Exact results for the Barabási queuing model

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Previous works on the queuing model introduced by Barabási to account for the heavy tailed distributions of the temporal patterns found in many human activities mainly concentrate on the extremal dynamics case and on lists of only two items. Here we obtain exact results for the general case with arbitrary values of the list length L and of the degree of randomness that interpolates between the deterministic and purely random limits. The statistically fundamental quantities are extracted from the solution of master equations. From this analysis, scaling features of the model are uncovered.

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I. INTRODUCTION

Many human activities, such as mail and electronic-mail exchanges, library loans, stock market transactions [1], or even motor activities [2], display heavy tailed interevent and waiting time distributions. To account for these heavy tails [1], a priority queuing model has been proposed by Barabási [3] that since then stimulated an active field of research with potential practical applications (e.g., see Refs. [4–8]).

Within Barabási priority queuing model (BPQM), each item in a list of fixed length L has a priority value. At each time step, the maximal priority task is executed with probability p, otherwise, a randomly selected one is accomplished. Once a task is executed, it is substituted by a new one (or the same) that adopts a new randomly selected priority value drawn from a probability density function (PDF) $\rho(x)$. This simple model yields power-law tailed distributions of interevent times, mimicking the empirical histograms of many human activities.

Besides the value of queuing models for diverse practical questions, another issue that makes BPQM attractive is its connection with diverse other physical problems such as invasion percolation [8,9] or self-organized evolutionary models [10–12] as soon as the roles of priorities and fitness can be identified.

Variants of the BPQM with variable list length have been analytically treated in the literature [5,9]. However, special attention has been given to the particular and more tractable case of extremal dynamics when $p\to 1$ [1,5,8,9], while non-null degree of randomness $(1-p\neq 0)$ may also display interesting features. Moreover, for the fixed length BPQM, exact results both for steady [6] and transient [8] regimes have been obtained for the simplest instance $L=2$ only. Although lists of two items already display the power-law decay of the distribution of waiting times when p approaches unity, naturally, other features are missed in the simplest case. Then, in the present work, we tackle the BPQM with arbitrary values of p and L.

The paper is organized as follows. In Sec. II we show exact results for the PDFs of priorities in lists of arbitrary length L by recourse to a master equation (ME). In Sec. III we obtain an approximate expression for the waiting time distribution. Section IV deals with exact results for "avalanches," which provide the time that higher priority tasks (above a threshold) remain in the list, and is also related to waiting time duration. Section V contains final remarks.

II. EXACT TREATMENT

A fundamental quantity is the probability, $P_{n,t}(x)$, that there are n tasks with priority higher than a given value x at time t. Its time evolution is ruled by a ME of the form

$$P_{n,t+1} = M_{n,n+1}P_{n+1,t} + M_{n,n}P_{n,t} + M_{n,n-1}P_{n-1,t} \qquad (1)$$

for $n=0,1,\ldots,L$.

If at a given time t there are n tasks $(0\leq n\leq L)$ with priorities above x, in the following step that quantity $t+1$ can either (a) decrease by one unit, (b) remain the same, or (c) increase by one unit, with transition probabilities $M_{n-1,n}$, $M_{n,n}$, and $M_{n+1,n}$, respectively. The non-null matrix elements can be obtained as follows.

(a) The transition $n\to n-1$ occurs if (i) the selected task has priority above x [be either the largest one, selected with probability p, or one among the n, randomly chosen with probability $(1-p)/L$] and (ii) it is replaced by a new task with priority below x [which arises with probability $R(x)$]. Then $M_{n-1,n}=\text{prob(i)prob(ii)}=[p+(1-p)n/L]R(x)$ for $0<n\leq L$.

(b) The number n remains constant if either (i) a task with priority above x is selected [as in (a)] and it is replaced by a new task that acquired a priority also above x or (ii) a task with priority below x [one among the $L-n$ tasks, randomly chosen with probability $(1-p)/L$] is selected and replaced by a new one also having priority below x. This leads to $M_{n,n} = \text{prob(i)}[1-R(x)] + \text{prob (ii)}R(x)=[p+(1-p)n/L][1-R(x)] +(1-p)[(L-n)/L]R(x)$ for $0<n\leq L$. Exception occurs for $n=0$, in which case the largest priority is necessarily below x, thus $M_{0,0}=pR(x)+(1-p)R(x)=R(x)$.

(c) The transition $n\to n+1$ occurs if a task with priority below x [that can only be chosen randomly among $L-n$ tasks, with probability $(1-p)/L$] is replaced by a new task with priority above x. Hence, $M_{n+1,n}=(1-p)(L-n)/L[1-R(x)]$ for $0<n<L$. Also in this case, exception occurs for $n=0$ for the same reason as in (b), thus $M_{1,0}=p[1-R(x)] +(1-p)[1-R(x)]=1-R(x)$.

