(\theta,\phi)=4I-[a(0,0)+a(1,0)e^{i\theta}+a(-1,0)e^{-i\theta}+a(0,1)e^{i\phi}+a(0,-1)e^{-i\phi}+a(1,1)e^{i(\theta+\phi)}+a(-1,-1)e^{-i(\theta+\phi)}], \tag{4}$$

$a(n,\acute{n})$ are the 2×2 cell adjacency matrices describing the connectivity between sites of the unit cells n, $\acute{n}$,

$$a(0,0)=\begin{pmatrix}0 & 1+\epsilon\\ 1+\epsilon & 0\end{pmatrix},\quad a(0,1)=a^T(0,-1)=\begin{pmatrix}0 & 0\\ 1+\epsilon & 0\end{pmatrix},$$

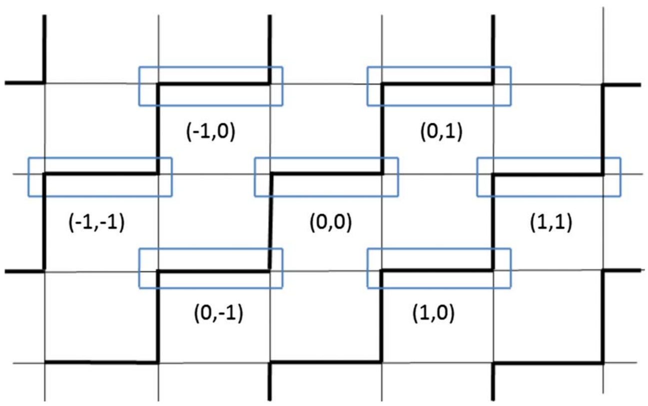

FIG. 2. (Color online) Unit cells of the patterned ASM.

$$a(-1,0) = a(-1,-1) = a^T(1,0) = a^T(1,1) = \begin{pmatrix} 0 & 1-\epsilon \\ 0 & 0 \end{pmatrix}. \tag{5}$$

With a straightforward calculation one finds

$$f = \frac{1}{8\pi^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln[12 - 4\epsilon^2 - 4(1-\epsilon^2)\cos\theta - 4 \times (1+\epsilon^2)\cos\phi - 2(1-\epsilon^2) \times \cos(\theta+\phi) - 2(1-\epsilon^2)\cos(\theta-\phi)]. \tag{6}$$

Now it is seen that the model is equivalent to a free fermion eight-vertex model with weights $\{w(1),\ldots,w(8)\}$ [22] that are related to ϵ with the following relations:

$$12 - 4\epsilon^2 = w(1)^2 + w(2)^2 + w(3)^2 + w(4)^2,$$

$$2(1-\epsilon^2) = w(2)w(4) - w(1)w(3),$$

$$(1-\epsilon^2) = w(5)w(6) - w(3)w(4),$$

$$2(1+\epsilon^2) = w(2)w(3) - w(1)w(4),$$

$$0 = w(5)w(6) - w(7)w(8).$$

The critical properties of the free fermion model are well known [22]. It is found that for all values of ϵ the free energy function is analytical and the model shows no phase transition. It means although by inserting this kind of anisotropy some symmetries of the lattice are broken, but the broken symmetry operator is irrelevant and takes the system to the original critical fixed point. This fact can be checked by numerical simulations. We have simulated the model on a square lattice with sizes $L=64$, 128, 256, and 512. After the system arrives at recurrent configurations, we began to collect data. At each size 10^6 avalanches have been considered to derive the wave statistics. Figure 3 displays the wave toppling distributions for different system sizes and three different values of ϵ. A power-law fit to these curves determines the critical exponent $\tau_s^{(w)}$ defined as $P_s^w(s) \sim s^{-\tau_s^{(w)}}$. In Fig. 4, the extrapolated value of τ for $L\to\infty$ is obtained: $\tau(\infty) = 1.00 \pm 0.01$ for $\epsilon=0.1$, $\tau(\infty) = 0.99 \pm 0.01$ for $\epsilon=0.4$, and $\tau(\infty) = 1.01 \pm 0.01$ for $\epsilon=0.8$. As we see, the wave exponents are independent of ϵ and are consistent with the exact value of $\tau_s^{(w)}=1$ for $\epsilon=0$ [8].

