Analytical results of the Nagel-Schreckenberg model with stochastic open boundary conditions

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The deterministic Nagel-Schreckenberg model with stochastic open boundary conditions is investigated in a mostly analytical way. By means of the Markov Chain, we model the working process of the stochastic open boundaries. First, the analytical expression of the free-flow density profiles is derived. Then, we discuss theoretically how the right boundary determines the traffic capacity, global density, and density profiles. For these features, the analytical and numerical results agree well. This paper implies that the deterministic Nagel-Schreckenberg model with stochastic open boundaries is almost totally analyzable.

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I. INTRODUCTION

For years, cellular automata (CA) are used to simulate both freeway and urban transportation systems [1–6]. The Nagel-Schreckenberg model (NS model) [7] is a typical one-dimension CA model for the freeway traffic flow. In this model, the road is modeled as a one-dimensional lattice with L sites that are labeled in sequence by $1 \sim L$ from the left. Each site can either be empty or occupied by a car with velocity $0 \ldots v_{\max}$($v_{\max}$ is assumed to be the maximum velocity that one car can reach). All sites are updated according to four rules follows.

(a) Acceleration: $v = \min(v+1, v_{\max})$.

(b) Slow down: $v = \min(v, g)$(where g is the number of empty sites in front of the car).

(c) Randomization: $v = \max(v-1, 0)$ with probability p.

(d) Movement: move v sites forward, $x = x + v$.

Besides the rules above, the model's behaviors also depend on the boundary conditions. There are mainly two kinds of boundary conditions: the periodic boundary conditions and the open boundary conditions. In a periodic system, cars move on a ring and the car density keeps constant. Differently, open boundary conditions describe a road with entrance and exit. Cars enter the road from the left boundary, and leave via the right boundary. The most representative open boundary conditions are the stochastic open boundaries introduced in Ref. [8], which are defined as follows:

The stochastic left boundary (SLB) is implemented by an additional site (site 0) on the left of site 1, and works according to the "standard injection rule:" with probability α, a car with velocity $v = v_{\max}$ is created at site 0, this car immediately moves according to the NS rules. If site 1 is occupied by another car, the injected car is deleted then. This left boundary condition is simple in the implementation, but complex in its behaviors [8–10]. Thus it is not as popular as the "expanded stochastic left boundary" (ESLB) raised later in Ref. [11].

The stochastic right boundary (SRB) is realized by an additional site $L+1$ next to the site L. When updated, this site is occupied with probability β. That means if a car's velocity is large enough to move out of the road, then with probability β, it will run out; with probability $1-\beta$, it stops at the last site. Besides, there is also a "traffic light right boundary (TLRB)" [16–18], which opens and closes periodically like a traffic light in front of it.

Obviously, the model with open boundaries is closer to the real road with inflow and outflow. Thus, the influences and mechanisms of open boundaries attract much scientific attention. The most popular tool of such researches is numerical simulation. On the other hand, it is desirable and useful to have a better qualitative understanding of the numerical results using analytic approaches, which are more difficult but can tell us much more than simulations. The analytical results of periodical-boundary models have been searched for years, many exact or approximate results are gained [12–15]. Unfortunately, a quantitative understanding of the open boundary models remains elusive. For the left boundary, Ref. [17] proves that ESLB to be completely analyzable. Recent research [19] shows that the exact functional relation between α and the inflow under standard SLB can also be figured out. But there is another unanswered question: Ref. [8] reported that in the free-flow phase of the deterministic NS model, SLB creates a periodical-structural density profiles. This is a quite interesting phenomenon without a convincing explanation found. There are even fewer theoretical results on the right boundary: Refs. [17,18] give out some results of the TASEP (totally asymmetric simple exclusion process) related models with TLRB. But the SRB still lacks analytical studies.

To get a deeper insight of the open boundary conditions, our works concentrate on the deterministic NS model with stochastic open boundaries. As an expansion of the works in Ref. [19], we proved that both the car-injection and the car-removal procedures can be described by Markov chain models. First, we find out an analytical way to calculate the free-flow density profiles. Second, we discuss how the stochastic right boundary determines the traffic capacity, global density, and density profiles in the jamming regime.

This paper is organized as follows: in Sec. II, we analytically explain why the free-flow density profiles are characterized by a periodic structure. In Sec. III, we model the car-removal procedure and put out the exact results of traffic capacity. In Secs. IV and V, we show that the global density and the jamming density profiles can be also predicted theoretically. Section VI is the conclusion and discussion.

II. DENSITY PROFILES IN THE FREE-FLOW PHASE

This section aims to show an analytical explanation for the periodical-structure of the density profiles in DNS($v_{\max}$

≤ 5)(means "deterministic NS model with $v_{\max} \leq 5$"). In DNS($v_{\max} > 5$), the situation is much more complex, without effective approaches found yet. As we concentrate on a phenomenon in the free-flow phase, the right boundary influence is reasonably ignored.

A. Modeling Cars' Movements

Given an arbitrary step t, every car on road has a unique state. We can describe the state by a triad (l, v_t, v_{t+1}), where l is the location of the car at step t, v_t is its speed at the beginning of step t, and v_{t+1} is its speed at the beginning of step $t+1$.

For a single car, suppose its state at step t is C_1, on next step it changes to C_2. Then the state transition $C_1 \rightarrow C_2$ tells the car's behavior in successive two steps. We call C_1 the source-state of C_2, and C_2 is the target-state of C_1.

If a car is inserted at step $t-1$, then at step t this car gets a new state C_{new}. It can be easily concluded that C_{new} has no source state. We call this kind of car states as "root states." It is clear that all the states that can be possibly observed in the system must be derived from a root states. Moreover, according to NS rules, for a car-state $C=(l, v_t, v_{t+1})$, if $l = v_t \Leftrightarrow C$ is a root states.

Let $\mathbb{C}$ denote the set of all the legal car states that may appear in the system, and we denote the probability of site l is occupied by $o(l)$, then

$$o(l) = \sum_{c \in \mathbb{C}, c[0]=l} P(C), \tag{1}$$

where $C[0]$ is the first element of C, and $P(C)$ is defined as: suppose that the system runs for n (n is a sufficiently large number) steps, and the state C appears for m times, then $P(C)=m/n$.

Suppose that a car-state C can evolve to C' with probability p, then $P(C')=P(C)p$. If among all root states, there are n and only n states $C_1 \sim C_n$, which can evolve to state C' with probabilities $p_1 \sim p_n$, respectively, then $P(C') = \Sigma_{i=1}^{n} P(C_i) p_i$. Furthermore, if all the state transitions are determined (each car-state has only one target state), then $P(C') = \Sigma_{i=1}^{n} P(C_i)$.

