Quantum transport in a periodic symmetric potential of a driven quantum system

Satyabrata Bhattacharya,[1] Pinaki Chaudhury,[2] Sudip Chattopadhyay,[1,*,†] and Jyotipratim Ray Chaudhuri[3,*,‡]
[1]*Department of Chemistry, Bengal Engineering and Science University, Shibpur, Howrah 711103, India*
[2]*Department of Chemistry, University of Calcutta, Kolkata 700009, India*
[3]*Department of Physics, Katwa College, Katwa, Burdwan 713130, India*
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A system-reservoir nonlinear coupling model has been proposed to the situation when the system is driven externally by a random force and the associated bath is kept in thermal equilibrium, in an attempt to put forth a microscopic approach to quantum state-dependent diffusion and multiplicative noises in terms of a quantum Langevin equation in the overdamped limit (quantum Smoluchowski equation). We then obtain the analytical expression for phase induced quantum current in a periodic potential when the external noise has finite correlation time and explore the dependence of the current on various parameters related to the external noise, for example, the noise strength.

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I. INTRODUCTION

For a heavy particle immersed in a fluid of lighter particles, Brownian motion is the fact of the former and is the prototype of a dissipative system coupled to a thermal bath with infinitely many degrees of freedom. In contrast to classical Brownian motion, where right from the beginning, the work by Einstein and Smoluchowski had allowed us to consider both weak and strong friction; for a long time, the quantum-mechanical theory could handle the limit of weak dissipation only. In this case, the interaction between the particle and the bath can be treated perturbatively and one can derive a master equation for the reduced matrix of the particle [1]. This approach is based on so-called Born and Markovian approximations and has been very successful in quite a number of fields [1,2]. However, to study the nonequilibrium transport like in ratchet potential, the analysis in the opposite domain, i.e., in the strong friction limit is necessary. Quantum Brownian motion in the strong friction limit is much more involved than its classical analog since, in general, tractable equations of motion do not exist. Some progress has been made recently along this line [3] and opens the door to study the dynamics of strong condensed phase systems at low temperature.

The basic underlying physics of Brownian motor is that by extracting energy fluctuations, Brownian motors operate far from thermal equilibrium and generate work against external loads [4–6]. They present the physical analog of biomolecular motor that establishes intramolecular transport and controls the motion and sensation in cells [6,7]. The molecular-sized physical engines, however, depending on the nature of particles to be transported and their operating temperature, necessitate a description that accounts for the quantum features such as tunneling, decoherence, etc. For this class of quantum Brownian motors, recent theoretical studies [3,8–10] reveal that the transport becomes distinctly modified as compared to its classical counterpart. In particular, innate quantum effects such as tunneling induced current reversal, quantum Brownian heat engines, quantum thermodynamic machines [11], stochastic heat pump [12], transport in Bose-Einstein condensation (BEC) without any bias, etc. have been reported with recent experiments [13–15]. In the classical regime, the transport of macroscopic objects is well elaborated in literature and well established [4–7,16]. It has been observed that thermal diffusion in a periodic potential has a prominent role in various systems such as Josephsons's junction [17,18], system for diffusion in crystal surfaces [19], noisy limit oscillators [20], etc. There is a renewed interest in recent times in the field of transport of Brownian particles moving in a periodic potential [5] with special emphasis on coherent transport and giant diffusion [21]. These studies have been motivated in part by an attempt to understand the mechanism of movement of protein motors in biological systems [22]. Several physical models have been proposed to understand the transport phenomena in such systems, such as vibrational ratchet [23], rocking ratchet [24], flashing ratchets [25], etc. Such ratchet models have a wide range of applications in biology and nanoscale systems [26] because of their wonderful success in exploring experimental observations on biochemical motors, active in muscle contraction [27], observation on directed transport in photovoltaic and photoreflective materials [28], etc. In all the above models, the potential is taken to be asymmetric in space. One can also obtain a unidirectional current in presence of spatially symmetric potential. For such nonequilibrium systems one requires time asymmetric random forces [29] or space-dependent diffusion [30–32]. The common feature of an overwhelming majority of the ratchet systems is that the system is thermodynamically closed (in presence of external ratchet potential and thermal equilibrium bath), which means that the noise of the medium is of internal origin so that the dissipation and fluctuations get related through the fluctuation-dissipation relation. However, in a number of situations, the system is thermodynamically open, i.e., when the system is driven by an external noise which is independent of the system's characteristic damping [33] or, when the reservoir is modulated by external noise agency [34]. The system-bath model with random driving force was

*Corresponding author.
†sudip_chattopadhyay@rediffmail.com
‡jprc_8@yahoo.com

employed by Mollow and Miller [35] forty years ago in the context of quantum optics. The distinctive feature of the dynamics in this case is the absence of any fluctuation-dissipation relation. While in the former case a zero current steady-state situation is characterized by an equilibrium Boltzmann distribution, the corresponding situation in the latter case is defined only by a steady-state condition, if attainable. It may, therefore, be anticipated that the independence of fluctuations and dissipation tend to make the steady-state distribution function depend on the strength and correlation time of external noise as well as on the directed transport of the system.

In contrast to the success in the classical regime, the quantum properties of directed transport are only partially elaborated in such a motor system [10]. One of the main goals of stochastic method is to provide accurate theoretical models to compute quantum current in recent times. Challenges arise in the quantum region because the transport usually strongly depends on the mutual interplay of pure quantum effects such as tunneling and particle wave interference with dissipation process, nonequilibrium fluctuation, and external driving. The present state of the art of the theory is characterized by specific restrictions such as an adiabatic driving regime, a tight-binding description, a semiclassical analysis, or combinations thereof [8,36]. As such, the study of quantum transport is far from being complete and there exists an emergent requirement for further developments. The analytic study of quantum Brownian transport for noise-driven open system, even in a symmetric periodic potential presents such a challenge. This goal has already been addressed in one of our pervious works [34], where the underlying quantum dynamics was modeled by a recently proposed ingenious quantum generalization of Langevin dynamics [8]. The issue of quantum Brownian transport was addressed [34] by proposing a system-reservoir nonlinear coupling model when the associated bath is not in thermal equilibrium, and is rather modulated by an external noise to present a microscopic approach to quantum-state-dependent diffusion and multiplicative noise in terms of quantum Langevin description. Consequently, we explored the possibility of observing the phase-induced quantum current. Keeping in mind the fact that the direct driving of a system, instead of the reservoir may be experimentally realized more easily and is well documented in literature [33], we extend here our methodology to address the issue of quantum transport where the system is modulated by an external noise. We thus allow the Brownian particle in a potential field to be driven by external stationary and Gaussian noise fluctuations. Then, following the recently developed methodology by Ray and co-workers [8], starting from a microscopic Hamiltonian picture of an external-noise-driven quantum system, which is nonlinearly coupled with a harmonic bath, we derive the *C*-number analog of the quantum Langevin equation for the system mode in the overdamped limit. The general expression for the stationary quantum current is then obtained for δ-correlated external noise where the underlying potential is taken to be sinusoidal. We then extend our development for exponentially correlated external noise. Consequently, the various characteristics of the quantum current are explored.

II. MODEL AND THE C-NUMBER LANGEVIN EQUATION

A particle of unit mass is connected to a heat bath comprising of a set of harmonic oscillators of unit mass with frequency set $\{\omega_j\}$. The system is driven externally by a random force $\epsilon(t)$. The total system-bath Hamiltonian can be written as

$$\hat{H} = \frac{\hat{p}^2}{2} + V(\hat{q}) + \sum_{j=1}^{N}\left[\frac{\hat{p}_j^2}{2} + \frac{1}{2}\left(\omega_j\hat{x}_j - \frac{c_j}{\omega_j}f(\hat{q})\right)^2\right] - \hat{q}\epsilon(t), \tag{1}$$

where $\hat{q}$ and $\hat{p}$ are the coordinate and momentum operators of the system, respectively, and $\{\hat{x}_j,\hat{p}_j\}$ are the set of coordinate and momentum of the bath oscillators. The potential function $V(\hat{q})$ is due to external force field exerted on the system. The system is coupled to the heat bath oscillators nonlinearly through the coupling function $f(\hat{q})$ and c_j is the coupling constant. The coordinate and momentum operators follow the commutation relation $[\hat{q},\hat{p}]=i\hbar$; and $[\hat{x}_j,\hat{p}_k]=i\hbar\delta_{jk}$. $\epsilon(t)$ is an external classical noise, nonthermal in nature, with the statistical properties that $\epsilon(t)$ is a Gaussian noise process and

$$\langle\epsilon(t)\rangle = 0; \quad \langle\epsilon(t)\epsilon(t')\rangle = 2D\delta(t-t'), \tag{2}$$

where D is the strength of the noise and the average is taken over each realization of $\epsilon(t)$. By the elimination of bath variables, one obtains the generalized operator Langevin equation for the system variable as

$$\dot{\hat{q}} = \hat{p},$$

$$\dot{\hat{p}} = -V'(\hat{q}(t)) - f'(\hat{q}(t))\int_0^t \gamma(t-t')f'(\hat{q}(t'))\hat{p}(t')dt' + f'(\hat{q}(t))\hat{\eta}(t) + \epsilon(t), \tag{3}$$

where the noise operator $\hat{\eta}(t)$ and the memory kernel $\gamma(t)$ are given by

$$\gamma(t) = \sum_{j=1}^{N}\frac{c_j^2}{\omega_j^2}\cos(\omega_j t), \tag{4}$$

$$\hat{\eta}(t) = \sum_{j=1}^{N}\left\{\left[\frac{\omega_j^2}{c_j}\hat{x}_j(0) - f(\hat{q}(0))\right]\frac{c_j^2}{\omega_j^2}\cos(\omega_j t) + \frac{c_j}{\omega_j}\hat{p}_j(0)\sin(\omega_j t)\right\}. \tag{5}$$