Then, for $\rho(x)=1$, the elements of the tridiagonal matrix $\mathbf{M}$ are given by

$$M_{n-1,n}(x) = px + (1-p)xn/L,$$

$$M_{n,n}(x) = p(1-x) + (1-p)[x(L-n) + (1-x)n]/L,$$

$$M_{n+1,n}(x) = (1-p)(1-x)(L-n)/L \qquad (2)$$

for $n=1,\ldots,L$ and additionally $M_{1,0}(x)=1-x$, $M_{0,0}(x)=x$. Of course, $M_{n,m}=0$ if at least one of the subindices is smaller than 0 or larger than L. For notation simplicity, we will consider the expressions in Eq. (2), however, generality can be recovered at any time, if desired, simply by redefining the threshold through $x \to R(x)=\int_0^x \rho(x')dx'$.

Notice that the ME [Eqs. (1) and (2)] signals a biased random walk with reflecting boundaries at $n=0$ and $n=L$, setting the basis to write a continuum limit approximation. However, for arbitrary L, drift and diffusion coefficients are state dependent and the approach of biased diffusion successfully applied [5] to determine the scaling of the waiting time distribution, in other queuing systems with constant coefficients, becomes more tricky in the nondeterministic case $p \neq 1$.

Then, let us find the exact steady solution of the ME [Eqs. (1) and (2)] for arbitrary length L. By recursion, one gets

$$P_n(x) = \frac{L\,!\,\Gamma(a+1)(1-x)^n}{(L-n)\,!\,\Gamma(a+n+1)(1-p)x^n} P_0(x) \qquad (3)$$

for $1 \le n \le L$, where $a=pL/(1-p)$, and from normalization

$$P_0(x) = \left(1 + \sum_{n=1}^{L} P_n(x)/P_0(x)\right)^{-1}. \qquad (4)$$

The distribution P_n, given by Eqs. (3) and (4), can be used now to evaluate diverse meaningful quantities. In particular, the PDF of the nth largest priority value, p_n, can be extracted from the relation $\int_0^x p_n(x')dx' = \sum_{m=0}^{n-1} P_m(x)$, hence

$$p_n(x) = \frac{\partial}{\partial x}\sum_{m=0}^{n-1} P_m(x). \qquad (5)$$

Figure 1 shows the exact PDFs of the two largest priorities in the list, $p_1(x)=P_0'(x)$ and $p_2(x)=P_0'(x)+P_1'(x)$, for $L=5$ and different values of p, compared to the results of numerical simulations of the BPQM.

In the fully random case $p=0$, Eqs. (3) and (4) yield $P_n(x)=\binom{L}{n}(1-x)^n x^{L-n}$, hence $p_n(x)=L\binom{L-1}{n-1}(1-x)^{n-1}x^{L-n}$, in accord with straightforward combinatorial analysis. In the opposite limit $p\to 1$, $p_1(x)$ gets closer to a unit step function at $x=0$ while $p_2(x)$ approaches the Dirac delta function $\delta(x)$. This is expected since those tasks that have entered the list more recently and adopted priority values uniformly distributed in [0, 1] have more chances to be chosen again, while the older tasks are more and more likely to remain in the list forever as p tends to 1; then the second priority value (and together with it the remaining ones) collapses to zero.

For large enough L [namely, $L/(1-p) \gg 1$], Eqs. (3) and (4) lead to $p_1 \simeq H(x-1+p)/p$, where H is the Heaviside unit step function and $p_2 \simeq (1-p)(1/x^2-1)H(x-1+p)/p^2$. In fact, the matrix elements of the ME [Eqs. (1) and (2)] become independent of n either in the limit of large L for fixed n (hence neglecting terms of order n/L in the matrix elements) or also when $p\to 1$. In such case, from the recurrence relation, one obtains