B. Density profiles in DNS($v_{\max}$=4)

Ref. [19] reports an important feature caused by standard injection rule—the so-called "injection-produced slow down" (IPSD for short). It indicates that a car may be forced to slow down even without the influence of right boundary. Ref. [19] also proved that IPSDs do not exist in DNS with $v_{\max} \leq 4$.

The nonexistence of IPSDs in DNS($v_{\max}$=4) guarantees that in the free-flow regime, all the cars can drive freely. It follows that for any car-state (l, v_t, v_{t+1}), $v_{t+1} = \min(v_t + 1, v_{\max})$ holds. Thus, the car-state could be simplified as (l, v_t), meanwhile the rule of state transition becomes

$$(l, v_t) \rightarrow (l + v_{t+1}, v_{t+1}) \quad \text{where} \quad v_{t+1} = \min(v_{\max}, v_t + 1).$$

According to the definition of root states, we can list out all root states in DNS($v_{\max}$=4)

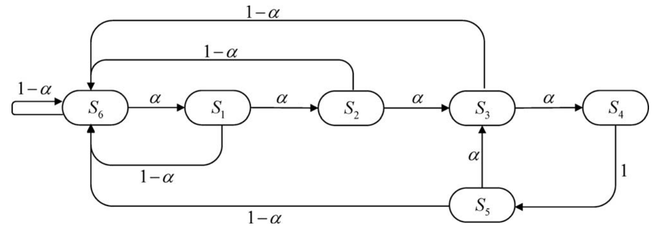

FIG. 1. The state transition graph of DNS($v_{\max}$=4).

$$(1,1), \ (2,2), \ (3,3), \ (4,4).$$

Moreover, by the state transition rule, all the state transitions are determined. We can list all the state transition sequences in DNS($v_{\max}$=4)

$$(1,1) \rightarrow (3,2) \rightarrow (6,3) \rightarrow (10,4) \rightarrow (14,4) \rightarrow \ \ldots,$$

$$(2,2) \rightarrow (5,3) \rightarrow (9,4) \rightarrow (13,4) \rightarrow \ \ldots,$$

$$(3,3) \rightarrow (7,4) \rightarrow (11,4) \rightarrow \ \ldots,$$

$$(4,4) \rightarrow (8,4) \rightarrow (12,4) \rightarrow \ldots$$

Clearly seen that from site 4 on, cars move $v_{\max}$ sites forward each step, hence we have

$$(l, v_t) \rightarrow (l+4, 4), \quad \text{if } l \geq 4.$$

In order to calculate the probabilities of the root states, let us define the "road state" first. Suppose the state of the first car in the road is $\hat{C}=(l, v_t, v_{t+1})$. If $l \leq 4$, then the road state is $\hat{C}$. Otherwise, the road state is a constant value $\bar{C}$ (Because if $l > 4$, then its target state is (4,4) with probability α and is $\bar{C}$ with $1-\alpha$, Thus, all states satisfies $l > 4$ could be regarded as the same). Thus, for DNS($v_{\max}$=4), the time evaluation of the system can be described by a Markov chain with state space $S=\{S_1 \ldots S_6\}$, where $S_1=(4,4)$, $S_2=(3,3)$, $S_3=(2,2)$, $S_4=(1,1)$, $S_5=(3,2)$, and $S_6=\bar{C}$. The state transition graph is show by (Fig. 1).

It is easy to prove it is a homogeneous Markov chain with aperiodicity and irreducibility. Thus, its limiting distribution exists. Let P_i denote the limiting probability of state S_i, and $\vec{P}=[P_1 \cdots P_6]$, then p_i can be obtained by solving the linear equation system

$$\vec{P} = \vec{P}T,$$

$$\sum_{j=1}^{n} P_j = 1,$$

where T is the one-step transition probability matrix, which indicate the probabilities of all the one-step state transitions. Finally we get

$$P(4,4) = \frac{\alpha(1-\alpha^2)}{1+\alpha+\alpha^4}, \quad P(3,3) = \frac{\alpha^2(1-\alpha^2)}{1+\alpha+\alpha^4},$$

$$P(2,2) = \frac{\alpha^3}{1+\alpha+\alpha^4}, \quad P(1,1) = \frac{\alpha^4}{1+\alpha+\alpha^4}.$$

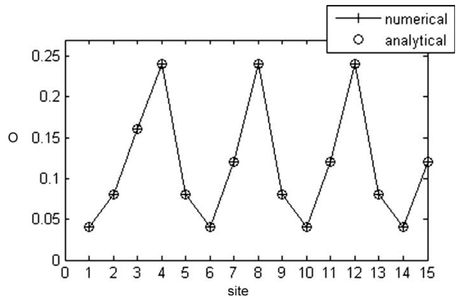

FIG. 2. Analytical and numerical results of the density profiles in DNS(v_{max}=4).

We will verify this result by the case of α=0.5. After substituting α=0.5 into the expressions above, we get the results follows:

$$P(1,1) = 0.04, \quad P(2,2) = 0.08, \quad P(3,3) = 0.12,$$

$$P(4,4) = 0.24.$$

Recall (1), the density profiles could be figured out

$$o(1) = 0.04,$$

$$o(2) = 0.08,$$

$$o(3) = P(3,3) + P(3,2) = P(3,3) + P(1,1) = 0.16,$$

$$o(4) = P(4,4) = 0.24,$$

$$o(5) = P(2,2) = 0.08,$$

$$o(6) = P(1,1) = 0.04,$$

$$o(7) = P(3,3) = 0.12.$$

And if $l>7$, then $o(l)=o(l-4)$.

The results highly agree with those by simulations (α =0.5) (Fig. 2).

The cases for other value of α could be easily verified by readers, and the density profiles in DNS(v_{max}=2), DNS(v_{max}=3) can be figured out in the same way.

C. Density profiles in DNS(v_{max}=5)

Though IPSDs exist, their influence is very limited in DNS(v_{max}=5). The following conclusion will greatly simplify our research on its free-flow density profiles.

Theorem 1. In DNS(v_{max}=5), only cars at site 4,5 might be effected by IPSD.

