Stability measures in metastable states with Gaussian colored noise

Alessandro Fiasconaro* and Bernardo Spagnolo
Dipartimento di Fisica e Tecnologie Relative, Group of Interdisciplinary Physics, Università di Palermo and CNISM-INFM, Viale delle Scienze, I-90128 Palermo, Italy

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We present a study of the escape time from a metastable state of an overdamped Brownian particle in the presence of colored noise generated by Ornstein-Uhlenbeck process. We analyze the role of the correlation time on the enhancement of the mean first passage time through a potential barrier and on the behavior of the mean growth rate coefficient as a function of the noise intensity. We observe the noise-enhanced stability effect for all the initial unstable states used and for all values of the correlation time τ_c investigated. We can distinguish two dynamical regimes characterized by weak and strong correlated noises, depending on the value of τ_c with respect to the relaxation time of the system.

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I. INTRODUCTION

The problem of the lifetime of a metastable state has been addressed in a variety of areas, including first-order phase transitions, Josephson junctions, field theory, and chemical kinetics [1,2]. Recent experimental and theoretical results show that long-live metastable states are observed in different areas of physics [3,4]. Experimental and theoretical investigations have shown that the average escape time from metastable states in fluctuating potentials presents a nonmonotonic behavior as a function of the noise intensity with the presence of a maximum [5–7]. This is the noise-enhanced stability (NES) phenomenon: the stability of metastable states can be enhanced and the average life time of the metastable state increases nonmonotonically with the noise intensity. This resonancelike behavior contradicts the monotonic behavior of the Kramers theory [8]. The occurrence of the enhancement of stability of metastable states by the noise has been observed in different physical and biological systems [2,5–7,9–15]. Very recently NES effect was observed in an ecological system [16], an oscillator chemical system (the Belousov-Zhabotinsky reaction) [17], and magnetic systems [18]. Interestingly, in Ref. [17] the stabilization of a metastable state due to noise is experimentally detected and a decreasing behavior of the maximum Lyapunov exponent as a function of the noise intensity is observed.

A generalization of the Lyapunov exponent for stochastic systems has been recently defined in Ref. [19] to complement the analysis of the transient dynamics of metastable states. This measure of stability is the "mean growth rate coefficient" (MGRC) Λ, and it is evaluated by a similar procedure used for the calculation of the Lyapunov exponent in stochastic systems [20]. By linearizing the Langevin equation of motion [see next Eq. (4)], we consider the evolution of the separation $\delta x(t)$ between two neighboring trajectories of the Brownian particle starting at x_0 and reaching x_F

$$\delta\dot{x}(t) = -\frac{d^2U(x)}{dx^2}\delta x(t) = \lambda_i(x,t)\,\delta x(t), \tag{1}$$

and we define $\lambda_i(x,t)$ as an instantaneous growth rate. We note that, in Eq. (1), $d^2U(x)/dx^2$ is calculated onto the noisy trajectory $x[\xi(t)]$ [19]. The growth rate coefficient Λ_i (for the i_{th} noise realization) is then defined as the long-time average of the instantaneous λ_i coefficient over $\tau(x_0,x_F)$ [19–21]

$$\Lambda_i = \frac{1}{\tau(x_0,x_F)}\int_0^{\tau(x_0,x_F)} \lambda_i(x,s)\,ds. \tag{2}$$

In the limit $\tau(x_0,x_F)\to\infty$, Eq. (2) coincides formally with the definition of the maximum Lyapunov exponent, and therefore, the Λ_i coefficient has the meaning of a finite-time Lyapunov exponent. This quantity is useful to characterize a transient dynamics in nonequilibrium dynamical systems [17,19]. The mean growth rate coefficient Λ is then defined as the ensemble average of the growth rate coefficient Λ_i,

$$\Lambda = \langle \Lambda_i \rangle, \tag{3}$$

over the noise realizations. The mean growth rate coefficient has a nonmonotonic behavior as a function of the noise intensity for Brownian particles starting from unstable initial positions [19]. This nonmonotonicity with a minimum indicates that Λ can be used as a new suitable measure or signature of the NES effect.

The inclusion of realistic noise sources, with a finite correlation time, impacts both the stationary and the dynamic features of nonlinear systems. For metastable thermal equilibrium systems it has been demonstrated that colored thermal noise can substantially modify the crossing barrier process [8]. A rich and enormous literature on escape processes driven by colored noise was produced in the 1980s [22–24]. More recently many papers investigated the role of the correlated noise on different physical systems [25–30], which indicates a renewed interest in the realistic noise source effects.