The noise properties of $\hat{\eta}(t)$ can be derived by using suitable canonical thermal distribution of bath coordinates and momenta operators at $t=0$ to obtain

$$\langle\hat{\eta}(t)\rangle_{QS} = 0,$$

$$\frac{1}{2}\langle\hat{\eta}(t)\hat{\eta}(t')+\hat{\eta}(t')\hat{\eta}(t)\rangle_{\rm QS} = \frac{1}{2}\sum_{j=1}^{N}\frac{c_j^2}{\omega_j^2}\hbar\omega_j\coth(\hbar\omega/2k_BT)\cos\omega_j(t-t'). \quad (6)$$

Here, $\langle . \rangle_{\rm QS}$ implies quantum statistical average on bath degrees of freedom and is defined as

$$\langle\hat{O}\rangle_{\rm QS} = \frac{{\rm Tr}[\hat{O}\exp(-\hat{H}_B/k_BT)]}{{\rm Tr}[\exp(-\hat{H}_B/k_BT)]}$$

for any bath operator $\hat{O}(\hat{x}_j,\hat{p}_j)$ and

$$H_B = \sum_{j=1}^{N}\left[\frac{\hat{p}_j^2}{2}+\frac{1}{2}\left(\omega_j\hat{x}_j-\frac{c_j}{\omega_j}f(\hat{q})\right)^2\right] \text{ at } t=0.$$

Now to construct the C-number Langevin equation equivalent to Eq. (3), following a recently developed methodology of Ray *et al.* [8], we carry out a quantum-mechanical averaging of the operator Eq. (3) to get

$$\langle\dot{\hat{q}}\rangle_{\rm Q} = \langle\hat{p}\rangle_{\rm Q},$$

$$\langle\dot{\hat{p}}\rangle_{\rm Q} = -\langle V'(\hat{q})\rangle_{\rm Q} - \langle f'(\hat{q}(t))\int_0^t dt'\,\gamma(t-t') \times f'(\hat{q}(t'))\hat{p}(t')\rangle_{\rm Q} + \langle f'(\hat{q})\hat{\eta}(t)\rangle_{\rm Q} + \epsilon(t), \quad (7)$$

where the quantum-mechanical average $\langle . \rangle_{\rm Q}$ is taken over the initial product separable quantum states of the particle and the bath oscillators at $t=0$, $|\phi\rangle\{|\alpha_j\rangle\}$; $j=1,2,\cdots N$. Here, $|\phi\rangle$ denotes any arbitrary initial state of the system and $\{|\alpha_j\rangle\}$ corresponds to the initial coherent state of the bath oscillators. $\langle\hat{\eta}(t)\rangle$ is now a classical-like noise term, which because of the mechanical averaging, in general is a nonzero number and is given by

$$\langle\hat{\eta}(t)\rangle_{\rm Q} = \sum_{j=1}^{N}\left[\left\{\frac{\omega_j^2}{c_j}\langle\hat{x}_j(0)\rangle_{\rm Q} - \langle f(\hat{q}(0))\rangle_{\rm Q}\right\}\frac{c_j^2}{\omega_j^2}\cos(\omega_j t) + \frac{c_j}{\omega_j}\langle\hat{p}_j(0)\rangle_{\rm Q}\sin(\omega_j t)\right]. \quad (8)$$

It should be pointed out here that we have considered the uncorrelated system and reservoir at $t=0$ and thereby employed the so-called factorization assumption in what follows, though the factorization condition is strictly valid for Markovian case. However, this is a widely used assumption in the literature [1,37], particularly in the context of quantum optics and condensed-matter physics, and we consider the Markovian case at the end of the day.

To realize $\langle\hat{\eta}(t)\rangle_{\rm Q}$ as an effective C-number noise, we now introduce the ansatz [8] that the momenta $\langle\hat{p}_j(0)\rangle_{\rm Q}$ and the shifted coordinates $\{\frac{\omega_j^2}{c_j}\langle\hat{x}_j(0)\rangle_{\rm Q}-\langle f(\hat{q}(0))\rangle_{\rm Q}\}$ of the bath oscillators are distributed according to a canonical distribution of Gaussian form as

$$\mathcal{P}_j\left(\left\{\frac{\omega_j^2}{c_j}\langle\hat{x}_j(0)\rangle_{\rm Q}-\langle f(\hat{q}(0))\rangle_{\rm Q}\right\},\langle\hat{p}_j(0)\rangle_{\rm Q}\right) = \mathcal{N}\exp\left\{-\frac{\left[\langle\hat{p}_j(0)\rangle_{\rm Q}^2+\frac{c_j^2}{\omega_j^2}\left\{\frac{\omega_j^2}{c_j}\langle\hat{x}_j(0)\rangle_{\rm Q}-\langle f(\hat{q}(0))\rangle_{\rm Q}\right\}^2\right]}{2\hbar\omega_j\left(\bar{n}_j(\omega_j)+\frac{1}{2}\right)}\right\} \quad (9)$$

so that for any quantum-mechanical mean value, $\langle\hat{O}\rangle_{\rm Q}$ of the bath operators, its statistical average $\langle . \rangle_{\rm S}$ is

$$\langle\langle\hat{O}\rangle_{\rm Q}\rangle_{\rm S} = \int\left[\langle\hat{O}\rangle_{\rm Q}\mathcal{P}_j d\left\{\frac{\omega_j^2}{c_j}\langle\hat{x}_j(0)\rangle_{\rm Q}-\langle f(\hat{q}(0))\rangle_{\rm Q}\right\} \times d\langle\hat{p}_j(0)\rangle_{Q}\right]. \quad (10)$$

Here, $\bar{n}_j(\omega_j)$ indicates the average thermal photon number of the jth oscillator at temperature T and $\bar{n}_j(\omega_j) = [\exp(\hbar\omega_j/k_{\rm B}T)-1]^{-1}$ and $\mathcal{N}$ is the normalization constant. The distribution $\mathcal{P}_j$ given by Eq. (9) and the definition of statistical average imply that the C-number noise $\langle\hat{\eta}(t)\rangle_{\rm Q}$ given by Eq. (8) must satisfy

$$\langle\langle\hat{\eta}(t)\rangle_{\rm Q}\rangle_{\rm S} = 0$$

and

$$\frac{1}{2}\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_{\rm Q}\rangle_{\rm S} = \frac{1}{2}\sum_{j=1}^{N}\frac{c_j^2}{\omega_j^2}\hbar\omega_j\coth\left(\frac{\hbar\omega_j}{2k_BT}\right)\cos\omega_j(t-t'), \quad (11)$$

which are equivalent to Eq. (8).

Now to obtain a finite result in the continuum limit, the coupling function $c_j=c(\omega)$ is chosen as

$$c(\omega) = \frac{c_0\omega}{\sqrt{\tau_c}}.$$

With this choice $\gamma(t)$ reduces to the following form:

$$\gamma(t) = \frac{c_0^2}{\tau_c}\int_0^\infty d\omega \mathcal{D}(\omega)\cos(\omega t), \tag{12}$$

where c_0 is some constant and $\omega_c = \frac{1}{\tau_c}$ is the cutoff frequency of the bath oscillators. τ_c may be regarded as the correlation time of the bath and $\mathcal{D}(\omega)$ is the density of modes of the heat bath which is assumed to be Lorentzian,

$$\mathcal{D}(\omega) = \frac{2}{\pi}\frac{1}{\tau_c(\omega^2 + \tau_c^{-2})}.$$

With these forms of $\mathcal{D}(\omega)$ and $c(\omega)$, $\gamma(t)$ takes the form

$$\gamma(t) = \frac{c_0^2}{\tau_c}\exp(-t/\tau_c) = \frac{\Gamma}{\tau_c}\exp(-t/\tau_c), \tag{13}$$

where $c_0^2 = \Gamma$. For $\tau_c \to 0$ Eq. (13) reduces to

$$\gamma(t) = 2\Gamma\delta(t)$$

and the noise correlation function (12) becomes

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S = \frac{\Gamma}{2\tau_c}\int_0^\infty d\omega\hbar\omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) \times\cos \omega(t-t')\mathcal{D}(\omega). \tag{14}$$