Proof. See Appendix A

Consequently, given a car-state $C=(l,v_t,v_{t+1})$, if $l\neq 4$ and $l\neq 5$, then $v_{t+1}=\min(v_t+1,v_{max})$; If l=4 or l=5, then v_{t+1} is either 4 or 5. Thus, there exists in total seven root states in DNS(v_{max}=5)

$$(1,1,2),\ (2,2,3),\ (3,3,4),\ (4,4,4),$$

$$(4,4,5),\ (5,5,4),\ (5,5,5).$$

Among which (4,4,4) and (5,5,4) are consequences of IPSDs. By investigating the system behaviors, we can find that all the state transitions in DNS(5) are also determined. The state transition sequences in DNS(v_{max}=5) are given follows:

$$(1,1,2) \to (3,2,3) \to (6,3,4) \to (10,4,5) \to (15,5,5) \to \ldots,$$

$$(2,2,3) \to (5,3,4) \to (9,4,5) \to (14,5,5) \to \ldots,$$

$$(3,3,4) \to (7,4,5) \to (12,5,5) \to \ldots,$$

$$(4,4,4) \to (8,4,5) \to (13,5,5) \to \ldots,$$

$$(4,4,5) \to (9,5,5) \to (14,5,5) \to \ldots,$$

$$(5,5,4) \to (9,4,5) \to (14,5,5) \to \ldots,$$

$$(5,5,5) \to (10,5,5) \to (15,5,5) \to \ldots$$

From site 7 on, cars move v_{max} sites forward each step, hence

$$(l,v_t,v_{t+1}) \to (l+5,5,5) \ \text{if} \ l \geq 7.$$

Similarly, we define the system state first. Suppose the state of the first car in the road is $\hat{C}=(l,v_t,v_{t+1})$, if $l>5$ and $v_{t+1}=5$, then we consider the road state is a constant value $\bar{C}$ (because if $l>5$ and $v_{t+1}=5$, then its target state is (5,5,5) with probability α and is $\bar{C}$ with $1-\alpha$, which can be easily verified by readers. Thus, all states satisfying "$l>5$ and $v_{t+1}=5$" could be regarded as the same). Otherwise, the road state is $\hat{C}$. Thus, in DNS(5), there are in total 11 system states, denoted by $S_1 \sim S_{11}$

$$S_1 = (5,5,5), \quad S_2 = (5,5,4), \quad S_3 = (4,4,5),$$

$$S_4 = (4,4,4), \quad S_5 = (3,3,4), \quad S_6 = (2,2,3),$$

$$S_7 = (1,1,2), \quad S_8 = (5,3,4), \quad S_9 = (6,3,4),$$

$$S_{10} = (3,2,3), \quad S_{11} = \bar{C}.$$

Then the time evaluation of the system can be described by a Markov chain with state space $S=\{S_1 \ldots S_{11}\}$. The state transition graph is given follows (Fig. 3).

Now, we can figure out probabilities of all root states, using the limiting probabilities of the Markov chain above

$$P(5,5,5) = \frac{\alpha(\alpha^6 - \alpha^5 + \alpha^4 - \alpha^3 - \alpha^2 + 1)}{\alpha^5 + \alpha^4 - \alpha^3 + \alpha + 1},$$

$$P(5,5,4) = \frac{\alpha^6(1-\alpha)}{\alpha^5 + \alpha^4 - \alpha^3 + \alpha + 1},$$

$$P(4,4,5) = \frac{\alpha^2(\alpha^6 - \alpha^5 + \alpha^4 - \alpha^3 - \alpha^2 + 1)}{\alpha^5 + \alpha^4 - \alpha^3 + \alpha + 1},$$

$$P(4,4,4) = \frac{-\alpha^5(\alpha^3 - \alpha^2 + \alpha - 1)}{\alpha^5 + \alpha^4 - \alpha^3 + \alpha + 1},$$

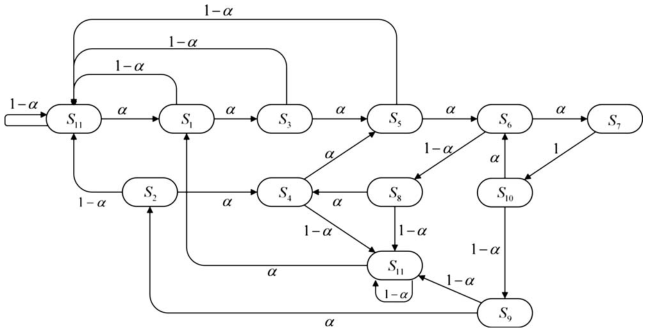

FIG. 3. The state transition graph of DNS($v_{\max}$=5).

$$P(3,3,4) = \frac{-\alpha^3(\alpha^2-1)}{\alpha^5+\alpha^4-\alpha^3+\alpha+1},$$

$$P(2,2,3) = \frac{\alpha^4}{\alpha^5+\alpha^4-\alpha^3+\alpha+1},$$

$$P(1,1,2) = \frac{\alpha^5}{\alpha^5+\alpha^4-\alpha^3+\alpha+1}.$$

We will verify this result by the case of α=0.65. After substituting α=0.65 into the expressions above, we have the results as follows:

$$P(1,1,2) = 0.0695,$$

$$P(2,2,3) = 0.107,$$

$$P(3,3,4) = 0.094,$$

$$P(5,5,4) = 0.0158,$$

$$P(4,4,4) = 0.0346,$$

$$P(4,4,5) = 0.111,$$

$$P(5,5,5) = 0.172.$$

Recall Eq. (1) and the state transition sequences, only $o(1) \sim o(11)$ need calculating

$$o(1) = P(1,1,2) = 0.0695,$$

$$o(2) = P(2,2,3) = 0.107,$$

$$o(3) = P(3,3,4) + P(1,1,2) = 0.094,$$

$$o(4) = P(4,4,4) + P(4,4,5) = 0.147,$$

$$o(5) = P(5,5,5) + P(5,5,4) + P(2,2,3) = 0.295,$$

$$o(6) = P(6,3,4) = P(1,1,2) = 0.0695,$$

$$o(7) = P(7,4,5) = P(3,3,4) = 0.094,$$

$$o(8) = P(8,4,5) = P(4,4,4) = 0.0346,$$

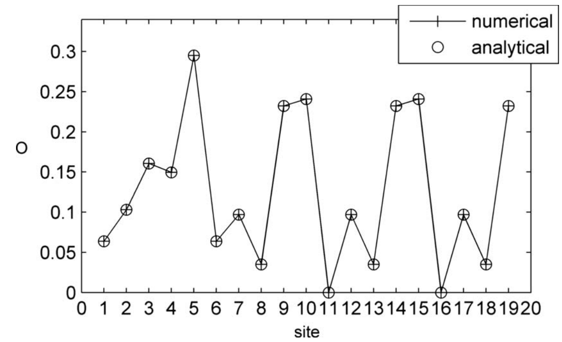

FIG. 4. Analytical and numerical results of the density profiles in DNS($v_{\max}$=5).

$$o(9) = P(9,4,5) + P(9,5,5) = P(2,2,3) + P(4,4,5) + P(5,5,4) = 0.234,$$

$$o(10) = P(10,4,5) + P(10,5,5) = P(1,1,2) + P(5,5,5) = 0.241.$$

Since no root state can evolve to a car-state (l, v_t, v_{t+1}) with l=11, hence $o(11)$=0.