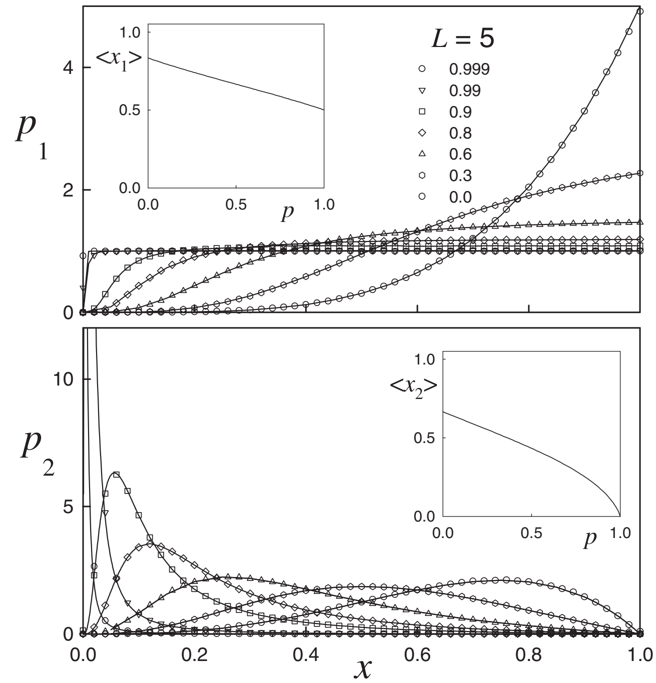

FIG. 1. PDFs of the largest (upper panel) and the second largest (lower panel) priority values for $L=5$, different values of p indicated on the figure, and $R(x)=x$. Solid lines correspond to exact results and symbols to numerical simulations of the BPQM performed as in previous figures. In the insets, the average values are displayed as a function of p.

$$P_n(x) \simeq \frac{(1-p)^{n-1}(1-x)^n}{p^n x^n} P_0(x) \quad \text{for } 0 < n \le L, \qquad (6)$$

which, for $x > 1-p$, can be summed up to obtain

$$P_0(x) \simeq (x-1+p)/p. \qquad (7)$$

For $x \le 1-p$, all P_n's tend to vanish in the large L limit. Figure 2 illustrates the performance of this approximation in comparison with exact results. The assumption $n \ll L$ fails as soon as the probability that $n > \mathcal{O}(1)$ becomes non-negligible. For each $x < 1-p$, the exact P_n is peaked around $n \simeq (x-1+p)L/(1-p)$. The approximate expression for P_n becomes exact both in the limits of $L\to\infty$ and $p\to 1$ and will be used later.

The PDF of all priorities x in the list, $p(x)$, represents a kind of average that verifies $\sum_{n=1}^{L} p_n(x) = Lp(x)$. Its time evolution is given by

$$p(x,t+1) = p(x,t) + [\rho(x) - pp_1(x,t) - (1-p)p(x,t)]/L, \qquad (8)$$

which in the long-time limit leads to the relation

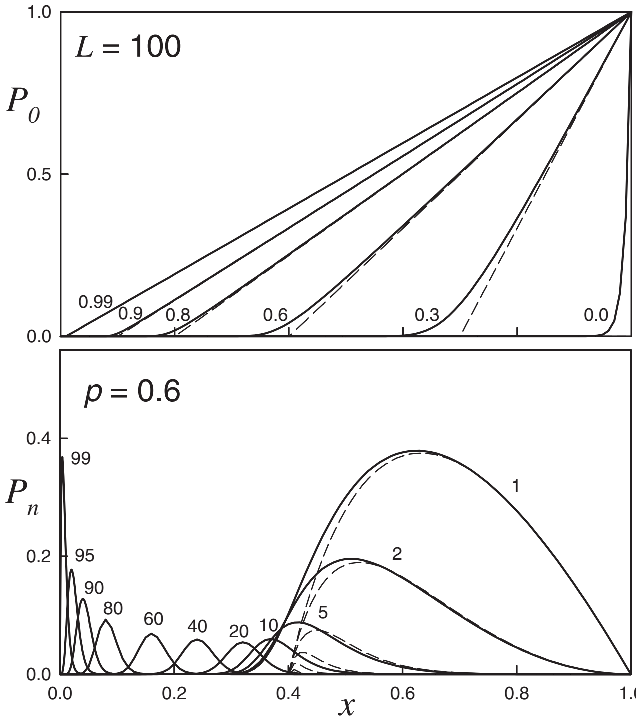

FIG. 2. Probabilities $P_0(x)$ (for different values of p, upper panel) and $P_n(x)$ (for different values of n and $p=0.6$, lower panel) at $L=100$ and $R(x)=x$. Solid lines correspond to exact results; dashed lines to the large $L/(1-p)$ approximation.

$$pP_0(x) + (1-p)P(x) = R(x), \tag{9}$$

where $P(x)=\int_0^x p(x')dx'$.