It is possible to reformulate the partition function or the number of the spanning trees on the lattice in terms of fermionic path integrals. We place a two-component Grassmannian variable $\psi_n = (\psi_1, \psi_2)$ on each unit cell n of the lattice. In this representation, the action of the field theory is written in the following form:

$$S = \sum_{\langle n,\acute{n}\rangle} \sum_{i,j=1}^{2} \psi_i^\dagger(n) a_{ij}(n,\acute{n}) \psi_j(\acute{n}), \tag{7}$$

where $a(n,\acute{n})$ are the adjacency matrices defined in Eq. (5). In the continuum limit, this action is obtained to be

$$S = \int dxdy \sum_{\alpha,\beta=1}^{2} [4(-1)^{\alpha+\beta}\psi_\alpha^\dagger(x)\psi_\beta(y) + 2\epsilon^{\alpha\beta} \times (1-\epsilon)\partial_x\psi_\alpha^\dagger(x)\psi_\beta(y) + 2\epsilon^{\alpha\beta}\psi_\alpha^\dagger(x)\partial_y\psi_\beta(y)], \tag{8}$$

where $\epsilon^{\alpha\beta}$ is the Levi-Civita antisymmetric tensor. At the first sight it may look strange that we have an action that there is only first derivative in it, in contrast with the $c=-2$ action that has second derivative terms. Even if we take the $\epsilon\to 0$ limit, it seems that the problem still exists. But if we look more closely, we will see that at least in the above limit one can write ψ_2 in terms of ψ_1 and its derivative and then the second derivative terms emerge.

It may be argued that the above defined patterned system actually has a preferred direction; the zigzag paths join the down-left corner to the up-right corner. This is true, in fact the system has an elliptical anisotropy at large scales and we

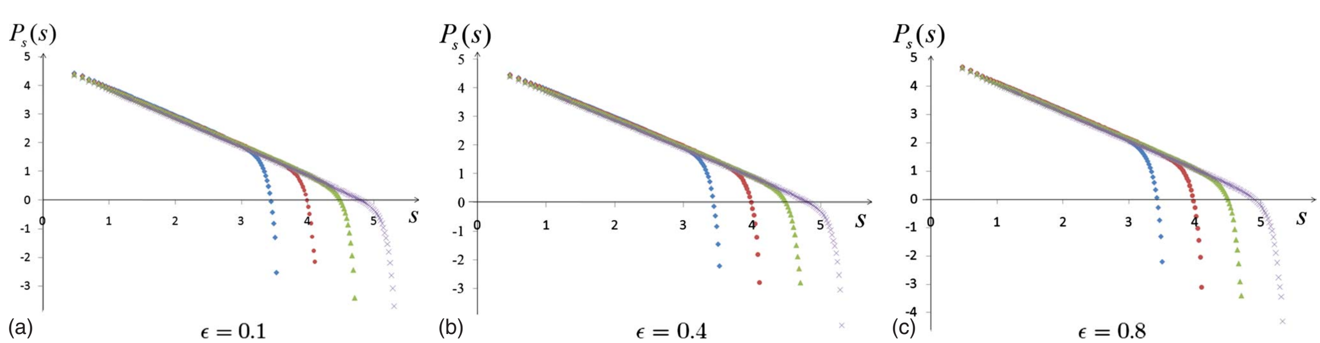

FIG. 3. (Color online) Wave size distributions for $\epsilon=0.1$, 0.4, and 0.8 and for lattice sizes $L=64$, 128, 256, and 512.

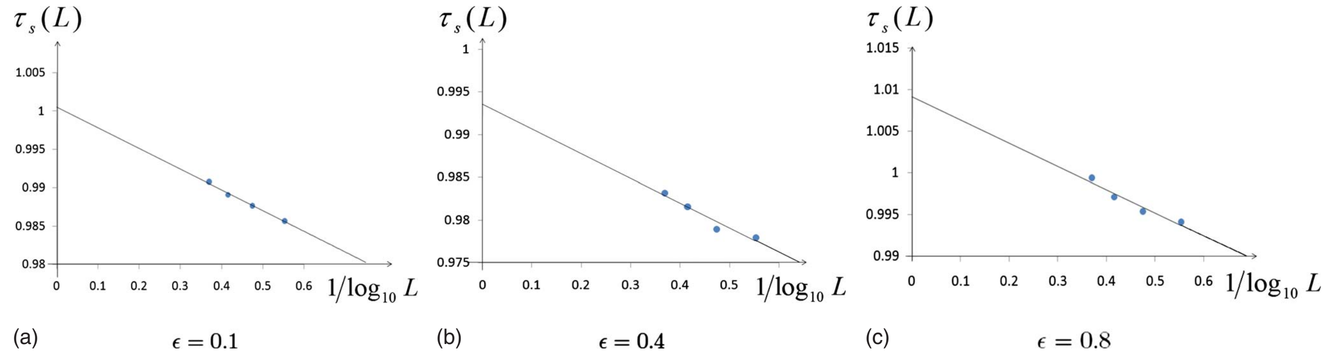

FIG. 4. (Color online) The exponent $\tau_s(L)$ is a linear function of $1/\log_{10} L$. The intersection with vertical axis gives $\tau_s(\infty)$.

know that the elliptical anisotropy does not change the universality class [12]. It is possible to introduce other patterns in a way that the system is symmetric on large scales. Figure 5 shows such a pattern. In this model the thick lines characterize bonds that carry $1+\epsilon$ amount of sand and the thin lines carry $1-\epsilon$ amount of sand after a site topples.