In this work we present a study of the average decay time of an overdamped Brownian particle subject to a cubic potential with a metastable state. We focus on the role of different unstable initial conditions and of colored noise in the

*afiasconaro@gip.dft.unipa.it; http://gip.dft.unipa.it

average escape time. The effect of the correlation time τ_c on the transient dynamics of the escape process is related to the characteristic time scale of the system, that is the relaxation time inside the metastable state τ_r. For $\tau_c < \tau_r$, the dynamical regime of the Brownian particle is close to the white-noise dynamics. For $\tau_c > \tau_r$, we obtain: (i) a big shift of the increase in the average escape times toward higher noise intensities; (ii) an enhancement of the value of the average escape time maximum with a broadening of the NES region in the plane (τ, D), which becomes very large for high values of τ_c; (iii) the shift (towards lower values) of the peculiar initial position x_c, found in our previous studies [7,19], which separates the set of the initial unstable states producing divergency, for D tending to zero, from those which give only a nonmonotonic behavior of the average escape time; (iv) the entire qualitative behaviors (i)–(iii) can be applied to the standard deviation of the escape time; (v) the shift of the minimum values in the curves of the mean growth rate coefficient Λ; (vi) trend to the disappearance of the minimum in the curves of Λ, with a decreasing monotonic behavior for increasing τ_c; (vii) trend to the disappearance of the divergent dynamical regime in τ, with increasing τ_c. The paper is organized as follows. In the next section we introduce the model. In the third section we show the results and in the final section we draw the conclusions.

II. MODEL

The starting point of our study is the Langevin equation

$$\dot{x} = -\frac{\partial U(x)}{\partial x} + \eta(t), \tag{4}$$

where $\eta(t)$ is the Ornstein-Uhlenbeck process

$$d\eta = -k\eta dt + k\sqrt{D}dW(t) \tag{5}$$

and $dW(t)=\xi(t)dt$ is the increment of the Wiener process. $\xi(t)$ is the white Gaussian noise with the usual statistical properties: $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t+\tau)\rangle = \delta(\tau)$. The system of Eqs. (4) and (5) represents a two-dimensional Markovian process, which is equivalent to a non-Markovian Langevin equation driven with additive Gaussian correlated noise, with $\eta(t)$ obeying the following statistical properties $\langle \eta(t)\rangle = 0$ and $\langle \eta(t)\eta(t+\tau)\rangle = (kD/2)e^{-k\tau}$, for $t\to\infty$ and $\eta(0)=0$. Here $1/k=\tau_c$ is the correlation time of the process. The integration of Eq. (5) yields in the limit $\tau_c \to 0$ the white-noise term

$$\lim_{\tau_c\to 0} \eta(t) = 2\sqrt{D}\int_0^t \lim_{\tau_c\to 0} \frac{e^{-(t-t')/\tau_c}}{2\tau_c} dW(t') = \sqrt{D}\xi(t), \tag{6}$$

and the stationary correlation function of the Ornstein-Uhlenbeck process gives in the limit $\tau_c \to 0$ the correlation function of the white noise: $\lim_{\tau_c\to 0}\langle \eta(t)\eta(t+\tau)\rangle = D\delta(\tau)$. The potential $U(x)$ used in Eq. (4) is $U(x)=ax^2-bx^3$, with $a=0.3$ and $b=0.2$. The potential profile has a local stable state at $x=0$ and an unstable state at $x=1$ (see Fig. 1). The relaxation time for the metastable state at $x=0$ is $\tau_r = [d^2U(x)/dx^2]_{x=0} = 2a$, which is the characteristic time scale of our system. For our potential profile we have $\tau_r = 0.6$.

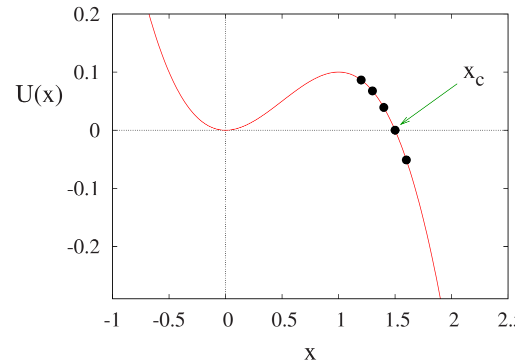

FIG. 1. (Color online) The cubic potential $U(x)=ax^2-bx^3$ with the various initial positions investigated (dots), namely, $x_o=1.2, 1.3, 1.4, 1.5, 1.6$. The parameters of the potential are $a=0.3$, $b=0.2$. For the white-noise case, $x_c=1.5$ is the critical initial position which separates the set of the initial unstable states producing divergency for D tending to zero from those which give only a nonmonotonic behavior of the average escape time [6,19].