Equation (14) is an exact expression for two-time correlation. We now make the following assumption. As $\hbar\omega \coth(\frac{\hbar\omega}{2k_BT})$ is a much more smooth function of ω, at least for not too low temperature, the integral in Eq. (14) can be approximated as [10]

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S \simeq \frac{\Gamma}{2\tau_c}\hbar\omega_0 \coth(\hbar\omega_0/2k_BT)\int_0^\infty d\omega \times\cos \omega(t-t')\mathcal{D}(\omega),$$

where ω_0 is the average bath frequency. Thus, we have in the limit $\tau_c \to 0$,

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S = 2D_0\delta(t-t'), \tag{15}$$

where

$$D_0 = \frac{\Gamma}{2}\hbar\omega_0\left(\bar{n}(\omega_0) + \frac{1}{2}\right), \tag{16}$$

with $\bar{n}(\omega_0) = [\exp(\hbar\omega_0/k_BT) - 1]^{-1}$. It is pertinent to mention here that the form of D_0, given by Eq. (16) can be obtained if and only if one extracts the $\hbar\omega \coth(\hbar\omega/2k_BT)$ term out of the integral. Clearly, at very high temperature, $2k_BT \gg \hbar\omega$, the integrand in Eq. (14) reduces to $2k_BT \cos \omega(t-t')\mathcal{D}(\omega)$ and in this case Eq. (15) is strictly valid with $D_0 = \frac{\Gamma k_BT}{2}$, which is the classical result. On the other hand, with our assumption made below Eq. (14), D_0 will be given by Eq. (16). Here also $k_BT \gg \hbar\omega_0$, and D_0 reduces to its classical expression namely, $D_0 = \frac{\Gamma k_BT}{2}$. Here, it is important to note that our above assumption is valid only at high temperature. In this regard our development does not account for the dynamics which are fully quantum mechanical. Nevertheless, the ansatz (9), which is the canonical thermal Wigner distribution function for a shifted harmonic oscillator and always remains positive definite, contains some quantum information of the bath comprised of a set of quantum-mechanical harmonic oscillators.

Now writing $q = \langle\hat{q}\rangle_Q$ and $p = \langle\hat{p}\rangle_Q$, we can rewrite Eq. (7) as

$$\dot{q} = p,$$

$$\dot{p} = -\langle V'(\hat{q})\rangle_Q - \Gamma\langle[f'(\hat{q})]^2\hat{p}\rangle_Q + \langle f'(\hat{q})\rangle_Q\eta(t) + \epsilon(t), \tag{17}$$

where $\eta(t) = \langle\hat{\eta}(t)\rangle_Q$ and is a classical like noise term. In writing Eq. (17) we have made use of the fact that the correlation time of the reservoir is very short, i.e., $\tau_c \to 0$.

We now add $V'(q)$, $\Gamma[f'(q)]^2p$, and $f'(q)\eta(t)$ on both sides of Eq. (15) and rearrange it to obtain

$$\dot{q} = p,$$

$$\dot{p} = -V'(q) + Q_V - \Gamma[f'(q)]^2p + Q_1 + f'(q)\eta(t) + Q_2 + \epsilon(t), \tag{18}$$

where

$$Q_V = V'(q) - \langle V'(\hat{q})\rangle_Q,$$

$$Q_1 = \Gamma[f'(q)]^2p - \langle[f'(\hat{q})]^2\hat{p}\rangle_Q,$$

$$Q_2 = \eta(t)[\langle f'(\hat{q})\rangle_Q - f'(\hat{q})]. \tag{19}$$

Referring to the quantum nature of the system in the Heisenberg picture we now write the system operator $\hat{q}$ and $\hat{p}$ as

$$\hat{q} = q + \delta\hat{q},$$

$$\hat{p} = p + \delta\hat{p}, \tag{20}$$

where $q(=\langle\hat{q}\rangle_Q)$ and $p(=\langle\hat{p}\rangle_Q)$ are the quantum-mechanical mean values and $\delta\hat{q}$ and $\delta\hat{p}$ are the operators which are quantum fluctuations around the respective mean values. By construction $\langle\delta\hat{q}\rangle_Q = \langle\delta\hat{p}\rangle_Q = 0$ and they also follow the usual commutation relation $[\delta\hat{q}, \delta\hat{p}] = i\hbar$. Using Eq. (20) in $V'(\hat{q})$, $[f'(\hat{q})]^2\hat{p}$, and in $f'(\hat{q})$, a Taylor-series expansion in $\delta\hat{q}$ around q, Q_V, Q_1, and Q_2 can be obtained as

$$Q_V = -\sum_{n\geq 2}\frac{1}{n!}V^{n+1}(q)\langle\delta\hat{q}^n\rangle_Q, \tag{21}$$

$$Q_1 = -\Gamma[2pf'(q)Q_f + pQ_3 + 2f'(q)Q_4 + Q_5], \tag{22}$$

and

$$Q_2 = \eta(t)Q_f, \tag{23}$$

where

$$Q_f = \sum_{n\geq 2}\frac{1}{n!}f^{n+1}(q)\langle\delta\hat{q}^n\rangle_Q,$$

$$Q_3 = \sum_{m\geq 1}\sum_{n\geq 1}\frac{1}{m!}\frac{1}{n!}f^{m+1}(q)f^{n+1}(q)\langle\delta\hat{q}^m\delta\hat{q}^n\rangle_Q,$$

$$Q_4 = \sum_{n\geq 1} \frac{1}{n!} f^{n+1}(q)\langle \delta\hat{q}^n \delta\hat{p}\rangle_Q,$$

$$Q_5 = \sum_{m\geq 1}\sum_{n\geq 1} \frac{1}{m!}\frac{1}{n!} f^{m+1}(q) f^{n+1}(q)\langle \delta\hat{q}^m \delta\hat{q}^n \delta\hat{p}\rangle_Q. \quad (24)$$

From the above expression it is evident that Q_V represents quantum correction due to nonlinearity of the system potential, Q_1 and Q_2 represent quantum corrections due to nonlinearity of the system-bath coupling function. Using Eqs. (21)–(23), we arrive at the dynamical equations for system variable from Eq. (18),

$$\dot{q} = p,$$

$$\dot{p} = - V'(q) + Q_V - \Gamma[f'(q)]^2 p - 2\Gamma p f'(q) Q_f - \Gamma p Q_3 - 2\Gamma f'(q) Q_4 - \Gamma Q_5 + f'(q)\eta(t) + Q_f \eta(t) + \epsilon(t). \quad (25)$$

The above equations contain a quantum multiplicative noise term $Q_f\eta(t)$ in addition to the usual classical contribution $f'(q)\eta(t)$. The classical limit, apart from the term $\epsilon(t)$, was obtained earlier by Lindenberg and Seshadri [38].

In the overdamped limit, the adiabatic elimination of the fast variable is usually done by simply putting $\dot{p}=0$. This adiabatic elimination provides the correct equilibrium distribution only when the dissipation is state independent. But for state-dependent dissipation or when the fluctuation is state dependent, which is a manifestation of the nonlinear nature of system-bath coupling function $f(q)$, the conventional adiabatic elimination of fast variable in overdamped limit does not provide correct result. To obtain a correct equilibrium distribution, an alternative approach proposed by Sancho *et al.* [39] must be followed to obtain the dynamical equation of motion for position coordinate in the case of state-dependent dissipation. The methodology of Sancho *et al.* [39] consists of a systematic expansion of the relevant variables in powers of Γ^{-1} and rejection of the terms smaller than $O(\Gamma^{-1})$. We follow the same procedure in our context. In this limit, the transient correction terms Q_4 and Q_5 do not affect the dynamics of the position, which varies over a much slower time scale in the overdamped limit [8]. So the equations governing the dynamics of the system variables are

$$\dot{q} = p,$$

$$\dot{p} = - V'(q) + Q_V - \Gamma h(q) p + g(q)\eta(t) + \epsilon(t), \quad (26)$$

where

$$h(q) = [f'(q)]^2 + 2f'(q)Q_f + Q_3,$$

$$g(q) = f'(q) + Q_f. \quad (27)$$

The function $g(q)$ arises due to nonlinearity of the system-bath coupling function $f(q)$. One can now easily identify Eq. (26) as the C-number analog of the quantum Langevin equation equivalent to the operator Langevin equation, Eq. (3), where $\Gamma h(q)$ is the state-dependent damping, $\eta(t)$ is the thermal noise and $\epsilon(t)$ is the external nonthermal noise. Following the method of Sancho *et al.*, we obtain the Fokker-Planck-Smoluchowski equation in position space corresponding to the Langevin Eq. (26),

$$\frac{\partial P(q,t)}{\partial t} = \frac{\partial}{\partial q}\left[\frac{V'(q) - Q_V}{\Gamma h(q)}\right] P(q,t) + D_0 \frac{\partial}{\partial q}\left[\frac{g(q) g'(q)}{\Gamma h^2(q)}\right] P(q,t) + D_0 \frac{\partial}{\partial q}\left[\frac{g(q)}{\Gamma h(q)} \frac{\partial}{\partial q} \frac{g(q)}{\Gamma h(q)}\right] P(q,t) + D \frac{\partial}{\partial q}\left[\frac{1}{\Gamma h(q)} \frac{\partial}{\partial q} \frac{1}{\Gamma h(q)}\right] P(q,t). \quad (28)$$

According to Stratonovich description, the Langevin equation corresponding to the above Fokker-Planck-Smoluchowski equation,

$$\dot{q} = - \frac{V'(q) - Q_V}{\Gamma h(q)} - D_0 \frac{g(q) g'(q)}{\Gamma h^2(q)} + \frac{g(q)}{\Gamma h(q)}\eta(t) + \frac{\epsilon(t)}{\Gamma h(q)}. \quad (29)$$

Equation (29) is the C-number quantum Langevin equation for multiplicative noise with state-dependent dissipation in the overdamped limit when the system is externally driven by a fluctuating δ-correlated Gaussian noise and thus can be regarded as the Langevin equation for open quantum system.