Following the standard procedure, the free energy of this system can be shown to be

$$f = \frac{1}{16\pi^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln\{132 - 136\epsilon^2 + 4\epsilon^4 + 2(1-\epsilon^2)^2 \times(\cos 2\theta + \cos 2\phi) - 64(1-\epsilon^2)(\cos\phi + \cos\theta) - 4(1-\epsilon^2)^2[\cos(\theta+\phi) + \cos(\theta-\phi)]\}, \tag{9}$$

which is again a smooth function and similar to the previous model, the self-organized criticality has the same universality class as that of the undirected sandpile model.

Up to now, we have observed that the patterns that do not produce a preferred direction in large scales do not change the universality class. In Sec. III we will consider a quenched random anisotropy to see if the universality class is changed or not.

III. RANDOM DIRECTED CONTINUOUS SANDPILE MODEL

In directed continuous sandpile model (DCSM) introduced in [12], it has been assumed that after toppling of a site, $1+\epsilon$ amount of sand moves to left (and up) and $1-\epsilon$ amount of sand moves to right (and down); that is there exists a preferred direction for the transportation of sands. In other words, the rotational symmetry is broken in this model. In the continuum limit, It turns out that the action of the theory assigned to the directed model is the action of $c=-2$ conformal field theory perturbed by the relevant scaling fields $\phi=-2\theta\partial\bar{\theta}$ and $\bar{\phi}=-2\theta\bar{\partial}\bar{\theta}$. As these operators are relevant, they grow under renormalization and take the system to a fixed point [12].

In DCSM, ϵ determines the strength of anisotropy and is in the interval $(-1,1)$. Positive ϵ means that the sand grains are pushed to the up-left corner and negative ϵ means that they are pushed to the down-right corner. In this model the value of ϵ is considered to be uniform throughout the lattice. However, we may assume a statistical distribution for ϵ, such that it can take both positive and negative values on different sites. The assumption that the mean value of ϵ vanishes means that there will be no preferred direction statistically and the rotational symmetry will be restored to the model. The question is if such a modification takes the system to a new universality class or not. The assumption of a weak randomness allows us to determine the critical behavior of the model based on the pertubative renormalization group technique.

In the continuous limit, the action of perturbed theory is given as

$$S = S_0 + \int_z \epsilon(z,\bar{z})[\phi(z,\bar{z}) + \bar{\phi}(z,\bar{z})], \tag{10}$$

where S_0 is the action of $c=-2$ logarithmic conformal field theory. One can obtain the effective action using the replica method; that is, we have to take the average of ϵ on N copies of the system and then find its limit when $N\to 0$. We assume that the values of $\epsilon(z)$ at different sites are independent and have a Gaussian distribution on each site with a standard deviation equals g_0,

$$\langle \epsilon(z_1)\epsilon(z_2)\rangle = g_0\delta(z_1 - z_2). \tag{11}$$

The effective action then is expressed as

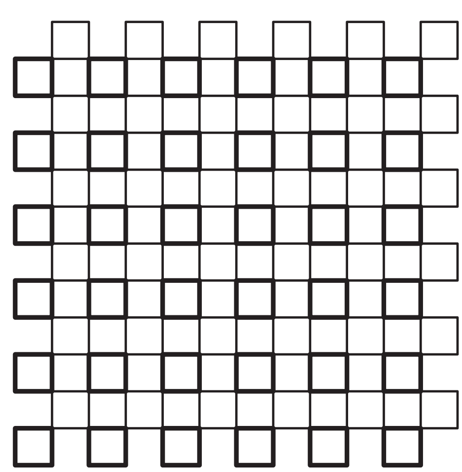

FIG. 5. Order and symmetric patterned ASM.