III. RESULTS

The calculations of the average escape time as a function of the colored noise intensity have been performed by averaging over $N=20\,000$ realizations the numerical solution of the stochastic differential Eq. (4). The absorbing boundary for the escape process is put on $x_F=20$, and the maximum simulation time is $T_{\max}=10\,000$ a.u.. For all the initial unstable states (see Fig. 1) and all the correlation times considered we find an enhancement of the mean first passage time (MFPT) τ with respect to the deterministic time.

In Fig. 2 the calculations performed with low colored noise ($\tau_c=0.01$) for the mean first passage time τ and the mean growth rate coefficient Λ are shown. We see as signatures of the NES effect a maximum in the curve of τ and a minimum in that of Λ. In the inset of Fig. 2(a) the standard deviation of the first passage time as a function of noise intensity is reported. We note that the behaviors of τ and Λ in this low colored noise regime ($\tau_c=0.01$) coincides with those obtained in the white-noise case [19]. Moreover by comparing the theoretical predictions of τ [see Eq. (3) of Ref. [19]] with direct numerical simulations of the Langevin equation, a very good agreement is obtained (see Fig. 3 of Ref. [19]). In Fig. 3 the semilogarithmic plots of the fraction of particles N_i/N reaching the threshold position $x_t=0.5$ into the potential well, within the $T_{\max}$, as a function of noise intensity D, with the same initial conditions of Fig. 1, are shown. This threshold position x_t corresponds to the concavity change in the potential and is considered for this reason as a reference indicator for the effective entrance of the particle into the well. It is possible to observe that for very low noise intensity none particle enters into the well within the $T_{\max}$ considered, and the estimation of the stability measures take their deterministic values. We note that the behavior of the mean growth rate coefficient as a function of the noise intensity is strongly affected by the characteristic potential shape of a metastable state. The curves shown in Fig. 3 clarify the behavior of Λ in the limit of $D\to 0$. In fact the position $x_t=0.5$ is the flex point of the potential, where the instanta-

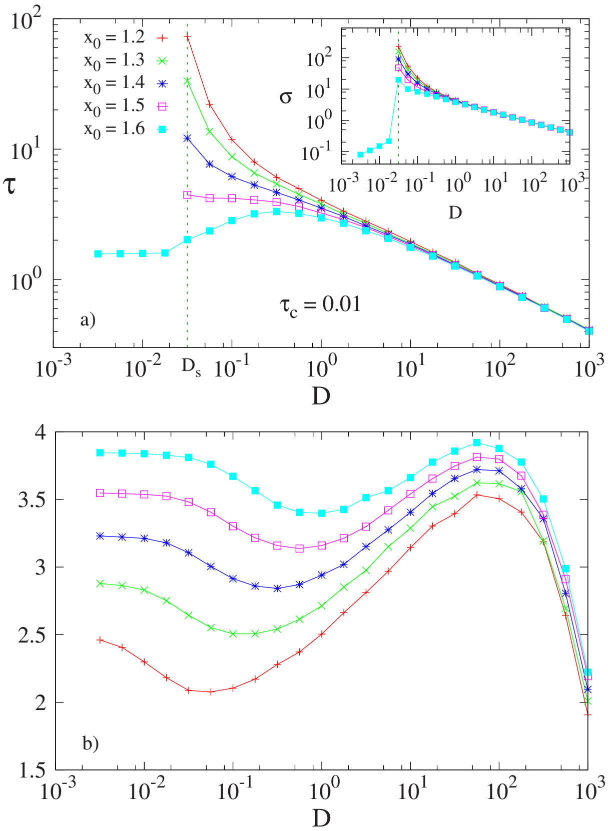

FIG. 2. (Color online) Panel a: log-log plot of the mean first passage time τ as a function of noise intensity D in the case of correlated noise with $\tau_c=0.01$ for the four initial positions investigated (see Fig. 1). Inset: the related standard deviation as a function of the noise intensity D. The dotted straight line at $D=D_s$ separates the simulation data representing the Brownian particles escaped within the maximum simulation time $T_{\max}$ for $D>D_s$ from those representing the particles partially trapped within the well for a time greater or equal to $T_{\max}$ for $D<D_s$. Panel b: mean growth rate coefficient Λ as a function of the noise intensity D, with the same initial positions of Fig. 1.