Let us call old tasks, those items, whose priority has not been assigned at a given step. The cumulative PDF of old task priorities, $O(x)$, can be obtained from the relation

$$LP(x) = R(x) + (L-1)O(x) \tag{10}$$

and, by means of Eq. (9), it can be expressed as

$$O(x) = \frac{(L+p-1)R(x) - pLP_0(x)}{(L-1)(1-p)}. \tag{11}$$

In the particular case $L=2$, Eqs. (3) and (4) give $P_0(x)=(1+p)x^2/(1-p+2px)$ and recalling that its derivation was carried out for uniform $\rho(x)$ but the general case is recovered simply through the mapping $x \to R(x)$, then, Eq. (11) allows us to reobtain the result of Vazquez [6], namely, $O(x)=(1+p)R(x)/[1-p+2pR(x)]$.

Figure 3 illustrates the behavior of $O(x)$ for different values of p and $L=20$. The distribution of the bulk of old values for arbitrary L is qualitatively similar to that obtained for $L=2$ in Ref. [6].

For $L>2$, however, $P(x)$ and $O(x)$ are average or mean-field quantities, while more meaningful is the distribution of the largest old priority $O_1(x)$ [that, of course, for $L=2$ coincides with $O(x)$]. It verifies

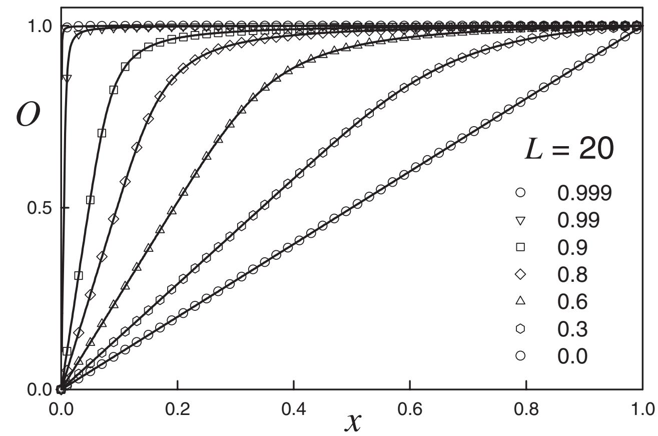

FIG. 3. Cumulative PDFs of old task priorities for $L=20$, different values of p, and $R(x)=x$. Symbols correspond to numerical simulations of the BPQM performed as in previous figures; black lines correspond to the exact results from Eq. (11).

$$P_0(x) = R(x)O_1(x) \tag{12}$$

because the probability that there are no tasks above x, $P_0(x)$, is the product of the probability that the freshly assigned (new) priority value is below x times the probability that the highest old task priority (hence also the remaining ones) is below x. For the particular case $p=0$, $O_1(x)=R^{L-1}(x)$, while in the opposite limit $p\to 1$, it tends to a unit step function at $x=0$.

Let us call o_n the nth largest old priority value. The probability that there is only one task above x, $P_1(x)$, is prob(new $\geq x$)prob($o_1 \leq x$)+prob(new$\leq x$)prob($o_2 \leq x \leq o_1$), where prob($o_2 \leq x \leq o_1$)=prob($o_2 \leq x$)−prob($o_2 \leq x \wedge o_1 \leq x$) $=O_2(x)-O_1(x)$. Thus, the distribution of the second largest old priority $O_2(x)$ can be extracted from the identity

$$P_1 = (1-R)O_1 + R(O_2 - O_1). \tag{13}$$

More generally, setting $O_0(x)\equiv 0$ and $O_L(x)\equiv 1$, then, prob$(o_{n+1} \leq x \leq o_n)=O_{n+1}(x)-O_n(x)$ for $0 \leq n \leq L-1$ and Eq. (13) can be straightforwardly generalized to obtain

$$P_n = (1-R)(O_n - O_{n-1}) + R(O_{n+1} - O_n) \tag{14}$$

for $1 \leq n \leq L-1$. Therefore, by knowing $\{P_n\}$ given by Eqs. (3) and (4), the whole family of stationary old task distributions $\{O_n\}$ can be build from Eq. (14) by recursion.

Exact results for $O_1(x)$ and $O_2(x)$ are compared to the outcomes of numerical simulations of the BPQM in Fig. 4 for $L=20$, different values of p, and $R(x)=x$. Observe that $O_2(x)$ is bounded from below by $O_1(x)$ and more generally $O_1(x) \leq O_2(x) \leq \cdots \leq O_{L-1}(x)$.

III. WAITING TIME DISTRIBUTION

The family of distributions of old values $\{O_n, 1 \leq n \leq L-1\}$ should in principle allow us to compute exactly the distribution of waiting times $P_w(\tau)$ in the steady regime.

