

Analytical calculation of the frequency shift in phase oscillators driven by colored noise: Implications for electrical engineering and neuroscience

Roberto F. Galán*

Department of Neurosciences, Case Western Reserve University, 10900 Euclid Avenue, Cleveland, Ohio 44106, USA

(Received 23 June 2009; published 17 September 2009)

We provide an analytical expression for the mean frequency shift in phase oscillators as a function of the standard deviation, σ and the autocorrelation time, τ of small random perturbations. We show that the frequency shift is negative and proportional to σ^2 . Its absolute value increases monotonically with τ , approaching an asymptote determined by the L^2 -norm of the phase-response curve. We validate our theoretical predictions with computer simulations and discuss their implications for the design of electronic oscillators and for the encoding of information in biological neural networks.

DOI: [10.1103/PhysRevE.80.036113](https://doi.org/10.1103/PhysRevE.80.036113)

PACS number(s): 05.40.Ca, 84.30.Ng, 87.19.lc

I. INTRODUCTION

The effects of stochastic forces on the frequency of nonlinear oscillators are beginning to be understood [1,2]. A thorough characterization of this phenomenon is in fact crucial for several applications. In electrical engineering, for instance, the implementation of radiofrequency devices requires precise electrical oscillators. In this context, thermal noise is an unavoidable perturbation to electrical circuits with resistive elements that according to Nyquist's theorem [3] can be described as low-pass filtered white noise (colored noise). In the last few years, several studies have been devoted to mathematically characterize the effects of thermal noise on the frequency of electronic oscillators [4–7]. Computer simulations of simplified radiofrequency circuits have revealed a monotonic decrease in the oscillator's mean frequency when perturbed with colored noise of increasing amplitude [7]. Specifically, the frequency decrease is proportional to the variance of the noise. However, an analytical expression for the dependence of the frequency shift on the amplitude and correlation time of the noise is still lacking.

Neuroscience is another field in which oscillator models are a useful tool. Neuronal networks, for instance, frequently display global oscillations that enable the firing of an individual neuron only at a certain phase of the global rhythm [8]. This way, the oscillation acts as a clock signal providing a temporal reference to multiplex and route information in the network. In fact, this mechanism is thought to encode information in time with millisecond precision in several areas of the brain [8]. For this neuronal code to be reliable, however, the robustness of global oscillations with respect to stochastic perturbations has to be demonstrated.

Recent work by Teramae *et al.* provided a rigorous phase reduction approach to investigate limit cycle oscillators driven by noise [2]. Their analysis reveals that the average frequency of a general nonlinear oscillator is affected by the time constant of the noise, τ as well as by the relaxation time of the oscillation's amplitude, τ_p , i.e., the time constant characterizing the return to the limit cycle after a transient deviation from the periodic trajectory. Their mathematical ap-

proach involves a perturbation analysis of the Fokker-Planck equation and an analytical expression for the mean frequency shift is restricted to the case of relatively small τ . In this paper, we have mathematically investigated how colored noise of arbitrarily large τ affects the mean frequency of phase oscillators. By avoiding the Fokker-Planck formalism, we can provide an analytical expression for the mean frequency shift as a function of the amplitude and correlation time of the noise, assuming that the limit cycle underlying the periodic oscillation has infinite attraction ($\tau_p=0$). Using computer simulations of various oscillator types, we corroborate our analytical results and discuss the relevance of this phenomenon for electrical engineering and neuroscience.

II. RESULTS

Let $\vec{x}(t)$ be a set of variables describing the evolution of a nonlinear system, e.g., an electronic circuit or a neuronal network, with dynamics $d\vec{x}/dt = \vec{F}(\vec{x})$. This system is an oscillator if the dynamics converges to a stable limit cycle. Let $\sigma\vec{u}(t)$ be a small stochastic perturbation of size σ added to the right-hand side. Teramae *et al.* showed that the motion of an oscillator perturbed by noise is well-described by Kuramoto's celebrated phase model [9], provided that the limit cycle has infinite attraction [2]. One thus has $d\varphi/dt = \omega + \sigma\vec{Z}(\varphi) \cdot \vec{u}(t)$, where φ is the phase, ω is the angular frequency, and $\vec{Z}(\varphi)$ is the phase-dependent sensitivity of the oscillator, also known as phase response and infinitesimal phase-resetting curve. In many applications, like those studied below, only one variable is directly perturbed. As a result, only one component of $\vec{u}(t)$ is nonzero and the dot product takes the simpler form $Z(\varphi)u(t)$.