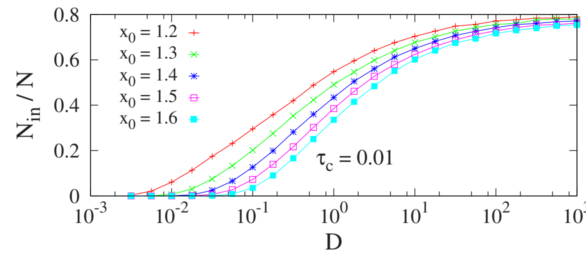

FIG. 3. (Color online) Semilogarithmic plot of the fraction of particles N_i/N reaching the threshold position $x_t=0.5$ into the potential well, within the $T_{\max}$, as a function of noise intensity D. This threshold position x_t corresponds to the flex point of the potential, where the instantaneous growth rate $\lambda_i(x,t)$ is equal to zero. The correlation time of the noise is $\tau_c=0.01$ with the same initial conditions of Fig. 1.

neous growth rate $\lambda_i(x,t)$ is equal to zero. We see that for low noise intensities the fraction N_i/N goes to zero, producing an increasing behavior of the MGRC [see Fig. 2(b)].

The behaviors of the MFPTs as a function of the noise intensity D with other values of τ_c are shown in Fig. 4. We clearly observe two dynamical regimes depending on the value of τ_c with respect to the relaxation time of the system ($\tau_r=0.6$): (a) weak colored noise ($0<\tau_c<\tau_r$) and (b) strong colored noise ($\tau_c>\tau_r$). By observing Fig. 4(a) ($\tau_c=0.1$) we can see that the qualitative behavior of MFPT shown in Fig. 2(a) is recovered. In the weak color noise regime we can still observe the divergent behavior of MFPTs for $x_{\max}<x_0<x_c$ and a nonmonotonic behavior for $x_0\geq x_c$, with $x_c=1.5$. By increasing the value of the correlation time ($\tau_c\geq\tau_r$) we observe a large displacement of the maximum of MFPT toward higher values of noise intensity and a shift of the peculiar initial position x_c toward lower values. For $\tau_c=\tau_r=0.6$, $x_c^*\simeq 1.4$, and for $\tau_c=1$, $x_c^*\simeq 1.3$ [see Figs. 4(b) and 4(c)], where x_c^* is the peculiar initial position of the Brownian particle in the presence of colored noise. We note that $x_c=1.5$ is a fixed value for the white-noise case [19], while the position x_c^* is a variable quantity for colored noise and it is depending on the value of the correlation time of the noise. Moreover, we observe a broadening of the NES region, which becomes very large for high values of the correlation time τ_c. The NES region is the area where enhanced stability of a metastable state is observed. In other words it is the area under each curve of τ vs D [see Figs. 2(a) and 4], where the values of τ are greater than the deterministic dynamical time related to the particular initial position investigated (see also Fig. 1 in *Mantegna and Spagnolo*, 1998, Ref. [5]).

The asymmetry of the potential profile with respect to the x coordinate makes more effective the correlation of the noise for Brownian particles moving from left to right. This means that, at very low noise intensities of the colored noise,

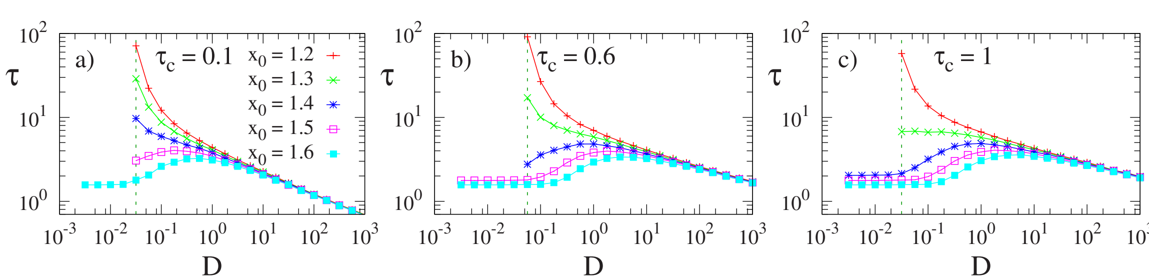

FIG. 4. (Color online) Log-log plot of the MFPT τ as a function of noise intensity D for the same initial positions of Fig. 1 and for different values of the correlation times τ_c, namely, $\tau_c=0.1,0.6,1$, corresponding, respectively, to the weak, intermediate, and strong colored noise dynamical regimes. The dotted straight line at $D=D_s$ separates the noise values for which all the Brownian particles escape ($D>D_s$) from those for which the particles are partially trapped into the well within the $T_{\max}$ ($D<D_s$).