$P_w(\tau)$ can be obtained as

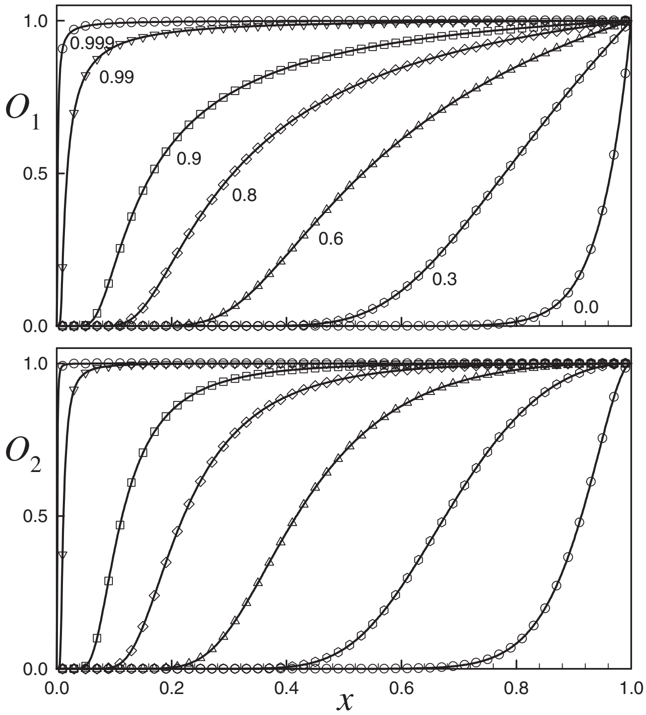

FIG. 4. Cumulative PDFs of the first (upper panel) and second (lower panel) largest old task priorities for $L=20$, different values of p, and $R(x)=x$. Symbols correspond to numerical simulations of the BPQM performed as in previous figures; black lines correspond to the exact results from Eqs. (12) and (13).

$$P_w(\tau) = \int_0^1 dR(x) r_\tau(x), \tag{15}$$

where $r_\tau(x)$ is the probability that a task (let us call it X) with freshly acquired priority value x at a given time $t=t_0$ (once attained the steady state) is again selected for the first time at $t=t_0+\tau$.

The task X can be selected again at the following step, $t=t_0+1$, either (i) with probability p if it has the largest priority value [which occurs with probability $O_1(x)$] or (ii) by random selection with probability $(1-p)/L$. Then, for $\tau=1$,

$$r_1(x) = pO_1(x) + (1-p)/L. \tag{16}$$

Hence $P_w(1)=1/L$ when $p=0$ and it tends to 1 in the opposite case $p\to 1$. Moreover, by means of the approximate Eq. (7) for $P_0(x)$, one has $P_w(1)\simeq p+(1-p)\ln(1-p)+(1-p)/L$.

For $\tau=2$, X must not be selected at $t=t_0+1$ and it must be selected at $t=t_0+2$. To obtain r_2, recall that prob$(o_{n+1}\leq x \leq o_n)=O_n(x)-O_{n+1}(x)$ for $0\leq n\leq L-1$, with $O_0(x)=0$ and $O_L(x)=1$, and let us consider the different possible relative ordering positions of the new priority x in the list at $t=t_0$. Then one has the following possibilities.

(1) For $o_1<x$: given that any of the $L-1$ old tasks is selected at t_0+1 [with probability $\gamma\equiv(1-p)/L$], X can be selected at t_0+2 with probability $pR+\gamma$. Then the desired probability is $r_2^{(1)}=(O_2-O_1)\gamma(L-1)(pR+\gamma)$.

(2) For $o_2\leq x\leq o_1$: given that the first old task is selected at t_0+1 (with probability $p+\gamma$), X can be selected at t_0+2 with probability $pR+\gamma$, while given that one of the $L-2$ remaining tasks is selected at t_0+1 (with probability γ), X can be selected at t_0+2 with probability γ. Then $r_2^{(2)}=(O_2-O_1)[(pR+\gamma)(p+\gamma)+(L-2)\gamma^2$.

(3) For $o_n\leq x\leq o_n-1$, with $n>2$: given that the first old task is selected (with probability $p+\gamma$) or one of the $L-2$ remaining ones is selected (with probability γ) at t_0+1, X can be selected at t_0+2 with probability γ. Then $r_2^{(n)}=(O_{n+1}-O_n)[(p+\gamma)\gamma+(L-2)\gamma^2$.

Summing up over all the cases, one obtains $r_2=\Sigma_{n=1}^{L} r_2^{(n)}$, namely,

$$r_2(x) = \gamma(1-r_1) + pR[(p+\gamma)O_2 + \{\gamma(L-2)-p\}O_1]. \tag{17}$$

Recalling that $O_L\equiv 1$, one recovers the expression found in Ref. [6] for $L=2$, namely, $r_2=(1-r_1)[pR+(1-p)/2]$.