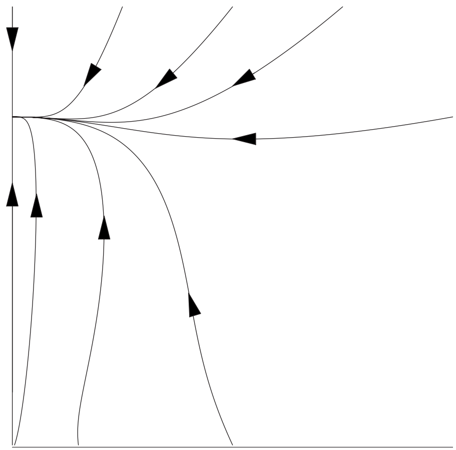

FIG. 6. The RG flow of the model with quenched randomness.

$$S = \sum_{a=1}^{N} S_{0,a} + g_0 \int_z \sum_{a\neq b}^{N} [\phi_a(z,\bar{z})\phi_b(z,\bar{z}) + \bar{\phi}_a(z,\bar{z})\bar{\phi}_b(z,\bar{z}) + \phi_a(z,\bar{z})\bar{\phi}_b(z,\bar{z})]. \tag{12}$$

Although the coupling constants of the field operators $\phi\phi$, $\bar{\phi}\bar{\phi}$, and $\phi\bar{\phi}$ are the same, as we will see, they have different RG equations. Therefore we distinguish between the coupling constants of these field operators and rewrite them as $g_{0\phi\phi}$, $g_{0\bar{\phi}\bar{\phi}}$, and $g_{0\phi\bar{\phi}}$, respectively,

$$\int_z \sum_{a\neq b}^{N} [g_{0\phi\phi}\phi_a(z,\bar{z})\phi_b(z,\bar{z}) + g_{0\bar{\phi}\bar{\phi}}\bar{\phi}_a(z,\bar{z})\bar{\phi}_b(z,\bar{z}) + g_{0\phi\bar{\phi}}\phi_a(z,\bar{z})\bar{\phi}_b(z,\bar{z})] \equiv \int_z \sum_{a\neq b}^{N} \Phi_{ab}(z), \tag{13}$$

where the second line is an abbreviation of the first line. It is easy to see that the coupling constants g are dimensionless; that is, they are marginal. Therefore to see if they are marginally relevant or not, we have to expand the partition function to the second order of g. If it is marginally relevant we would like to see if it grows to infinity or will introduce a fixed point. This means that we have to consider at least up to the third order of the coupling constants,

$$\int_z \sum_{a\neq b} \Phi_{ab}(z) + \frac{1}{2!}\int_{z_1,z_2} \sum_{a\neq b} \Phi_{ab}(z_1)\sum_{c\neq d} \Phi_{cd}(z_2) + \frac{1}{3!}\int_{z_1,z_2,z_3} \sum_{a\neq b} \Phi_{ab}(z_1)\sum_{c\neq d} \Phi_{cd}(z_2)\sum_{e\neq f} \Phi_{ef}(z_3) + \cdots$$
$$= g_{\phi\phi}\int_z \sum_{a\neq b} \phi_a(z,\bar{z})\phi_b(z,\bar{z}) + g_{\bar{\phi}\bar{\phi}}\int_z \sum_{a\neq b} \bar{\phi}_a(z,\bar{z})\bar{\phi}_b(z,\bar{z}) + g_{\phi\bar{\phi}}\int_z \sum_{a\neq b} \phi_a(z,\bar{z})\bar{\phi}_b(z,\bar{z}). \tag{14}$$

To proceed, we have to know the contraction of fields in different possible ways. The calculation is done using operator product expansion (OPE) relations of the perturbing operators,

$$\phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2) = \frac{1}{(z_1-z_2)^2} + \partial\phi(z_2,\bar{z}_2) + 2T(z_2,\bar{z}_2) + \cdots, \tag{15}$$

$$\bar{\phi}(z_1,\bar{z}_1)\bar{\phi}(z_2,\bar{z}_2) = \frac{1}{(\bar{z}_1-\bar{z}_2)^2} + \partial\bar{\phi}(z_2,\bar{z}_2) + 2\bar{T}(z_2,\bar{z}_2) + \cdots, \tag{16}$$

$$\phi(z_1,\bar{z}_1)\bar{\phi}(z_2,\bar{z}_2) = \frac{1}{|z_1-z_2|^2} + \frac{\bar{\phi}(z_1,\bar{z}_1)}{z_1-z_2} - \frac{\phi(z_1,\bar{z}_1)}{\bar{z}_1-\bar{z}_2} + \cdots, \tag{17}$$

where T and $\bar{T}$ are the components of the energy-momentum tensor.