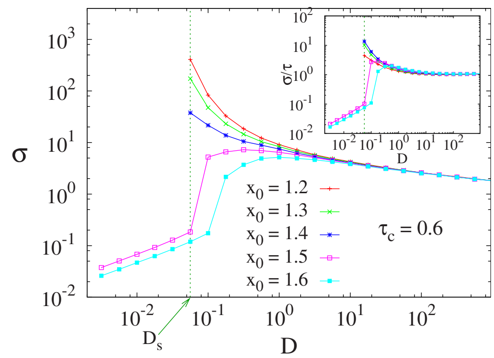

FIG. 5. (Color online) Log-log plot of the standard deviation σ as a function of the noise intensity D for τ_c=0.6 and the same initial positions of Fig. 1. The dotted straight line indicates the value of the noise intensity D_s which separates the simulation data representing the trapped Brownian particles from those escaped within $T_{\max}$. Inset: log-log plot of the ratio σ/τ as a function of noise intensity D.

the particles inside the potential well will escape more easily with respect to the white-noise case. Therefore, the trapping effect, which is responsible for the divergent behavior for any initial unstable state within the range $x_{\max}<x_o<x_c$ will happen in a restricted range of initial positions that is $x_{\max}<x_o<x_c^*$ with $x_c^*<x_c$. Specifically this peculiar position x_c^* is shifted toward decreasing values of the x coordinate for increasing correlation time τ_c of the noise source. In Fig. 2(a) and all panels of Fig. 4 the dotted straight line at $D=D_s$ separates the simulation data representing the Brownian particles escaped within the maximum simulation time $T_{\max}$ for $D>D_s$, from those representing the particles partially trapped within the well for a time greater or equal to $T_{\max}$ for $D<D_s$. This means that the simulation data obtained for $D<D_s$ underestimate the real data in the divergent dynamical regime. In fact if we prolong the maximum simulation time $T_{\max}$ we obtain more approximate values for τ and σ and the divergent behavior will be visible at lower noise intensities. As a consequence D_s will be shifted toward lower values.

For high values of the noise intensity all the plots show a monotonic decrease behavior as a function of noise intensity collapsing in a unique curve. Moreover the slope of this limit curve becomes flatter by increasing the correlation time. This means that the NES effect involves more and more orders of magnitude of the noise intensity. The effect of the colored noise is therefore to delay the escape process or in other words to enhance more and more the stability of the metastable state for increasing values of the noise intensity.

In Fig. 5 the standard deviation σ of the first passage time distribution for τ_c=0.6 is shown. We see a huge increase in the σ for low values of noise intensity, demonstrating a strong enlargement of the distribution when the particle feels a noise intensity comparable with the height of potential barrier. Similarly to the MFPTs, *color* induces a shift in the divergent behavior of σ. The relative measure of the width with respect to the mean value is shown in the inset of Fig. 5, where the ratio σ/τ is plotted. This ratio reveals a nonmonotonic behavior with a minimum, demonstrating the existence of a noise intensity for which the width of the first passage time distribution is the minimum related to its mean. In other words this value corresponds to a maximum of precision in the measure of τ. This optimal noise intensity is shifted toward high noise values by increasing τ_c.