Alternatively, notice that the probability that instead of X the first old task is selected at $t=t_0+1$ is $p[1-O_1(x)]+\gamma$, while the probability that any other old task is selected (for $L>2$) is γ. Given each one of these $L-1$ cases, one has a different probability of selection of X at the second step ($t=t_0+2$) which will be a function of O_1 and O_2. More generally, the exact calculation of $P_w(\tau)$, for $\tau>1$, will require to consider a branching process, with $L-1$ paths at each node, such that for $L>2$, $r_\tau(x)$ does not factorize. This tree generalizes the binary one considered in analogy to invasion percolation for $p=1$ and $L=2$ [8]. As already remarked in Ref. [8], L plays the role of the coordination number while p represents a sort of temperature, but a mapping exists between the branching processes in invasion percolation and queued tasks.

At further time steps the branching will lead to an exact but nonreadily manageable form. Then, let us proceed to find an approximate expression for $P(\tau)$ by analyzing its integrand r_τ, whose behavior is illustrated in Fig. 5.

Notice, in Fig. 5, that for small τ, $P(\tau)$ is dominated by the integral of r_τ in the interval $x\gtrsim 1-p$, the more the larger L. This is due to the propensity (the higher, the closer p to 1) of such values to be rechosen early. That is, in the first steps (up to $\tau\sim L$), the statistics will be conditioned by the

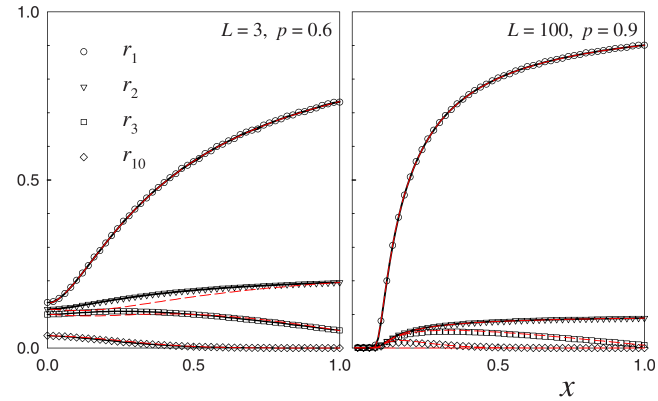

FIG. 5. (Color online) Integrands r_τ of the distribution of waiting times for values of τ indicated on the figure and two couples of parameters (L,p). Symbols correspond to numerical simulations, solid lines correspond to exact results, and dashed red lines correspond to the approximate analytical expressions.

memory of previous selections (aging regime).

Contrarily, for large enough τ, r_τ (and hence P_w) will gain the main contribution from the purely random (unconditioned) selection from the bulk of relatively small x values [as can be seen in Fig. 5, for large τ, $r_\tau(x)$ becomes a decreasing function of x]. This is expected to apply also when $p\to 1$ at any $\tau \gtrsim 2$.

For such cases, one can assume that there is an effective (or average) probability that task X is selected at some given step t (with $t>t_0+1$) and it is reasonably given by $f(x) = pP_0(x)+(1-p)/L$ as soon as $P_0=RO_1$ is the probability that there are no tasks with priorities higher than x (notice that it does not discern between old and new tasks). Then, although the exact probability does not factorize, one would approximately have

$$r_\tau(x) \simeq [1-r_1(x)][1-f(x)]^{\tau-2}f(x). \qquad (18)$$

A comparison between numerical and approximate forms of r_τ is displayed in Fig. 5. In particular, for $p=0$, $f(x)=(1-p)/L$ and it correctly yields the pure exponential decay $P_w(\tau)=(1-1/L)^{\tau-1}/L$ for all τ [3]. In the opposite case $p\to 1$ and using the approximation given by Eq. (7) for P_0 (with the aid of MAPLE), Eq. (15) leads to the asymptotic behavior,

$$P_w(\tau) \sim \frac{1}{\tau}\exp(-\tau/\tau_0), \qquad (19)$$

where $\tau_0 = 1/\ln[L/(L-1+p)] \sim L/(1-p)$.

This expression for the characteristic time τ_0 applies for any p, indicating that the characteristic exponential decay time τ_0 is shifted to larger τ when $p\to 1$ as well as when L increases.