III. STATIONARY CURRENT

Equation (28) can be written in a more compact form as

$$\frac{\partial P(q,t)}{\partial t} = \frac{\partial}{\partial q}\frac{1}{\Gamma h(q)}\left[V'(q) - Q_V + \frac{D_0}{\Gamma}\frac{\partial}{\partial q}\frac{g^2(q)}{h(\omega)} + D \frac{\partial}{\partial q}\frac{1}{h(q)}\right] P(q,t). \quad (30)$$

Equation (30) is the required Smoluchowski equation corresponding to the quantum Langevin equation where the noise is multiplicative and the dissipation is state dependent and where the system-reservoir combination is not thermodynamically closed; rather the system is externally driven by a noise.

Under stationary condition, $\frac{\partial P}{\partial t}=0$ and Eq. (30) reduces to

$$\frac{d}{dq}\left[\left(\frac{D_0}{\Gamma}\frac{g^2(q)}{h(q)} + \frac{D}{h(q)}\right) P_{st}(q)\right] + [V'(q) - Q_V] P_{st}(q) = 0 \quad (31)$$

from which we have the stationary probability distribution in the overdamped limit as

$$P_{st}(q) = \frac{N}{R(q)} \exp\left[- \int_0^q \frac{V'(q) - Q_V}{R(q)} dq\right], \quad (32)$$

where N is the normalization constant and

$$R(q) = \left[\frac{D_0}{\Gamma} g^2(q) + D\right] / h(q). \quad (33)$$

It is easy to observe that all quantum corrections Q_V etc. vanish in the classical regime, where the quantum fluctuations around their mean value is zero. When the exact noise is absent and for linear system-reservoir coupling, the above expression for stationary probability density reduces to the usual Boltzmann distribution in the classical limit

$$P_{st}(q) = N \exp\left[-\frac{V(q)}{k_B T}\right].$$

The stationary distribution (32) is thus essentially a generalization of Boltzmann distribution for state-dependent dissipation in a quantum open system. The space-dependent dissipation stems from the inhomogeneity of the medium and can be described phenomenologically. In inhomogeneous medium, the diffusion term for Brownian particle may have several structure. The microscopic origin of these terms, in general, do not have a common Hamiltonian. Thus, the physics of diffusion in inhomogeneous media is somewhat model dependent. Also, the various forms notwithstanding, the generalization of Boltzmann factor $\exp[-V(q)/k_BT]$ for state-dependent dissipation in the steady-state assumes a common structure

$$P_{st}(q) \sim \exp[-\phi(q)]$$

with

$$\phi(q) = \int_0^q \frac{\tilde{V}'(q)}{R(q)} dq,$$

$\tilde{V}$ is the potential field and $R(q)$ is the state-dependent diffusion term. In our context, the effective potential $\phi(q)$ becomes

$$\phi(q) = \int_0^q \frac{V'(q) - Q_V}{R(q)} dq, \tag{34}$$

where $R(q)$ is given by Eq. (33). The above structure of the diffusion term and steady-state distribution implies that the effective potential $\phi(q)$ is nonlocal in space. The generality in the structure of $\phi(q)$ is such that it may include the spatial variation in temperature, diffusion, or drift coefficient as specific cases as considered by several authors [40]. In the Langevin scheme of description, the state-dependent diffusion has received attention under multiplicative noises [37]. The microscopic origin of multiplicative noise within the framework of standard paradigm of system-reservoir Hamiltonian that includes a variety of model calculations lies in the nonlinear coupling between the system and the bath coordinates, which leads to nonlinear dissipation. A thermodynamically consistent approach was put forward by Lindenberg and co-worker [38] and exact Fokker-Planck equation for time- and space-dependent friction was derived by Pollak *et al.* [41]. Tanimura and co-workers [42] extensively used nonlinear coupling in modeling elastic and inelastic relaxation mechanisms and their interplay in vibrational and Raman spectroscopy. The role of inhomogeneous dissipation in reducing quantum decay rate has also been explored in the recent past [43]. Along with the formal developments, the theories of multiplicative noise and state-dependent dissipation have found wide applications in several areas like stochastic resonance [44], signal processing [45], noise-induced transitions [46], etc. We extend the above studies to a thermodynamically open system in the context of directed transport in the quantum regime. The thermodynamical openness of the system is incorporated by subjecting the system to an external noise.

From Eq. (30) the stationary current can be obtained as

$$J = -\frac{1}{\Gamma h(q)}\left\{V'(q) - Q_V + \frac{d}{dq}\left[\frac{D_0}{\Gamma}\frac{g^2(q)}{h(q)} + \frac{D}{h(q)}\right]\right\}P_{\rm st}(q). \tag{35}$$

Integrating Eq. (35) we obtain the expression for the stationary probability density function in terms of stationary current as

$$P_{\rm st}(q) = \frac{e^{-\phi(q)}h(q)}{\frac{D_0}{\Gamma}g^2(q) + D}\left[\frac{\frac{D_0}{\Gamma}g^2(0) + D}{h(0)}P_{\rm st}(0) - J\int_0^q h(q')e^{\phi(q')}dq'\right], \tag{36}$$

where

$$\phi(q) = \int_0^q \frac{[V'(q') - Q_V]h(q')}{\frac{D_0}{\Gamma}g^2(q') + D} dq' \tag{37}$$

is the generalized potential in which the particle is moving. We now consider symmetric periodic potential with periodicity 2π and periodic derivative of coupling function with the same periodicity, i.e., $V(q+2\pi)=V(q)$ and $f'(q+2\pi)=f'(q)$. Consequently, Q_V, $h(q)$, and $g(q)$ are also periodic functions of q with period 2π. Now applying the periodic boundary condition on $P_{st}(q)$, i.e., $P_{st}(q+2\pi) = P_{st}(q)$, we have from Eq. (36)

$$\frac{\frac{D_0}{\Gamma}g^2(0) + D}{h(0)}P_{st} = J[1 - e^{\phi(2\pi)}]^{-1}\int_0^{2\pi} h(q)e^{\phi(q)}dq. \tag{38}$$

By applying the normalization condition on stationary probability distribution given by

$$\int_0^{2\pi} P_{st}(q)dq = 1,$$

we get from Eq. (36)

$$\int_0^{2\pi} \frac{\Gamma e^{-\phi(q)}h(q)}{D_o g^2(q) + \Gamma D} \times \left[\frac{\frac{D_0}{\Gamma}g^2(0) + D}{h(0)}P_{st}(0) - J\int_0^q h(q')e^{\phi(q')}dq'\right]dq = 1. \tag{39}$$

Now eliminating $[(\frac{D_0}{\Gamma}g^2(0)+D)/h(0)]P_{st}(0)$ from Eqs. (38)

and (39) one obtains the stationary current as

$$J=[1-e^{\phi(2\pi)}]/\left[\left(\left(\int_0^{2\pi}\frac{\Gamma e^{-\phi(q)}h(q)}{D_0g^2(q)+\Gamma D}dq\int_0^{2\pi}h(q')e^{\phi(q')}dq'\right)-(1-e^{\phi(2\pi)})\int_0^{2\pi}\times\left(\frac{\Gamma e^{-\phi(q)}h(q)}{D_0g^2(q)+\Gamma D}\int_0^{q}h(q')e^{\phi(q')}dq'\right)dq\right]. \quad (40)$$

IV. EXPONENTIALLY CORRELATED EXTERNAL NOISE AND STATIONARY CURRENT

At this point we consider that the random external force is exponentially correlated and Gaussian in nature,

$$\langle\epsilon(t)\rangle=0;\quad \langle\epsilon(t)\epsilon(t')\rangle=\frac{D_e}{\tau_e}\exp\left(-\frac{|t-t'|}{\tau_e}\right), \quad (41)$$

where D_e is the strength and τ_e is the correlation time, respectively, of $\epsilon(t)$. Proceeding as in Sec. II, we arrive at the same Eq. (26),

$$\dot{q}=p,$$

$$\dot{p}=-V'(q)+Q_V-\Gamma h(q)+g(q)\eta(t)+\epsilon(t) \quad (42)$$

with $h(q)$ and $g(q)$ being defined by Eq. (27). The only difference is that here the additive nonthermal noise is not δ correlated, and direct use of the prescription of Sancho [39] cannot be applied. To construct the Fokker-Planck-Smulochowski equation valid in the overdamped case for state-dependent dissipation, we proceed as follows:

We rewrite Eq. (42) in the form

$$\dot{u}_1=F_1(u_1,u_2,t;\eta(t),\epsilon(t)),$$

$$\dot{u}_2=F_2(u_1,u_2,t;\eta(t),\epsilon(t)), \quad (43)$$

where we use the following abbreviation:

$$u_1=q,\quad u_2=p \quad (44)$$

and

$$F_1=p,$$

$$F_2=-V'(q)+Q_V-\Gamma h(q)+g(q)\eta(t)+\epsilon(t) \quad (45)$$

The vector u with components u_1 and u_2 thus represents a point in a two-dimensional "phase space" and the Eq. (43) determines the velocity at each point in this phase space. The conservation of phase points now asserts the following linear equation of motion for density $\rho(u,t)$ in phase space:

$$\frac{\partial}{\partial t}\rho(u,t)=-\sum_{n=1}^{2}\frac{\partial}{\partial u_n}F_n(u_1,u_2,t;\eta(t),\epsilon(t))\rho(u,t)$$

or more compactly,

$$\frac{\partial\rho}{\partial t}=-\nabla\cdot F\rho. \quad (46)$$

Our next task is to find out a different equation whose average solution is given by $\langle\rho\rangle$, where the stochastic averaging has to be performed over two noise processes $\eta(t)$ and $\epsilon(t)$. To this end we note that $\nabla\cdot F$ can be partitioned into two parts: a constant part $\nabla\cdot F_0$ and a fluctuating part $\nabla\cdot F_1(t)$, containing these noises. Thus, we write

$$\nabla\cdot F(u_1,u_2,t;\eta(t),\epsilon(t))=\nabla\cdot F_0(u_1,u_2)+\alpha\nabla\cdot F_1(u_1,u_2,t;\eta(t),\epsilon(t)), \quad (47)$$

where α is a parameter (we put it as an external parameter to keep track of the order of the perturbation expansion in $\alpha\tau_e$; we put $\alpha=1$ at the end of the calculation) and also note that $\langle f_1(t)\rangle=0$. Equation (46) therefore takes the following form:

$$\dot{\rho}(u,t)=(A_0+\alpha A_1)\rho(u,t), \quad (48)$$

where $A_0=-\nabla\cdot F_0$ and $A_1=-\nabla\cdot F_1$. The symbol ∇ is used for the operator that differentiates everything that comes after it with respect to u.

Making use of one of the main results for the theory of linear equation of the form of Eq. (48) with multiplicative noise, we derive an average for ρ $[\langle\rho\rangle=P(u,t)]$, the probability density of $u(t)$, as

$$\frac{\partial P}{\partial t}=\left\{A_0+\alpha^2\int_0^{\infty}d\tau\langle A_1(t)\times\exp(\tau_eA_0)A_1(t-\tau_e)\rangle\exp(-\tau_eA_0)\right\}P. \quad (49)$$

The above result is based on second-order cumulant expansion and is valid in the case of small but rapid fluctuations and the correlation time τ_e is short but finite, i.e.,

$$\langle A_1(t)A_1(t')\rangle=0\ \text{ for }\ |t-t'|>\tau_e.$$

Equation (49) is exact in the limit $\tau_e\to 0$. Using the expressions for A_0 and A_1, we then obtain

$$\frac{\partial P(u_1,u_2,t)}{\partial t}=\left\{-\nabla\cdot F_0+\alpha^2\int_0^{\infty}d\tau\langle\nabla\cdot F_1(t)\times\exp(-\tau_e\nabla\cdot F_0)\nabla\cdot F_1(t-\tau_e)\rangle\exp(\tau_e\nabla\cdot F_0)\right\}P(u_1,u_2,t). \quad (50)$$

The operator $\exp(-\tau_e\nabla\cdot F_0)$ in the above equation provides the solution of the equation

$$\frac{\partial G(u,t)}{\partial t}=-\nabla\cdot F_0G(u,t). \quad (51)$$

G signifies the unperturbed part of ρ, which can be found explicitly in terms of characteristic curves. The equation

$$\dot{u}=F_0(u)$$

for fixed t determines a mapping from $u(\tau=0)$ to $u(\tau)$, i.e., $u\to u^{\tau_e}$ with inverse $(u^{\tau})^{-\tau}=u$. The solution of Eq. (51) is

$$G(u,t) = G(u^{-t},0)\left|\frac{d(u^{-t})}{d(u)}\right| = \exp(-t\nabla\cdot F_0)G(u,0), \tag{52}$$

where $\left|\frac{d(u^{-t})}{d(u)}\right|$ is a Jacobian determinant. The effect of $\exp(-t\nabla\cdot F_0)$ on $G(u)$ is as follows:

$$\exp(-t\nabla\cdot F_0)G(u,0) = G(u^{-t},0)\left|\frac{d(u^{-t})}{d(u)}\right|. \tag{53}$$

The above simplification when we put in Eq. (50) yields

$$\frac{\partial P(u_1,u_2,t)}{\partial t} = \nabla\cdot\left\{F_0 + \alpha^2\int_0^\infty d\tau\left|\frac{d(u^{-\tau})}{d(u)}\right| \times\langle F_1(u,t)\nabla_{-\tau}\cdot F_1(u^{-\tau},t-\tau)\rangle \times\left|\frac{d(u)}{d(u^{-\tau})}\right|\right\}P(u_1,u_2,t). \tag{54}$$

$\nabla_{-\tau}$ denotes differentiation with respect to $(u_{-\tau})$. We put $\alpha=1$ for the rest of the treatment. We now identify

$$u_1 = q,\ \ u_2 = p,$$

$$F_{01} = p,\ \ F_{11} = 0,$$

$$F_{02} = -V'(q) + Q_V - \Gamma h(q),$$

$$F_{12} = g(q)\eta(t) + \epsilon(t). \tag{55}$$

In this notation, Eq. (54) now reduces to

$$\frac{\partial P(q,p,t)}{\partial t} = -\frac{\partial}{\partial q}(pP) + \frac{\partial}{\partial p}\{\Gamma h(q)p + V'(q) - Q_V\}P + \frac{\partial}{\partial p}\int_0^\infty d\tau\left\langle[g(q)\eta(t)+\epsilon(t)] \times\left[\frac{\partial}{\partial p^{-\tau}}\{g(q^{-\tau})\eta(t-\tau)+\epsilon(t-\tau)\}\right]\right\rangle P, \tag{56}$$

where we have used the fact that the Jacobian obeys the equation

$$\frac{d}{dt}\log\left|\frac{d(q^t,p^t)}{d(q,p)}\right| = \frac{\partial p}{\partial q} + \frac{\partial}{\partial p}[-\Gamma h(q)p + V'(q)] = -\Gamma h(q)$$

so that the Jacobian becomes $\exp(-\Gamma h(q)t)$. As a next approximation, we consider the "unperturbed" part of Eq. (56) and take the variation in p during τ into account to first order in τ. Thus we have

$$q^{-\tau} = q - \tau p,$$

$$p^{-\tau} = p + \Gamma h(q)\tau p + \tau V'(q). \tag{57}$$

Neglecting terms $O(\tau^2)$, Eq. (57) yields

$$\frac{\partial}{\partial p^{-\tau}} = [1-\Gamma h(q)\tau]\frac{\partial}{\partial p} + \tau\frac{\partial}{\partial x}. \tag{58}$$

With the help of Eq. (58), Eq. (56) can be simplified after some algebra to the following form:

$$\frac{\partial P(q,p,t)}{\partial t} = -\frac{\partial}{\partial q}(pP) + \frac{\partial}{\partial p}[\Gamma h(q)p + V'(q) - Q_V - 2g(q)g'(q)I_{nn}]P + [I_{ee}+g^2(q)I_{nn}]\frac{\partial^2 P}{\partial p\,\partial q}, \tag{59}$$

$$+[J_{ee} - \Gamma h(q)I_{ee} + g^2(q)J_{nn} - \Gamma h(q)g^2(q)I_{nn} - vg(q)g'(q)I_{nn}]\frac{\partial^2 P}{\partial p^2}, \tag{60}$$

where

$$I_{ee} = \int_0^\infty d\tau\langle\epsilon(t)\epsilon(t-\tau)\rangle\tau,$$

$$I_{nn} = \int_0^\infty d\tau\langle\eta(t)\eta(t-\tau)\rangle\tau,$$

$$J_{ee} = \int_0^\infty d\tau\langle\epsilon(t)\epsilon(t-\tau)\rangle,$$

$$J_{nn} = \int_0^\infty d\tau\langle\eta(t)\eta(t-\tau)\rangle.$$

The statistical properties of $\eta(t)$ and $\epsilon(t)$ ensure that $I_{nn}=0$, $J_{nn}=D_0$, $J_{ee}=D_e$, and $I_{ee}=D_e\tau_e$. Hence, Eq. (59) takes the simpler form

$$\frac{\partial P(q,p,t)}{\partial t} = -\frac{\partial}{\partial q}(pP) + \frac{\partial}{\partial p}[\Gamma h(q)p + V'(q) - Q_V]P, \tag{61}$$

$$+D_e\tau_e\frac{\partial^2 P}{\partial p\,\partial q} + [D_e - \Gamma h(q)D_e\tau_e + D_0 g^2(q)]\frac{\partial^2 P}{\partial p^2}. \tag{62}$$

For small τ_e we neglect the non-Markovian contribution $D_e\tau_e\frac{\partial^2 P}{\partial p\partial q}$ and the phase-space probability density function obeys the approximate Fokker-Planck equation,

$$\frac{\partial P(q,p,t)}{\partial t} = -\frac{\partial}{\partial q}(pP) + \frac{\partial}{\partial p}[\Gamma h(q)p + V'(q) - Q_V]P + [D_e - \Gamma h(q)D_e\tau_e + D_0 g^2(q)]\frac{\partial^2 P}{\partial p^2}. \tag{63}$$

The above assumption is equivalent to considering the exponentially correlated noise with the underlying dynamics is Markovian. As there is no fluctuation-dissipation relation for $\epsilon(t)$, our assumption is physically relevant.