The behavior of the mean growth rate coefficient Λ as a function of the noise intensity D for different values of the noise correlation time is shown in Fig. 6. In the weak color noise regime we observe a nonmonotonic behavior with a minimum for all the initial positions investigated with a shift in the position of the minimum toward higher noise intensities. In the strong color regime the minimum, which represents a trapping phenomenon for a finite time, is visible for the divergent behavior of MFPTs for $x_{\max}<x_0<x_c^*$, and it is shifted toward higher noise intensities by increasing the correlation time. For initial positions $x_0\geq x_c^*$, the minimum tends to disappear, but at the same time the Λ parameter decreases monotonically with increasing noise intensity, showing a trapping phenomenon at higher noise intensities. This trend to the disappearance of the minimum in the curves of Λ, corresponds to the trend to disappearance of the divergent behavior of τ, that is to a restricted range of initial positions for which we observe this divergent behavior. We note that the behavior of Λ as a function of the noise intensity D obtained in our analysis is in qualitative agreement with that obtained by the experimental investigation of the stabilization of a metastable state in an oscillatory chemical system (the Belousov-Zhabotinsky reaction) [17]. Specifically the decreasing behavior of the maximum Lyapunov exponent of Fig. 2 of Ref. [17] is in qualitative good agreement with the behavior of the curves in Figs. 6(b) and 6(c). This could be ascribed to the correlation time always present in noise sources used in any experimental setup.

In Fig. 7 we report, for all the initial positions investigated and for τ_c=0.6, the semilogarithmic plot of the fraction of

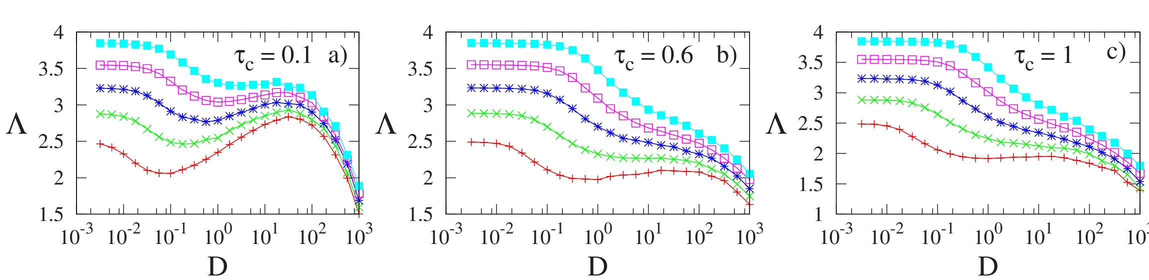

FIG. 6. (Color online) Semilogarithmic plot of mean growth rate coefficient Λ as a function of noise intensity D for the same initial positions x_o and the same values of the correlation times τ_c of Fig. 4, namely, τ_c=0.1, 0.6, 1, corresponding respectively to the weak, intermediate, and strong colored noise dynamical regimes.

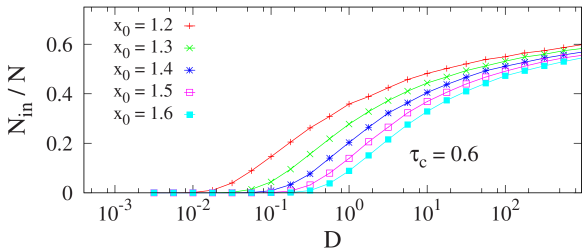

FIG. 7. (Color online) Semilogarithmic plot of the fraction of particles N_i/N reaching the threshold position x_t=0.5 into the potential well, within the T_{max}, as a function of noise intensity D. This threshold position x_t corresponds to the flex point of the potential, where the instantaneous growth rate $\lambda_i(x,t)$ is equal to zero. The correlation time of the noise is τ_c=0.6, with the same initial conditions of Fig. 1.

particles N_i/N entering into the potential well up to the position x_t=0.5, within the T_{max}, as a function of noise intensity D. At very low noise intensities and for increasing values of the correlation time τ_c, the particles have difficulty to enter into the potential well, within the T_{max} considered, shifting the entrance statistics toward higher values of the noise intensity.

IV. CONCLUSIONS

In this work we analyzed the effect of the colored noise, generated by an Ornstein-Uhlenbeck process, on the enhancement of the mean first passage time in a cubic potential with a metastable state and on the minimum of the mean growth rate coefficient as a function of the noise intensity. We analyze different initial unstable states. We obtain NES effect for all the initial positions investigated and an enhancement of the NES region for increasing values of correlation times. The results obtained for a particle moving in a cubic potential are quite general, because we always obtain NES effect when a particle is initially located just to the right of a local potential maximum and next to a metastable state in the escape region.

In experiments, real noise sources are correlated with a finite correlation time. As a consequence, the NES effect can be observed at higher noise intensities with respect to the idealized white-noise case. The enhancement and the shift of the NES region, toward higher values of the noise intensity, allows us to reveal experimentally the NES effect only by using a suitable correlation time τ_c in the noise source.

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