And if l>11, then $o(l)$=$o(l-5)$.

Compared to the simulation data (Fig. 4), the analytical results are with high precision.

D. Free-Flow Density profiles under ESLB

The expanded stochastic left boundary outlined in Ref. [11] is defined as: the left boundary is expanded from one single site to a minisystem of width $v_{\max}$+1 (Fig. 5). It works according to the expanded injection rule: when updated, if there is a car in the minisystem, it has to be emptied first. Then a vehicle with an initial velocity of $v_{\max}$ is inserted with probability α. Its initial position is the site at the right end point of the boundary if no car is present in the main system within the first $v_{\max}$ sites, otherwise its initial position is the site with $v_{\max}$ distance from the first car in the main system.

With the help of such a rule, high inflow can be achieved [11], so the whole spectrum of possible system states is accessible. As a result, ESLB has become more popular than SLB [20,21].

According to the injection rule of the ESLB, we can easily realize that all cars run with velocity $v_{\max}$ if the right boundary exerts no influence. Thus, the car state could be denoted by its position l only, the state transition rule then becomes

$$(l) \rightarrow (l+5).$$

Let us next number the sites in the minisystem (from left to right) with $-v_{\max} \ldots 0$. Noted that if a car is placed on site

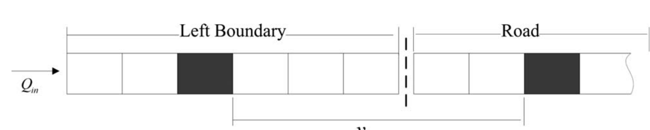

FIG. 5. The expanded stochastic left boundary.

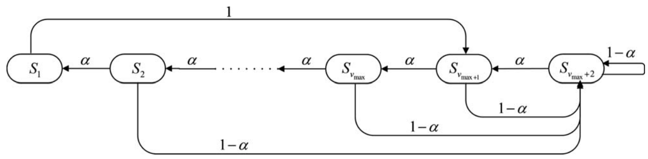

FIG. 6. The state transition graph of DNS with expanded left boundary.

$-v_{max}$, then it cannot enter the main road, thus we find that the number of root states is equal to v_{max}, and they are $-v_{max}+1 \sim 0$.

Suppose the state of the first car in the system is $\hat{C}=(l)$, we define the system state in the following way.

The system state is $\hat{C}$ if $l \leq 0$. Otherwise, the system state is a constant value $\bar{C}$. Thus, there are totally $v_{max}+2$ system states denoted by $S_1 \sim S_{v_{max}+2}$, where

$$S_i = \begin{cases} (i - v_{max} - 1) & \text{if} \quad 1 \leq i \leq v_{max} + 1, \\ \bar{C} & \text{if} \quad i = v_{max} + 2. \end{cases}$$

.

As shown in Ref. [17], the time evaluation of the system can be described exactly by a Markov chain with state space $S=\{S_1 \ldots S_{v_{max}+2}\}$. The state transition graph is given follows (Fig. 6).

Let P_i denote the limiting probability of system state S_i, we can find that

$$P_i = \frac{\alpha^{v_{max}+2-i}(1-\alpha)}{1-\alpha^{v_{max}} + \alpha^{v_{max}+1} - \alpha^{v_{max}+2}}.$$

Thus, for any site l in the main road

$$o(l) = \begin{cases} o(l - v_{max}) = P_{l+1} = \dfrac{\alpha^{v_{max}+1-l}(1-\alpha)}{1-\alpha^{v_{max}} + \alpha^{v_{max}+1} - \alpha^{v_{max}+2}} & \text{if} \quad l \leq v_{max}, \\ o(l - v_{max}) & \text{if} \quad l > v_{max}. \end{cases} \tag{2}$$

We can verify Eq. (2) by the case with $\alpha=0.65$ and $v_{max}=5$ (Fig. 7).

III. TRAFFIC CAPACITY

According to the works in Ref. [8], in the free-flow region, the outflow of the road equals to the inflow. In the jamming region, the system outflow equals to the maximum outflow that right boundary allows to pass through. From a more realistic view, such maximum outflow corresponds to the traffic capacity. The capacity may be the most attractive feature of the right boundary [17,18]. In this section, we concentrate on how the right boundary determines the traffic capacity, therefore, we just investigate the traffic flow in jamming phase.

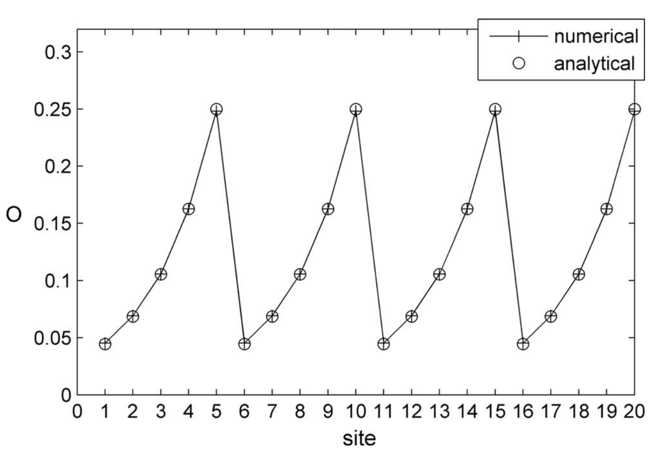

FIG. 7. Analytical and numerical results of the density profiles in DNS($v_{max}=5$) with expanded left boundary.

Figure 8 is a snapshot of the system before it is completely jammed. The left part of the road is full of cars running freely which are not influenced by the right boundary yet. But the right part of the traffic flow shows a periodical structure with alternating jamming blocks and driving sections. Here, we temporarily ignore the queuing cars at the right boundary, the leftmost jamming block is called a "leading block," and other blocks are named as "following blocks."

In the deterministic NS model, there exist enough analytical results for the dynamics of jam blocks. For a single jamming block, its right boundary moving from right to the left with a speed $v_{jr}=1$, and its left boundary propagates backward with a speed $v_{jl}=\frac{v_{max}J_{in}}{v_{max}-J_{in}}$, where J_{in} is the inflow of the jam block. Because the inflow of the leading block is the inflow created by car-injection procedure, thus the size of the leading block will keep decreasing except the case of using expanded injection rule with $\alpha=1$ [11].