In this part we now want to illustrate the relationship of our recent formulation with other allied methods bearing kinship with ours in vogue. The method conceptually closest to ours is that of Steffen and Tanimura [48] (but deviates in a number of details). In the absence of external noise, that is, for $D_e=0$, Eq. (63) reads

$$\frac{\partial P(q,p,t)}{\partial t} = -\frac{\partial}{\partial q}(pP) + \frac{\partial}{\partial p}[\Gamma h(q)p + V'(q) - Q_V]P + D_0 g^2(q)\frac{\partial^2 P}{\partial p^2}. \tag{64}$$

It is to be noted that we have treated the system part quantum mechanically and the dissipation part semiclassically. This situation is equivalent to the case that Caldeira and Leggett [47] discussed for the linear-linear system-bath coupling case. The nonlinear-linear system-bath coupling with same condition is also considered by Steffen and Tanimura [48], where they assumed the square-linear coupling (in our notation $f(q)=\frac{q^2}{2}$). To relate our method with that of Steffen and Tanimura [48] it is more appropriate to consider Eq. (64) as follows:

$$\frac{\partial P(q,p,t)}{\partial t} = -\frac{\partial}{\partial q}(pP) + \frac{\partial}{\partial p}[V'(q) - Q_V]P + q^2\Gamma\frac{\partial}{\partial p}\left(p + \frac{D_0}{\Gamma}\frac{\partial}{\partial p}\right)P. \tag{65}$$

In the work of Steffen and Tanimura [48], the dissipative part of the quantum Fokker-Planck is obtained as $\Gamma\frac{\partial}{\partial p}(p+k_BT\frac{\partial}{\partial p})P$ and $4q^2\Gamma\frac{\partial}{\partial p}(p+k_BT\frac{\partial}{\partial p})P+q\Gamma\hbar^2\frac{\partial^2 P}{\partial q\partial p}$ for linear-linear and square-linear system-bath couplings, respectively. In our case, for linear-linear coupling, the dissipative part reduces to the same form at high temperature, as is evident from Eq. (64). Also square-linear coupling, the first term of the dissipative part of quantum Fokker-Planck equation of Steffen and Tanimura [48] has the same q dependence as that of our model. As we have neglected the small non-Markovian contribution, the cross derivative term, i.e., $\frac{\partial^2 P}{\partial q\partial p}$, does not appear in our case. Thus, we may expect an innate relation between path-integral formulation of dissipative dynamics and our present development. It is worth noting that being quasiprobability distribution function, the Wigner distribution function may not always be positive definite, but it has been shown by Ray and co-workers [49] that the ansatz Eq. (9) always remains a positive-definite function. As a further development, one may consider a correlated noise (τ_c finite) instead of considering white-noise process as discussed in Ref. [50]. These aspects will be addressed in future works.

Now introducing the auxiliary function $G(q)$ and a Gaussian δ-correlated noise $\beta(t)$ we may check that the above Fokkerf-Planck Eq. (63) is equivalent to the Langevin equation

$$\dot{q} = p,$$
$$\dot{p} = -V'(q) + Q_V - \Gamma h(q)p + G(q)\beta(t), \tag{66}$$

where

$$G(q) = \sqrt{D_e + D_0 g^2(q) - \Gamma h(q) D_e \tau_e} \tag{67}$$

and

$$\langle\beta(t)\rangle = 0;\ \langle\beta(t)\beta(t')\rangle = 2\delta(t-t'). \tag{68}$$

In the above Langevin equation, the dissipation is state dependent and the noise term $\beta(t)$ appears multiplicatively. We now apply the method of Sancho to get the overdamped Langevin equation in space-dependent frictional medium as

$$\dot{q} = -\frac{V'(q) - Q_V}{\Gamma h(q)} - \frac{G(q)G'(q)}{\Gamma h^2(q)} + \frac{G(q)}{\Gamma h(q)}\beta(t). \tag{69}$$

The corresponding Fokker-Planck-Smoluchowski equation for the probability density $P(q,t)$ of a particle at q at a time t is

$$\frac{\partial P(q,t)}{\partial t} = \frac{\partial}{\partial q}\left[\frac{V'(q) - Q_V}{\Gamma h(q)}\right]P(q,t) + \frac{\partial}{\partial q}\left[\frac{G(q)G'(q)}{\Gamma h^2(q)}\right]P(q,t) + \frac{\partial}{\partial q}\left[\frac{G(q)}{\Gamma h(q)}\frac{\partial}{\partial q}\frac{G(q)}{\Gamma h(q)}\right]P(q,t), \tag{70}$$

which can be written in a more compact form as

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial q}\frac{1}{\Gamma h(q)}\left[V'(q) - Q_V + \frac{1}{\Gamma}\frac{\partial}{\partial q}\frac{G^2(q)}{h(q)}\right]P(q,t). \tag{71}$$

Equation (71) is the required Smoluchowski equation corresponding to the quantum Langevin equation, where the noise is multiplicative, dissipation is state dependent, and the system is externally driven by an exponentially correlated noise. In the overdamped limit, the stationary current can be obtained as

$$J = -\frac{1}{\Gamma h(q)}\left[V'(q) - Q_V + \frac{1}{\Gamma}\frac{d}{dq}\left(\frac{G^2(q)}{h(q)}\right)\right]P_{st}(q). \tag{72}$$

Proceeding as in the earlier case, the stationary current in a symmetric periodic potential $V(q)$ and for the periodic derivative of coupling function $f(q)$, will be given by

$$J = \frac{1}{\Gamma^2}\left\{[1 - e^{U(2\pi)}]/\left[\left(\int_0^{2\pi}\frac{h(q)}{G^2(q)}e^{-U(q)}dq\int_0^{2\pi}h(q)e^{U(q)}dq\right) - (1 - e^{U(2\pi)})\int_0^{2\pi}\left(\frac{h(q)}{G^2(q)}e^{-U(q)}\int_0^{q}h(q')e^{U(q')}dq'\right)dq\right\}. \tag{73}$$

In this case the generalized potential $U(q)$ is

$$U(q) = \Gamma\int_0^{q}\frac{V'(q) - Q_V}{G^2(q)/h(q)}dq. \tag{74}$$

It is clear from the expression (74) for $U(q)$ that the peaks of the steady-state probability $P_{st}\sim\exp[-U(q)]$ are determined not by minima of $V(q)$ alone but are determined as a combined effect of other dynamic parameters. $P_{st}(q)$ may even be peaked at positions which would be less likely to be populated in the stationary situation, i.e., when the dissipation is not state dependent. From Eq. (74) it is easy to verify that for $\tau_e \to 0$, $U(q)$ reduces to $\phi(q)$.

At this juncture we are in a position to make some comments regarding results of our recent work with respect to our previous work reported in Ref. [34]. In both the cases we studied the phase-induced quantum current in thermodynamically open systems. But the nature of mechanisms that make the systems open is completely different. As we have already mentioned that in present work the associated heat bath is kept in thermal equilibrium and the system is driven externally, contrary to our previous work where the bath was driven externally, instead of the system. It is pertinent to mention here that the expression of current J is similar to that of the current, Eq. (82) in Ref. [34]. The basic difference between Eq. (73) of this work and Eq. (82) of Ref. [34] lies in the structure of the effective potential through the function $G(q)$. The close kinship between the structures of the current appears due to the fact that in both the cases, the Brownian particle is driven by effective multiplicative noises. But it should be noted here that the nature of the state-dependent part of the noises are different. The difference in the state-dependent part makes the difference in the effective potential. Thus though the two expressions bear similar structure, they explain different physical phenomena.