Analytical predictions obtained by substituting Eq. (18) into Eq. (15) are compared to numerical simulations in Fig. 6. One observes that the approximate expression (18) accounts for the exponential cutoff in all cases as well as for the scaling regime in the limit $p\to 1$ although it fails to predict the $-3/2$ power law neatly observed in numerical simulations for $0<p<1$ as $L\to\infty$ (notice in the lower panel of Fig. 6 the deviation for $\tau \lesssim L$, leading to a spurious power-law exponent -2). This is due to the fact that the aging regime is overlooked by this approximation.

Let us remark that a $-3/2$ exponent is also found in classical queuing models with fluctuating length [3,5] and the return time distribution of a random walk is at its origin. We will solve next a closely related problem.

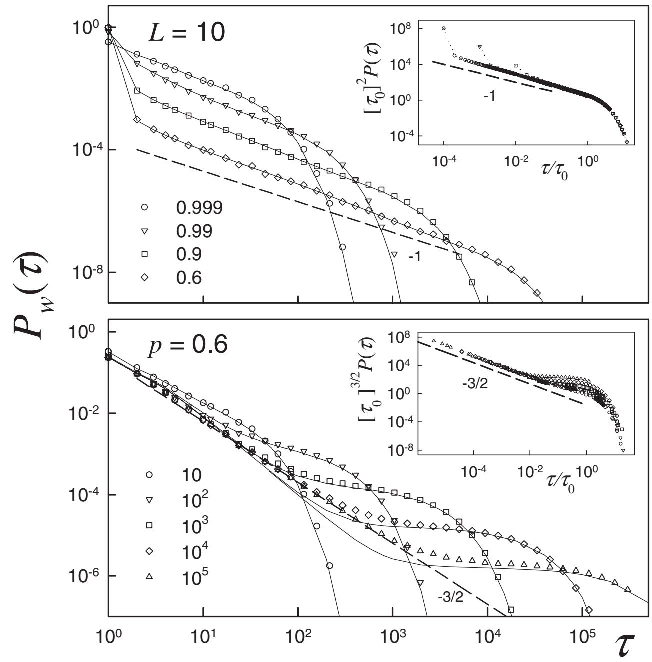

FIG. 6. Distributions of waiting times for fixed size $L=10$ and different values of p indicated on the figure (upper panel) and for fixed $p=0.6$ and increasing values of L indicated on the figure (lower panel), $R(x)=x$. Solid lines join the analytical results from Eqs. (15) and (18) and symbols correspond to numerical simulations of the BPQM. Insets: rescaled plots of the numerical histograms, where $\tau_0=1/\ln[L/(L-1+p)]$. Dashed lines with slopes -1 (upper inset) and $-3/2$ (lower inset) are drawn for comparison.

IV. AVALANCHES

We will consider now the events between two successive times when the number n of priorities above a given threshold x vanishes (avalanche). Avalanche duration is relevant in the present context as soon as it provides the duration of intervals in which there are queued tasks with priorities above a threshold to be executed. From the viewpoint of random walks, this is a first passage problem. Following the lines in Ref. [12], let us define $Q_{n,t}(x)$, the probability of having n values with priorities higher than x, given that the ongoing avalanche started t time units ago. $Q_{n,t}$ follows the same ME [Eqs. (1) and (2)] as $P_{n,t}$ does, except for $M_{0,1}=0$, and the initial conditions are $Q_{1,0}=1-x$ and $Q_{n,0}=0\ \forall\ n>1$. Thus, the probability that an avalanche, relative to threshold x, has duration t is

$$q_t(x) = xQ_{1,t-1}(x), \qquad (20)$$

where the factor x comes from the probability of hopping from $n=1$ to $n=0$.

The ME of Q_n could in principle be solved analytically through diverse standard methods [13,14]. Yet, we are interested in the large L limit. For large L and fixed n, the ME describes a simple biased random walk, with an absorbing boundary at $n=0$ and probabilities to step either to the right, to the left, or remain still, given by $m^+=(1-p)(1-x)$, $m^-=px$, and $m^0=1-m^+-m^-$, respectively. From this viewpoint, $q_t(x)$ is the probability that the first return to the origin occurs t time units after the avalanche started (that we take as being at $t=0$), while $Q_{1,t}(x)$ is the probability of reaching $n=1$ at time $t-1$, without having visited $n\le 0$. Thus, q just differs from Q_1 in appending the last step from 1 to 0. Therefore, for any n, $Q_{n,t}(x)$ can be found by first solving the unbounded random walk problem and then, to account for the absorbing barrier, by resorting to the reflection (or images) method (see, for instance, Ref. [16]). Moreover, if we are concerned with the asymptotic behavior, we can directly take advantage of the central limit theorem, which for the unbounded problem leads to the Gaussian approximation, $Q^0_{n,t|n_0}$