However, differently, for the following jam blocks, there exists $J_{in}=J_{out}$ (outflow of the jam block), thus the size of the

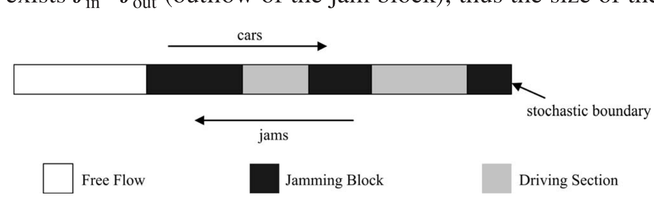

FIG. 8. The typical structure of the traffic flow in the jamming phase.

following blocks will be constant until the leading block completely vanishes. Consequently, the size of a driving section between two jam blocks also keeps steady. Moreover, the inflow of the right boundary equals to $J_{\text{out}}=\frac{v_{\max}}{v_{\max}+1}$ because it is actually the outflow from the rightmost jam block. We have to emphasize again that Fig. 8 only illustrates the situation before it is completely jammed. With the evolutionary of the system, the free-flow area keeps shrinking until the whole road is full of the jam-driving structure, when the global density becomes steady.

Next, we will introduce a Markov Chain-based method for the traffic capacity.

First, we construct the state space $\mathbb{S}$ of the Chain. The system state is defined as a two-tuple: (d,v), where d is the distance between the last car and right boundary, and v is the velocity. Let us recall the important conclusion: the inflow of right boundary equals $\frac{v_{\max}}{v_{\max}+1}$. This assertion implies that in each step, there must be a car arriving at or running through the right boundary. Thus, $\forall S=(d,v)\in\mathbb{S}$, S must satisfy $d \leq \min(v+1,v_{\max})$.

The above conclusion tells that for a DNS, its state space is finite with size $\sum_{k=2}^{v_{\max}+2}k-1$. Furthermore, all the states can be listed in the following way:

$$\begin{array}{lllll} (0,0) & (0,1) & \cdots & (0,v_{\max}-1) & (0,v_{\max}) \\ (1,0) & (1,1) & \cdots & (1,v_{\max}-1) & (1,v_{\max}) \\ & (2,1) & \cdots & (2,v_{\max}-1) & (2,v_{\max}) \\ & & \vdots & & \\ & & & (v_{\max},v_{\max}-1) & (v_{\max},v_{\max}). \end{array}$$

Second, we can prove that the state transitions between states obey several rules follows.

Given an arbitrary time step t, we denote the system state at t and $t+1$ by S and S', respectively. Suppose that $S=(d,v)$, then:

Rule 1: with probability 1, $S'=[0,\min(v+1,v_{\max})]=(0,d)$ if $d=\min(v+1,v_{\max})$.

Rule 2: with probability $1-\beta$, $S'=(0,d)$ if $d<\min(v+1,v_{\max})$.

Rule 3: with probability β, $S'=(d+1,v)$ if $d<\min(v+1,v_{\max})$.

Proof of these rules is shown in Appendix B.

Now we have got enough preparation to model the SRB. We will illustrate it by a simple case of DNS($v_{\max}=2$). Its state space contains eight states

$$(0,0),\ (0,1),\ (0,2),\ (1,0),\ (1,1),\ (1,2),\ (2,1),\ (2,2).$$

We marked them in sequence by $S_1 \sim S_8$, and the one-step transition matrix is obtained by the rules above

$$\left\{\begin{array}{cccccccc} 1-\beta & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 1-\beta & 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 1-\beta & 0 & 0 & 0 & 0 & 0 & \beta \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right\}.$$

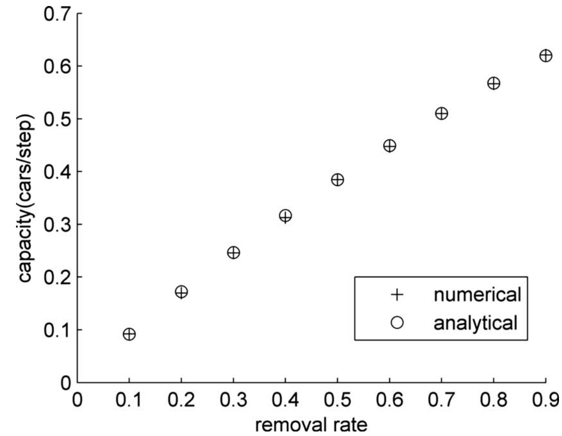

FIG. 9. Analytical and numerical results of the capacity under SRB in DNS($v_{\max}=2$).

Now, although $P_1 \sim P_8$ (the limiting probabilities of $S_1 \sim S_8$, respectively) can be gained, we actually need only a few values among them. Note that for an arbitrary step t, if the system state (d,v) satisfies $d=\min(v+1,v_{\max})$, then definitely no car can run out of the road at that step. Thus, the maximum outflow, such that the traffic capacity is

$$Q_{\text{out}} = \left(1-\sum P_i\right)\beta, \tag{3}$$

where P_i is the limiting probability of $S_i=(d,v)$ satisfying $d=\min(v+1,v_{\max})$. In this case, $i=4,7,8$. After substituting P_4, P_7, and P_8 into (3), the result is

$$Q_{\text{out}} = \frac{\beta^3+\beta}{\beta^3+\beta+1}.$$

The comparison between numerical and analytical results is shown in Fig. 9.

For higher velocity, large state space brings large state transition matrixes. However, because the state space is finite, and transition rules are all determined, we can calculate the limiting probabilities automatically by computer programs.

Following is the traffic capacity of DNS($v_{\max}=5$), which is calculated by a MATLAB program.

$$Q_{\text{out}} = \left(\frac{\beta^2-\beta+\beta^9-\beta^6-\beta^{15}}{\beta^{15}+\beta^{10}+\beta^6+\beta^3+\beta+1} + \frac{\beta^{14}+\beta^5+1}{\beta^{14}-\beta^{13}+\beta^{12}-\beta^{11}+\beta^{10}+\beta^5-\beta^4+\beta^3+1}\right)\beta.$$

Figure 10 tells that high precision is achieved.

IV. GLOBAL DENSITY

The steady global density in the free-flow phase is very easy to calculate, from the traffic flow theory, we know that density equals volume divided by velocity. Ref. [8] finds that all cars will run with velocity $v_{\max}$ after a sufficiently long time. Thus, we can calculate the global density

$D_g = \frac{Q_{\text{in}}}{v_{\max}}$, where Q_{in} is the volume of inflow.

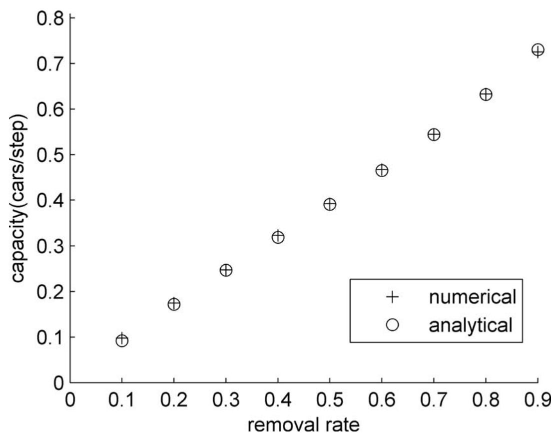

FIG. 10. Analytical and numerical results of the capacity under SRB in DNS($v_{\max}=5$).