V. RESULT AND DISCUSSION

Before proceeding to examine the various features of the current given by Eqs. (40) and (73) we calculate the quantum correction terms. Following Ray *et al.*, the details of the calculations of quantum correction terms are shown in the Appendix. Though the quantum dispersion terms $\langle \delta\hat{q}^n \rangle_Q$ can be obtained by direct numerical simulation of the coupled Eq. (A1) subject to appropriate boundary conditions, it is instructive to deal with quantum correction terms in an analytical way to find out the approximate value of quantum dispersion terms. For overdamped limit we neglect the $\delta\hat{p}$ term from Eq. (A2) to obtain

$$\frac{d}{dt}\delta\hat{q} = \frac{1}{\Gamma[f'(q)]^2}[-V''(q)\delta\hat{q} - 2\Gamma p f'(q) f''(q)\delta\hat{q} + \eta(t) f''(q)\delta\hat{q}] + O(\delta\hat{q}^2). \tag{75}$$

With the help of Eq. (75) we then obtain the equations for $\langle \delta\hat{q}^n \rangle_Q$

$$\frac{d}{dt}\langle \delta\hat{q}^2 \rangle_Q = \frac{2}{\Gamma[f'(q)]^2}[-V''(q)\langle \delta\hat{q}^2 \rangle_Q - 2\Gamma p f'(q) f''(q)\langle \delta\hat{q}^2 \rangle_Q] + \eta(t) f''(q)\langle \delta\hat{q}^2 \rangle_Q + O(\langle \delta\hat{q}^3 \rangle_Q), \tag{76}$$

$$\frac{d}{dt}\langle \delta\hat{q}^3 \rangle_Q = \frac{3}{\Gamma[f'(q)]^3}[-V''(q)\langle \delta\hat{q}^2 \rangle_Q - 2\Gamma p f'(q) f''(q)\langle \delta\hat{q}^3 \rangle_Q + \eta(t) f''(q)\langle \delta\hat{q}^3 \rangle_Q] + O(\langle \delta\hat{q}^4 \rangle_Q), \tag{77}$$

and so on. It is apparent from Eqs. (76) and (77) that in the overdamped limit, the higher-order quantum contributions are small since each successive order of correction is lower than the preceding one by a factor $\frac{1}{\Gamma}$. A simplified expression for the leading-order quantum correction term $\langle \delta\hat{q}^2 \rangle_Q$ can be estimated by neglecting the higher-order coupling terms in the square bracket in Eq. (76) and rewriting it as

$$d\langle \delta\hat{q}^2 \rangle_Q = -\frac{2}{\Gamma[f'(q)]^2} V''(q)\langle \delta\hat{q}^2 \rangle_Q dt.$$

On the other hand, the overdamped deterministic classical motion gives

$$dq = -\frac{V'(q)}{\Gamma[f'(q)]^2} dt.$$

These together yield after integration

$$\langle \delta\hat{q}^2 \rangle_Q = \Delta_q [V'(q)]^2, \tag{78}$$

where $\Delta_q = \langle \delta\hat{q}^2 \rangle_Q^0 / [V'(q^0)]^2$ and q^0 is the quantum-mechanical mean position at which $\langle \delta\hat{q}^2 \rangle_Q$ becomes minimum, i.e., $\langle \delta\hat{q}^2 \rangle_Q^0 = \frac{\hbar}{2\omega_0}$, where ω_0 is the average frequency of the bath as defined earlier.

For numerical implementation of our results, we consider a sinusoidal periodic and symmetric potential

$$V(q) = V_0[1 + \cos(q + \theta)], \tag{79}$$

where V_0 is the barrier height and θ is the phase factor, which can be controlled externally. The coupling function $f(q)$ is chosen as $f(q) = (q + \alpha \sin q)$ so that the derivative of the coupling function becomes $f'(q) = 1 + \alpha \cos q$, where α is the modulation parameter. Consequently, the second-order quantum correction in the overdamped limit becomes

$$\langle \delta\hat{q}^2 \rangle_Q = -\Delta_q V_0^2 \sin^2(q + \theta)$$

and the correction to the potential in the leading order are given by

$$Q_V = -\frac{1}{2}\Delta_q V_0^3 \sin^3(q + \theta). \tag{80}$$

The quantum corrections Q_f and Q_3 in the same order can be estimated as

$$Q_f = -\frac{1}{2}\Delta_q \alpha V_0^2 \cos q \sin^2(q + \theta), \tag{81}$$

$$Q_3 = \Delta_q \alpha^2 V_0^2 \sin^2 q \sin^2(q + \theta), \tag{82}$$

and the functions $h(q)$ and $g(q)$ are given by

$$h(q) = (1 + \alpha \cos q)^2 - \Delta_q \alpha V_0^2 \cos q \sin^2(q + \theta)(1 + \alpha \cos q) + \Delta_q \alpha^2 V_0^2 \sin^2 q \sin^2(q + \theta), \tag{83}$$

$$g(q) = 1 + \alpha \cos q - \frac{1}{2}\Delta_q \alpha V_0^2 \cos q \sin^2(q + \theta). \tag{84}$$

In the unit of $\hbar = k_B = 1$, we set the parameters $\langle \delta\hat{q}^2 \rangle_Q^0 = \frac{1}{2}$, the minimum uncertainty value, $\Delta_q = 0.5$, $V_0 = 1.0$, $\alpha = 1.0$, $T = 1.0$, $\Gamma = 1.0$;. In Fig. 1, we plot the variation in effective

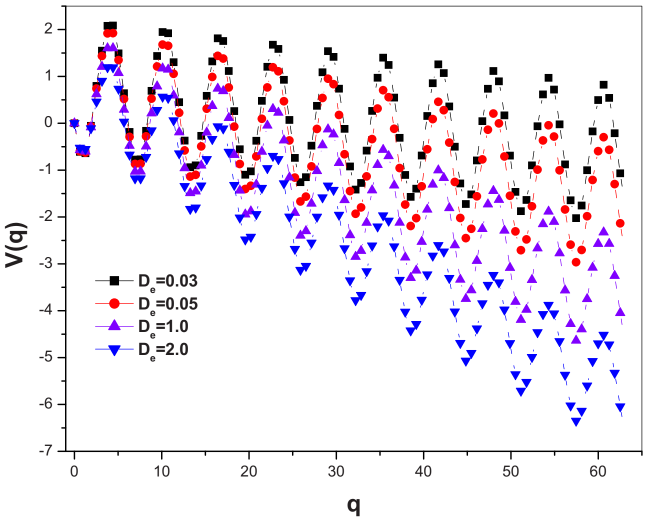

FIG. 1. (Color online) Variation in effective potential with q for τ_e=0.01.

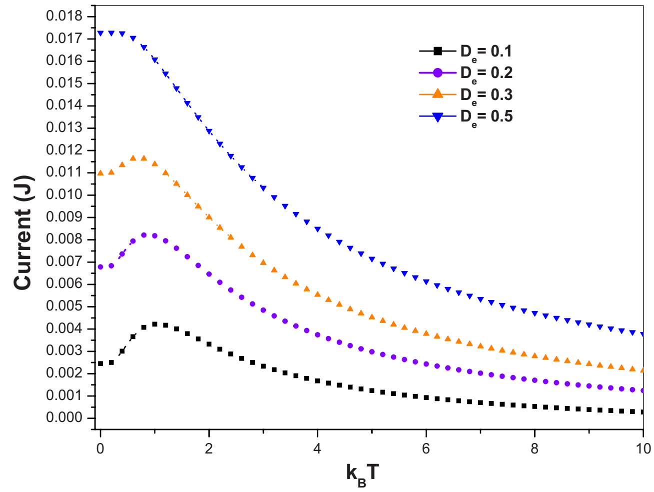

FIG. 3. (Color online) Variation in current with temperature for τ_e=0.01.

potential $U(q)$ for a particular correlation time τ_e=0.01, to observe the effect of D_e from which we see that a tilt to the effective potential has been generated. This asymmetry in the generalized potential makes the transition between left to right and right to left unequal and consequently unidirectional motion appears. The variation in current as a function of phase difference θ is shown in Fig. 2 for different values of D_e. In an attempt to get a better insight, we also plot the classical counter part of current (with $D_0=\frac{\Gamma k_B T}{2}$) in Fig. 2. Figure 2 clearly indicates the deviation of classical results with respect to the quantum current increase with decreasing value of D_e as the effective temperature increases with the value of D_e. The behavior of system(s) described by the quantum-mechanical theory reproduces the classical result in the limit of high temperature. The variation in current as a function of temperature is depicted in Fig. 3 for different D_e values.

Our view is that the primary feature of the current profile generated via our present formalism is very similar in nature as that of our previous work published in Ref. [34] as the basic Langevin equation used in both works are structurally similar. This aspect bolsters our belief that our proposed methodology to compute quantum current is quite promising and can be used as a potential tool to study the transport phenomena in nanoscale device like molecular motor.