$\propto e^{-(n-n_0-ct)^2/(2\sigma^2)}$, where in our case the initial value is $n_0=1$ and $c\equiv m^+-m^-=1-x-p$ and σ^2 are the mean and variance of each single step of the random walk, respectively. Finally by the reflection principle for the biased walk [15], one has $Q_{n,t}=(1-x)[Q^0_{n,t|1}-m^-/m^+Q^0_{n,t|-1}]$, where the prefactor $1-x$ corresponds to the initial step from $n=0$ to $n=1$. Hence,

$$Q_{n,t}(x) \sim (1-x)\frac{e^{-(n-1-ct)^2/2\sigma^2 t}-\frac{m^-}{m^+}e^{-(n+1-ct)^2/2\sigma^2 t}}{\sqrt{2\pi\sigma^2 t}}, \tag{21}$$

where, notice, the dependence on x is also embodied in the mean and variance.

In the limit of large t, Eq. (21) readily leads to the asymptotic behaviors

$$q_t(x) = xQ_{1,t-1}(x) \sim \begin{cases} t^{-3/2} & \text{if } c=0 \\ t^{-1/2}e^{-c^2t/2\sigma^2} & \text{otherwise,} \end{cases} \tag{22}$$

that is, an exponential decay dominates the long-time decay in the biased cases; meanwhile, if $c=0$ (hence $x=1-p$), a power-law arises, in agreement with the well known results for a driftless random walk [16].

Figure 7 illustrates the scaling that comes up as $L\to\infty$, for any p at the critical threshold $x=1-p$. Exact results were obtained by numerical integration of the ME for Q_n and compared to the results of numerical simulations of the BPQM. Notice that the scaling region increases with L and shifts toward larger times as $p\to 1$.

Let us remark that there is a connection with the avalanches observed in a variant of the BPQM with lists of growing length (related to invasion percolation) which present the same critical scaling [9]. There is also a correspondence with the random annealed Bak-Sneppen model, where the same scaling is observed for any K at the critical threshold $x=1/K$ [12]. Notice that in the Bak-Sneppen model the transition matrix for the associated ME has $2K$ non-null diagonals, and a generic univoque relation between p and K does not emerge. However, concerning avalanches, the equivalence between both models occurs for $K=1/(1-p)$. Due to the threshold being an upper or lower bound in each case, that relation is complementary to $K=1/p$ which arises by identifying ratios of deterministic and/or random sites [1].

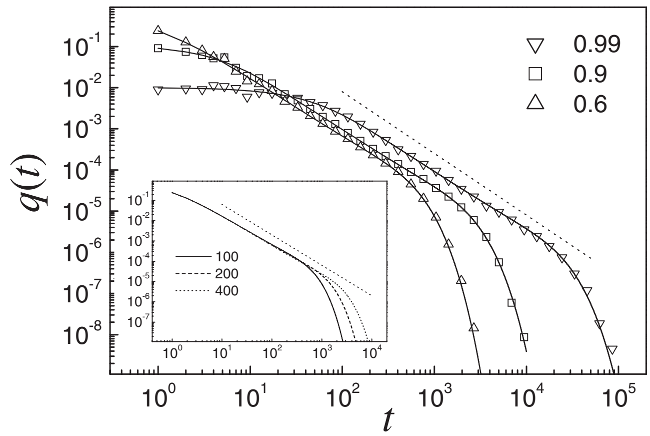

FIG. 7. Distribution of avalanche size $q(t)\equiv q_t(x=1-p)$ for $L=100$, different values of p indicated on the figure, and $R(x)=x$. Solid lines join the exact values and symbols correspond to numerical simulations of the BPQM. In the inset, exact results for $p=0.6$ and different values of L indicated on the figure are displayed. Dotted straight lines with slope $-3/2$ are drawn for comparison.

V. FINAL COMMENTS

Summarizing, we obtained analytical results for the BPQM with queues of arbitrary length. Exact expressions were shown to be in agreement with the outcomes of numerical simulations of the dynamics. Progress has still to be made to obtain the exact waiting time distribution for arbitrary L that besides the purely exponential decay (at $p=0$) and the power-law decay with unit exponent (at $p\to 1$) displays a 3/2 power-law regime when $L\to\infty$. However, an approximate expression has been shown to account for most of the distribution traits. Moreover, we have shown that avalanches, at the critical threshold $x=1-p$, constitute another scale-free feature of the BPQM for $p>0$. Besides the main applications here illustrated, the present results may allow us to estimate many other relevant statistical quantities of the BPQM and can be extended to other queuing systems. Furthermore, our exact results set the basis to further explore the correspondence between BPQM and other related models.

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