The steady density in the jamming phase is much more difficult to get. First, we turn to the dynamics near the right boundary. If the right boundary keeps close, arriving cars will queue at the right boundary, which causes a new jam block. Based on the conclusion in the previous section, the size of the evolving jam block will increase by 1 per step during the closed period because its inflow equals $\frac{v_{\max}}{v_{\max}+1}$. Thus, the size of the newly generated jam block is equal to the duration of the closed period. When the right boundary changes to the open state from the closed state, cars start to move out of the road through the boundary. Then, the rightmost jam block start to move backward, meanwhile the driving section that next to the rightmost jam block start to form. Moreover, the size of this driving section is equal to the duration of the open period (Fig. 11).

Let us next develop a scenario to help further analysis. Suppose that the system has run for T steps (T is a sufficiently large number) in jamming phase. According to the analysis in the previous section, the system has reached the steady global density, and a number of jam blocks and driving sections with random size have been generated during this period.

Let $N_j(n)$ be the total number of jam blocks with size n, and $N_f(n)$ be the total number of driving sections with size n.

Let $r(n)$ denote the number of vehicles that can leave the road in an open period with n steps.

Let $\rho_j(n)$ denote the number of cars contained in a jam block with size n. Obviously, $\rho_j(n)=n$. Let $\rho_f(n)$ denote the number of cars contained in a driving area with size n. Recall the definition of $r(n)$, there exists $\rho_f(n)=n-r(n)$.

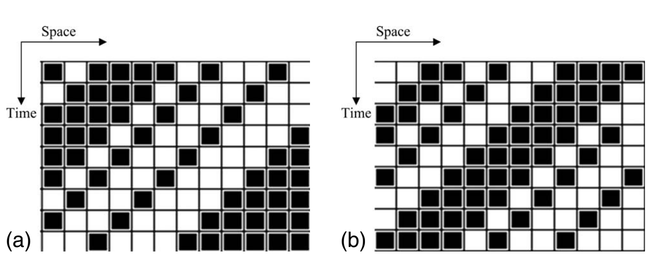

FIG. 11. Formation of the (a) jam block and the (b) driving section.

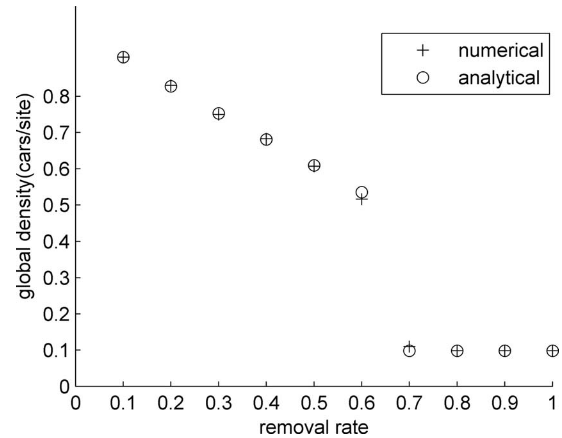

FIG. 12. Analytical and numerical results of global density with different removal rate in DNS($v_{\max}=5$).

Thus, the number of the leaved cars in the T steps is

$$N_{\text{out}}=\sum_{n=1}^{+\infty} N_f(n)r(n).$$

Consequently, the capacity can also be figured out by

$$Q_{\text{out}}=\frac{N_{\text{out}}}{T}=\sum_{n=1}^{+\infty}\frac{N_f(n)}{T}r(n).$$

The global density when the road is completely jammed could be calculated by

$$D_g=\frac{\sum_{n=1}^{+\infty}N_j(n)\rho_j(n)+\sum_{n=1}^{+\infty}N_f(n)\rho_f(n)}{\sum_{n=1}^{+\infty}N_j(n)n+\sum_{n=1}^{+\infty}N_f(n)n}$$

$$=1-\frac{\sum_{n=1}^{+\infty}N_f(n)r(n)}{T}=1-Q_{\text{out}}.$$

In summary, the steady global density satisfies

$$D_g=\begin{cases}\dfrac{Q_{\text{in}}}{v_{\max}} & \text{if} \quad Q_{\text{in}}<Q_{\text{out}},\\ 1-Q_{\text{out}} & \text{if} \quad Q_{\text{in}}>Q_{\text{out}}.\end{cases} \tag{4}$$

Formula (4) could be verified by an example (Fig. 12). In the simulations we use SLB and set injection rate α to a fixed value 0.5, $v_{\max}=5$.

Moreover, we can analyze the time evaluation of the global density. If $Q_{\text{in}}<Q_{\text{out}}$, the time evolution from step 0 (when the whole road is empty) can be divided into two stages:

Stage 1: this stage starts when the first car enters the road, and finishes when the first car reaches the right boundary. During this stage, the global density increases by Q_{in}. We denote the length of road by L, then this state lasts for $\frac{L}{v_{\max}}$ steps.

Stage 2: this stage starts when the first car reaches the right boundary. During this stage, the global density fluctuates around the equilibrium value $\frac{Q_{\text{in}}}{v_{\max}}$.

Correspondingly, if $Q_{\text{in}}>Q_{\text{out}}$, the time evolution can be divided into three stages:

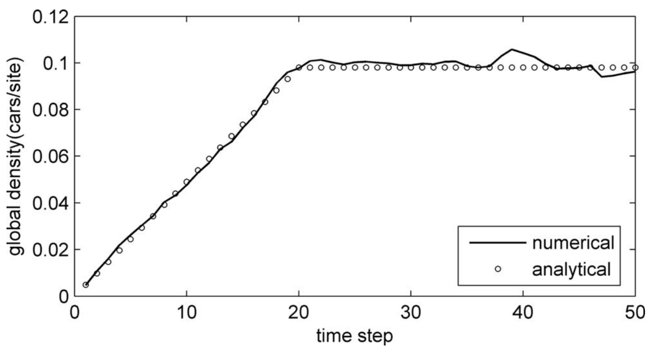

FIG. 13. The time evolution of global density in DNS($v_{\max}=5$), $\alpha=0.4$, $\beta=0.8$, and $L=1000$.

Stage 1: at this stage, only free-flow exists in the road. This stage starts when the first car enters the road, and finishes when the first car reaches the right boundary. During this stage, the global density increases by Q_{in}, and this stage lasts for $\frac{L}{v_{\max}}$ steps.