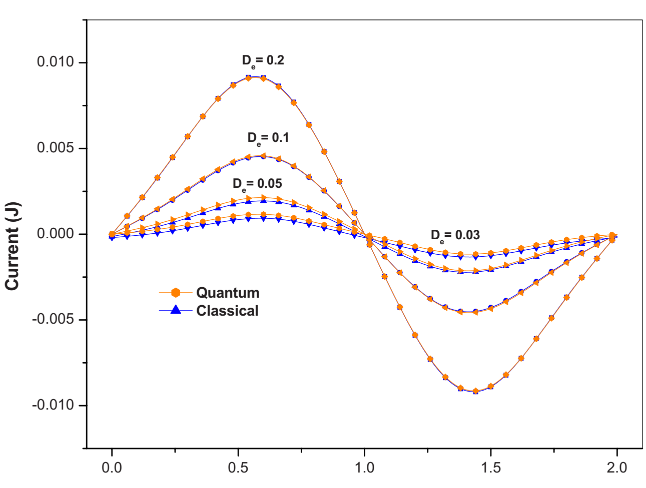

FIG. 2. (Color online) Variation in current as a function of phase difference for τ_e=0.01.

VI. CONCLUSION

We have formulated a theory for the transport of a quantum system when the system is driven by an external random force by making it thermodynamically open. Our approach is based on the system-reservoir model with nonlinear system-bath coupling. We then derive the quantum Langevin equation with multiplicative noise (and with the additive external noise) and a nonlinear dissipation. We then obtain the c number analog of the quantum Langevin equation in the Markovian limit. Following Sancho we then derive the quantum analog of the Smoluchowski equation for the state-dependent diffusion of stochastically driven quantum system. We applied our formulation to the problem of diffusion of a quantum particle in a periodic potential. Our tractable result holds true away from the semiclassical limit and, more interestingly, can be applied to an arbitrarily shaped ratchet potential, which we would like to address in near future. Our investigation can advantageously be put to work for quantum ratchets on the microscale and nanoscale. Furthermore, the structure of our quantum analogy of Smoluchowski equation can be generalized to higher dimensional overdamped situations as for quantum noise-induced directed transport on surfaces and to optimize transport properties in superconductors by controlling the motion of vortices and magnetic-flux quanta [51]. In passing, we point out that very recently, an experimental realization of quantum ratchets associated with quantum resonance of the kicked particle for arbitrary values of the quasimomentum has been reported [52,53]. However, the theoretical study of the phenomena of quantum ratchets remains wide open, and we hope that our study will be helpful to understand the various characteristics of such systems.

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APPENDIX: QUANTUM CORRECTION TERMS

Before embarking on our recent formulation based on the c-number approach of Ray and co-workers [8], it will be useful to first present a brief resumé of the scheme of calculation of quantum correction terms of Ray and co-workers [8]. This will motivate toward the essential ingredients which should be embedded in our case. Hence, we describe briefly here the essential aspects of the c-number approach of Ray and co-workers [8] insofar as it is pertinent to our formulation.

In the Heisenberg picture, one can write the system operators $\hat{q}$ and $\hat{p}$ as $\hat{q}=q+\delta\hat{q}$ and $\hat{p}=p+\delta\hat{p}$, respectively. $\delta\hat{q}$ and $\delta\hat{p}$ describe the quantum fluctuations around their respective mean values.

With the help of the operator Langevin Eq. (3) in the Markovian limit, the time evolution of these correction terms can be calculated via the following equations using quantum-mechanical average over the initial product separable coherent bath states:

$$\dot{\hat{q}} = \hat{p},$$

$$\dot{\hat{p}} = -V'(\hat{q}) - \Gamma[f'(\hat{q})]^2\hat{p} + f'(\hat{q})\eta(t) + f'(\hat{q})\pi(t) + \epsilon(t), \tag{A1}$$

$$\delta\dot{\hat{q}} = \delta\hat{p},$$

$$\begin{aligned}\delta\dot{\hat{p}} = &-V''(q)\delta\hat{q} - \sum_{n\ge 2}\frac{1}{n!}V^{n+1}(q)[\delta\hat{q}^n - \langle\delta\hat{q}^n\rangle_Q] \\ &- \gamma\left[2f'(q)f''(q)\delta\hat{q} + 2f'(q)\sum_{n\ge 2}\frac{1}{n!}f^{n+1}(q)[\delta\hat{q}^n - \langle\delta\hat{q}^n\rangle_Q] + \sum_{m\ge 1}\sum_{n\ge 1}\frac{1}{m!}\frac{1}{n!}f^{m+1}(q)f^{n+1}(q)[\delta\hat{q}^m\delta\hat{q}^n - \langle\delta\hat{q}^m\delta\hat{q}^n\rangle_Q]p\right] \\ &- \gamma\left[[f'(q)]^2\delta\hat{p} + 2f'(q)\sum_{n\ge 1}\frac{1}{n!}f^{n+1}(q)[\delta\hat{q}^n\delta\hat{p} - \langle\delta\hat{q}^n\delta\hat{p}\rangle_Q] + \sum_{m\ge 1}\sum_{n\ge 1}\frac{1}{m!}\frac{1}{n!}f^{m+1}(q)f^{n+1}(q)[\delta\hat{q}^m\delta\hat{q}^n\delta\hat{p} - \langle\delta\hat{q}^m\delta\hat{q}^n\delta\hat{p}\rangle_Q]\right] \\ &+ \eta(t)\left[f''(q)\delta\hat{q} + \sum_{n\ge 2}\frac{1}{n!}f^{n+1}(q)[\delta\hat{q}^n - \langle\delta\hat{q}^n\rangle_Q]\right].\end{aligned} \tag{A2}$$

From the work of Ray and co-workers [8], it is clear that the operator correction equations can be used to yield an infinite hierarchy of equations. Up to third order, we construct, for example, the following set of equations which are coupled to quantum Langevin equations from Eq. (8):

$$\frac{d}{dt}\langle\delta\hat{q}^2\rangle_Q = \langle\delta\hat{q}\,\delta\hat{p} + \delta\hat{p}\,\delta\hat{q}\rangle_Q,$$

$$\frac{d}{dt}\langle\delta\hat{q}\,\delta\hat{p} + \delta\hat{p}\,\delta\hat{q}\rangle_Q = -2\chi(q,p)\langle\delta\hat{q}^2\rangle_Q + 2\langle\delta\hat{q}^2\rangle_Q - \gamma[f'(q)]^2\langle\delta\hat{q}\,\delta\hat{p} + \delta\hat{p}\,\delta\hat{q}\rangle_Q - \zeta(q,p)\langle\delta\hat{q}^3\rangle_Q - 2\gamma f'(q)f''(q)\langle\delta\hat{q}^2\delta\hat{p} + \delta\hat{p}\,\delta\hat{q}^2\rangle_Q,$$

$$\frac{d}{dt}\langle\delta\hat{p}^2\rangle_Q = -2\gamma[f'(q)]^2\langle\delta\hat{p}^2\rangle_Q - \chi(q,p)\langle\delta\hat{q}\,\delta\hat{p} + \delta\hat{p}\,\delta\hat{q}\rangle_Q - \frac{1}{2}\zeta(q,p)\langle\delta\hat{q}^2\delta\hat{p} + \delta\hat{p}\,\delta\hat{q}^2\rangle_Q - 2\gamma f'(q)f''(q)\langle\delta\hat{q}\,\delta\hat{p}^2 + \delta\hat{p}^2\delta\hat{q}\rangle_Q,$$

$$\frac{d}{dt}\langle\delta\hat{q}^3\rangle_Q = \frac{3}{2}\langle\delta\hat{q}^2\delta\hat{p} + \delta\hat{p}\,\delta\hat{q}^2\rangle_Q,$$

$$\frac{d}{dt}\langle\delta\hat{p}^3\rangle_Q = -3\gamma[f'(q)]^2\langle\delta\hat{p}^3\rangle_Q - \frac{3}{2}\chi(q,p)\langle\delta\hat{q}\,\delta\hat{p}^2 + \delta\hat{p}^2\delta\hat{q}\rangle_Q,$$

$$\frac{d}{dt}\langle\delta\hat{q}^2\delta\hat{p} + \delta\hat{p}\,\delta\hat{q}^2\rangle_Q = -2\chi(q,p)\langle\delta\hat{q}^3\rangle_Q + 2\langle\delta\hat{q}\,\delta\hat{p}^2 + \delta\hat{p}^2\delta\hat{q}\rangle_Q - \gamma[f'(q)]^2\langle\delta\hat{q}^2\delta\hat{p} + \delta\hat{p}\,\delta\hat{q}^2\rangle_Q,$$

$$\frac{d}{dt}\langle \delta\hat{q}\,\delta\hat{p}^2 + \delta\hat{p}^2\,\delta\hat{q}\rangle_Q = 2\langle \delta\hat{p}^3\rangle_Q - 4\chi(q,p)\langle \delta\hat{q}^2\,\delta\hat{p} + \delta\hat{p}\,\delta\hat{q}^2\rangle_Q - 2\gamma[f'(q)]^2\langle \delta\hat{q}\,\delta\hat{p}^2 + \delta\hat{p}^2\,\delta\hat{q}\rangle_Q, \tag{A3}$$

where

$$\chi(q,p) = V''(q) + 2\gamma p f'(q) f''(q) - \eta(t) f''(q),$$

$$\zeta(q,p) = V'''(q) + 2\gamma p f'(q) f'''(q) + 2\gamma p[f''(q)]^2 - \eta(t) f'''(q).$$

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