At each order we contract all the fields using the above OPE relations and only keep a pair of ϕ or $\bar{\phi}$ fields. While doing the integrations, we have to perform the regularization. We do the regularization in cutoff scheme: we assume that the distance between any pair of integration variables is restricted to be between a, the lattice constant, and L, size of the lattice. Up to the third order, the renormalized couplings are obtained to be

$$g_{\phi\phi} = g_{0\phi\phi} + 2\alpha(N-2)g_{0\phi\phi}g_{0\phi\bar{\phi}} + 2\alpha^2(N-2)\times[g_{0\phi\phi}g_{0\phi\bar{\phi}}^2(5N-9) + g_{0\bar{\phi}\bar{\phi}}g_{0\phi\phi}^2(3N-7)], \tag{18}$$

$$g_{\bar{\phi}\bar{\phi}} = g_{0\bar{\phi}\bar{\phi}} + 2\alpha(N-2)g_{0\bar{\phi}\bar{\phi}}g_{0\phi\bar{\phi}} + 2\alpha^2(N-2)\times[g_{0\bar{\phi}\bar{\phi}}g_{0\phi\bar{\phi}}^2(5N-9) + g_{0\phi\phi}g_{0\bar{\phi}\bar{\phi}}^2(3N-7)], \tag{19}$$

$$g_{\phi\bar{\phi}} = g_{0\phi\bar{\phi}} + 2\alpha(N-3)(g_{0\phi\phi}g_{0\bar{\phi}\bar{\phi}} + g_{0\phi\bar{\phi}}^2) + 8\alpha\{g_{0\phi\bar{\phi}}^3[(N-2)\times(N-1) + 2(N-3)^2] + g_{0\phi\phi}g_{0\bar{\phi}\bar{\phi}}g_{0\phi\bar{\phi}}\times[3(N-2)(N-1) + 2(N-3)^2]\}, \tag{20}$$

where $\alpha=4\pi\ln\frac{L}{a}$ and by the symmetry reasons, $g_{\phi\phi}=g_{\bar{\phi}\bar{\phi}}$. In the limit $N=0$, we obtain the β functions up to third order,

$$\beta_{g_{\phi\phi}} = a\frac{\partial g_{\phi\phi}}{\partial a} = 16\pi g_{\phi\phi}g_{\phi\bar{\phi}} - 16\pi\alpha(9g_{\phi\phi}g_{\phi\bar{\phi}}^2 + 7g_{\phi\phi}^3), \tag{21}$$

$$\beta_{g_{\phi\bar{\phi}}} = a\frac{\partial g_{\phi\bar{\phi}}}{\partial a} = 24\pi(g_{\phi\phi}^2 + g_{\phi\bar{\phi}}^2) - 32\pi\alpha(5g_{\phi\bar{\phi}}^3 + 6g_{\phi\phi}^2 g_{\phi\bar{\phi}}). \tag{22}$$

It is clear from above equations that these fields are marginally relevant. However, the coefficients of the terms proportional to g^3 are negative. Hence the renormalization flow takes the system to a fixed point at $g_{\phi\phi}=g_{\bar{\phi}\bar{\phi}}=0$, $g_{\phi\bar{\phi}}=\frac{3}{20\alpha}$ (see Fig. 6). In the random fixed point, the rotational symmetry of the lattice is restored so it is expected that the system shows critical behaviors different from the deterministic directed model.

We can compare our results with what Pan *et al.* [14] have found. In the patterned case, the outflow and inflow of the sand were balanced and we found that the universality class is not changed in such cases. On the other hand in the model with quenched randomness, there is not such a balance hence it is expected that the random fixed point shall belong to another universality class such as the universality class of the directed Manna sandpile model. We say that it may correspond to Manna model because in this model there is randomness in the toppling rule, and we say it *may* because in Manna model the randomness is annealed but in our model it is quenched. Of course one may argue that the Manna model and Oslo model are in the same universality class [23] and it has been shown that the Oslo model is a quenched Edwards-Wilkinson equation [24]. Therefore Manna model is related to a model having quenched randomness and the connection we mentioned becomes more plausible.

IV. CONCLUSIONS

In this paper we have studied critical behaviors of the continuous sandpile model with some patterned anisotropies in the toppling matrices. Using the correspondence with the spanning trees, we obtained the free energy function for theses models. Both theoretic analysis and numerical simulations for the probability distribution of waves indicate that the anisotropic models are in the same universality class of the continuous sandpile model.

Also we have investigated analytically the effect of quenched randomness on the critical behavior as continuous directed sandpile model. Our calculations are based on the perturbed renormalization conformal field theory and replica technique. Up to the third order in the perturbation expansion, we obtained the renormalization group equations for the coupling constants of the perturbing fields. We showed that the perturbing fields are relevant and take the system to a fixed point.

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