Stage 2: at this stage, free-flow, jam blocks and driving sections co-exist in the road. This stage starts when the first car reaches the right boundary, and finishes when a leading jamming block reaches the left boundary. During this stage, the global density increases by $Q_{\text{in}}-Q_{\text{out}}$. Figure 8 is a snapshot of this stage.

Stage 3: at this stage, the whole road is full of the jam-driving structure, no free-flow exists in the road. In this state, the global density stops increasing but stochastically fluctuates around an equilibrium value. The equilibrium value of the global density at this stage is $1-Q_{\text{out}}$.

The above conclusion can be illustrated by two examples:

(1) $\alpha=0.5$, $\beta=0.8$, and $L=1000$ (Fig. 13). Under the given condition, $Q_{\text{in}}=0.49$, $Q_{\text{out}}=0.632$, and $Q_{\text{in}}<Q_{\text{out}}$. Thus, we can analyze the two stages, respectively.

Stage 1: the global density increases by 4.9×10^{-4} per step, and this stage lasts for 200 steps. Finally, the global density is 0.098.

Stage 2: the global density fluctuates around 0.098.

From Fig. 13, we can read that strong fluctuations exist, but in fact, it is a finite-size effect: small jam blocks emerge and disappear quickly near the right boundary, thus causing fluctuations. If we enlarge the size of the road, fluctuations will be weaker.

(2) $\alpha=0.5$, $\beta=0.4$, and $L=1000$ (Fig. 14). Under the given condition, $Q_{\text{in}}=0.49$, $Q_{\text{out}}=0.318$, and $Q_{\text{in}}>Q_{\text{out}}$. Thus, there are three stages:

Stage 1: the global density increases by 4.9×10^{-4} per step, and this stage lasts for 200 steps. Finally the global density is 0.098.

Stage 2: the global density increases by 1.72×10^{-4} per step, and finally the global density reaches $D_g=1-Q_{\text{out}}=0.682$. So, this stage lasts for $\frac{0.682-0.098}{1.72\times10^{-4}}\approx3395$ steps.

Stage 3: the global density fluctuates around $1-Q_{\text{out}}=0.682$.

V. DENSITY PROFILES IN THE JAMMING PHASE

Based on the results of the previous sections, we can step further to find the density profiles in the jamming phase analytically.

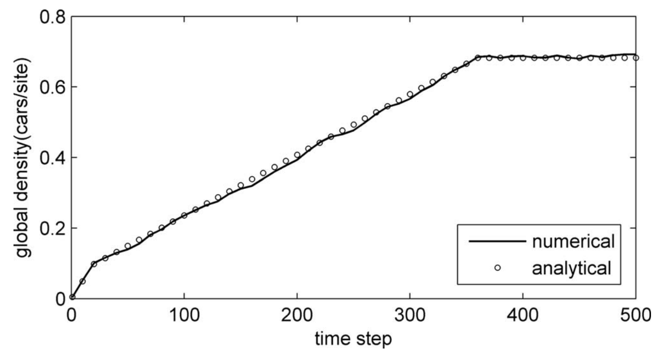

FIG. 14. The time evolution of global density in DNS($v_{\max}=5$), $\alpha=0.8$, $\beta=0.5$, and $L=1000$.

For any single site $1\leq l\leq L$, let $o(l)$ denote the probability of site l is occupied. If l is influenced by the left boundary, then $o(l)$ can be calculated by methods in the Sec. II. On the other hand, if it is influenced purely by the right boundary, we can first make an analysis.

Without loss of generality, let us consider an arbitrary site l which is exclusively under the influence of right boundary. Consequently, we know that jam blocks and driving sections keep running through site l, from right to the left, by velocity 1. Thus, it takes n steps for a jam block with length n to pass through site l completely. Obviously, during the n steps, site l is occupied for n steps. On the other hand, as for a driving section with length n, site l is occupied for $n-r(n)$ steps during n steps. Hence, we find

$$o(l)=\frac{\sum_{n=1}^{+\infty}N_j(n)n+\sum_{n=1}^{+\infty}N_f(n)[n-r(n)]}{\sum_{n=1}^{+\infty}N_j(n)n+\sum_{n=1}^{+\infty}N_f(n)n}$$
$$=1-\frac{\sum_{n=1}^{+\infty}N_f(n)r(n)}{T}=1-Q_{\text{out}}. \quad (5)$$

Note that in the jamming regime, every site has the same occupation rate, thus we can show the relationship between the removal rate and the occupation rate in one figure (Fig. 15). We have to emphasize that in the simulations, we use expanded left boundary, and the injection rate is 1. This setting is to ensure the stability of all the jam blocks, which

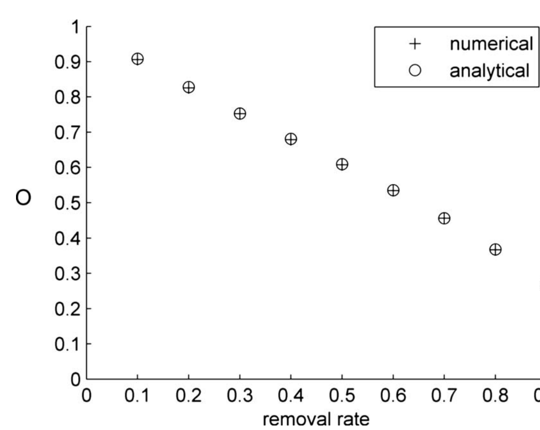

FIG. 15. Analytical and numerical results of jamming density profiles with different removal rate in DNS($v_{\max}=5$).

makes all the sites under the influence of the right boundary and have the same occupation rate. With lower injection rate, Eq. (5) is not exact for the sites that are not influenced only by the right boundary [8].

VI. CONCLUSION AND DISCUSSION

This paper has presented a theoretical study on the deterministic NS model with stochastic open boundary conditions. Following results are achieved.

We first find an analytical way to calculate the density profiles of the free-flow phase, both for the standard and expanded left boundary. Afterwards, the exact solution of traffic capacity under SRB is given. We also prove that the global density and the jamming density profiles are both analyzable. All these findings tell that most features of the deterministic NS model with stochastic open boundaries can be described exactly by analytical results.

However, more research is still required within the subjects of this paper:

(1) How to manage the standard injection rule in DNS with $v_{\max}>5$? With the IPSDs and the complicate behaviors, the Markov chain method is difficult to applied to DNS($v_{\max}>5$). But fortunately, it is not very necessary to consider those cases, because as shown in Refs. [8,9], DNS($v_{\max}=5$) includes all features that are characteristic for higher $v_{\max}$. Thus, the existing analytical results are enough to understand the mechanism of the standard injection rule. We can also find that the "gradual acceleration" is an important characteristic for reality, but it also makes the quantitative analysis difficult. The complex behaviors of SLB are proved to be a consequence of gradual acceleration. On the other hand, the behaviors of ELB are much simpler because they exclude such behaviors.

(2) How to deal with the nondeterministic NS model? No matter the method raised in Refs. [11,17–19], or those in this paper, they are based on the deterministic behaviors of the model. It seems that for the NS model with randomization, it is very hard to find exact analytical results [14,18,19].

ACKNOWLEDGMENT

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APPENDIX A: PROOF OF THEOREM 1

Lemma A1. If $v_t(c)<v_{\max}$ and $g_t^+(c)\le v_t(c)$, then $g_{t+1}^+[n^-(c)]\ge g_t^+(c)$ [where $v_t(c)$ is the velocity of c at the beginning of step t, $g_t^+(c)$ is the headway of c, $n^-(c)$ is the first car on the left side of c]

Proof. We denote the location of c at step t by $l_t(c)$. Since $g_t^+(c)\le v_t(c)$, then $l_{t+1}(c)=l_t(c)+g_t^+(c)$. Let $c'=n^-(c)$, $l_{t+1}(c')\le l_t(c)-1$ implies that $g_{t+1}^+(c')=l_{t+1}(c)-l_{t+1}(c')-1 \ge [l_t(c)+g_t^+(c)]-[l_t(c)-1]-1\ge g_t^+(c)$.

Lemma A2. If $v_t(c)<v_{\max}$ and $g_t^+(c)>v_t(c)$, then $g_{t+1}^+[n^-(c)]>v_t(c)$.

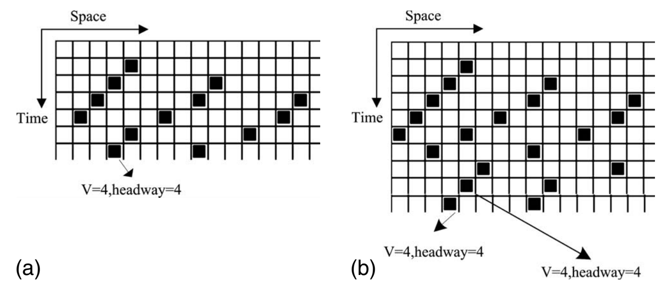

FIG. 16. Illustration of IPSDs at sites 4 and 5.

Proof. Under the given condition, car c is able to increase its velocity, thus $l_{t+1}(c)=l_t(c)+v_t(c)+1$. Let $c'=n^-(c)$, $l_{t+1}(c')\le l_t(c)-1$ implies that $g_{t+1}^+(c')=l_{t+1}(c)-l_{t+1}(c')-1 >v_t(c)$.

Theorem 1. In DNS(5), only cars at site 4,5 might be effected by IPSD.

Proof 1. Figure 16 shows the IPSDs at site 4 and site 5; it confirms that IPSDs may happen at site 4 and site 5.

Proof 2. IPSDs never happen at site $l\ge 6$

Suppose that there exists a car that is forced to be slowed down at site $l\ge 6$. Applying the procedure A.1. as follows, finally we can find car c (this car will definitely be found, because the first injected car always keeps free driving). Lemma A1 and A2 imply that c must satisfy $v_t(c)\le 3$, and $l_t(c)>6$.

Let $c=n^+(c')$ $t=t-1$.
While (c is slowed down at time t)
{$c=n^+(c)$
$t=t-1$}
return c

(Procedure A.1.)

The proposition 2 will hold if we prove that the car c with $v_t(c)\le 3$ and $l_t(c)>6$ does not exist.

If car c is injected into the road at time t_i ($t_i<t$), then there are only three possible ways for c to arrive at $l_t(c)$ at time t:

(1) "Direct Hit:"$t-t_i=1$, $v_t(c)=l_t(c)$.

(2) "Acceleration:"$t-t_i>1$, and $\forall \bar{t}$ $t_i\le\bar{t}-1<\bar{t}\le t$, $v_{\bar{t}} =\min(v_{\bar{t}-1}+1,v_{\max})$. That means c keeps free driving during the period $t_i\sim t$.

(3) "Mixed:"$t-t_i>1$, c has experienced IPSD.

If car c arrives at a site $l>6$ by Direct Hit or Acceleration, with no doubt that $v_t(c)>3$, which contradict to the assertion $v_t(c)\le 3$; And "Mixed" is also impossible, because it has been proved in Ref. [19] that IPASs cannot make cars' velocity lower than 4.

Thus, in DNS(5), the car satisfying the conditions above do not exist. Thus, IPSDs won't happen at site $l\ge 6$.

Proof 3. IPSDs never happen at site $l\le 3$. This proposition has been proved in Ref. [19].

Therefore, in DNS(5), IPSDs can only happen at site 4,5. This theorem holds.

APPENDIX B: PROOF OF THE STATE TRANSITION RULES

Given an arbitrary time step t, we denote the system state at t and $t+1$ by S and S', respectively. Suppose that $S=(d,v)$, then:

Rule 1: with probability 1, $S'=[0,\min(v+1,v_{\max})]=(0,d)$ if $d=\min(v+1,v_{\max})$.

This rule is obviously in accordance with the NS model rules.

Rule 2: with probability $1-\beta$, $S'=(0,d)$ if $d<\min(v+1,v_{\max})$.

If S satisfies $l<\min(v+1,v_{\max})$, and the boundary closes at this step, then the last car in the road must stop and wait at the last site. Thus, it is easy to know $S'=(0,d)$. Known that the probability of right boundary's blockage is $1-\beta$, we find this rule holds.

Rule 3: with probability β, $S'=(d+1,v)$ if $d<\min(v+1,v_{\max})$.

Under given condition, with probability β, the right boundary opens, then the last car runs out. We denote the last car in the road at step t by c_1, and the last car at step $t+1$ by c_2.

We denote the length of the road by L, then S and S' become

$$S=[L-l_t(c_1),v_t(c_1)], \quad S'=[L-l_{t+1}(c_2),v_{t+1}(c_2)].$$

Recall this conclusion again: the inflow of right boundary equals $\frac{v_{\max}}{v_{\max}+1}$. Then at step $t+1$, c_2 must arrive at the site which is just in front of site $l_t(c_1)$, such that $L-l_{t+1}(c_2)=L-l_t(c_1)+1$. Consequently, $v_{t+1}(c_2)=l_t(c_1)-l_t(c_2)-1=l_t(c_1)-[l_{t-1}(c_1)-1]-1=l_t(c_1)-l_{t-1}(c_1)=v_t(c_1)$.

The proof is hence completed.

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