The phase of an *unperturbed* oscillator grows linearly between 0 and 2π as the oscillator completes one round along the limit cycle of period $2\pi/\omega$. The phase of a *perturbed* oscillator, however, can be defined in different ways, and the functional form of $\vec{Z}(\varphi)$ obviously depends on that definition. A common definition of phase for a perturbed oscillator, i.e., in the neighborhood of the limit cycle, is the *asymptotic* phase [9], or the phase that the system will have when it converges back to the limit cycle after a perturbation. The advantage of using this definition is that the phase response

*rfgalan@case.edu

$\bar{Z}(\varphi)$ can be directly calculated from the dynamical variables using numerical techniques [10]. However, the asymptotic phase itself, and hence, its derivative the frequency, can only in exceptional cases be directly calculated from the dynamical variables because the mapping of \vec{x} onto φ is not trivial. An alternative definition of phase uses a Poincaré section, cutting the limit cycle transversally in the state space. Specifically, if t_k are the times of the crossings through the Poincaré section in a specified direction, then the instantaneous phase between two successive crossings is $\varphi(t) = 2\pi(t - t_k)/(t_{k+1} - t_k)$. These two definitions of phase, the asymptotic phase and the ‘‘Poincaré’’ phase, will be used below.

We now regard the perturbation $u(t)$ as colored noise with zero mean and unitary variance, i.e., its autocorrelation function reads $C(s) = \langle u(t)u(t-s) \rangle = \exp(-|s|/\tau)$, where the brackets denote temporal averaging. Thus, $u(t)$ can be regarded as an Ornstein-Uhlenbeck process [3] with time constant (correlation time) τ , and unitary variance. In the limit of $\tau \rightarrow 0$, $u(t)$ becomes white noise. As τ grows, the power for increasing frequencies fades down faster. We then focus on the stochastic dynamics of system

$$\begin{cases} \frac{d\varphi}{dt} = \omega + \sigma Z(\varphi)u(t) \\ \frac{du}{dt} = -\frac{u}{\tau} + \sqrt{\frac{2}{\tau}}\eta(t) \end{cases}, \quad (1)$$

where $\eta(t)$ is white noise. At first, one is tempted to investigate system (1) with its corresponding two-dimensional Fokker-Planck equation, along the lines of previous studies [2]. In order to obtain analytical expressions, however, this approach leads to a perturbation analysis with respect to τ as small parameter (adiabatic approximation). Since we are interested in the dynamics of Eq. (1) for a wide range of τ , we need an alternative to the Fokker-Planck equation.

We thus attempt to quantify the frequency change directly from Eq. (1). The mean frequency shift of the oscillator is given by

$$\langle \Delta\omega \rangle = \langle d\varphi/dt - \omega \rangle = \lim_{T \rightarrow \infty} \frac{\sigma}{T} \int_0^T Z[\varphi(t)]u(t)dt. \quad (2)$$

Since the perturbation is assumed to be small ($\sigma \ll \omega$), from the phase model in Eq. (1) we obtain a mapping of the phase, φ onto time, t to the lowest order in σ : $\varphi(t) \approx \omega t + \sigma \int_0^t Z(\omega s)u(s)ds$. Using this mapping, from Eq. (2) one obtains

$$\langle \Delta\omega \rangle = \lim_{T \rightarrow \infty} \frac{\sigma}{T} \int_0^T \left[Z(\omega t) + \frac{\sigma}{\omega} \frac{dZ(\omega t)}{dt} \int_0^t Z(\omega s)u(s)ds \right] u(t)dt.$$

Due to the symmetric distribution of $u(t)$ values around its zero mean, the first term of the integral vanishes. Note also that $C(s-t) = C(t-s) = u(s)u(t)$. We thus obtain

$$\langle \Delta\omega \rangle = \lim_{T \rightarrow \infty} \frac{\sigma^2}{T\omega} \int_0^T \frac{dZ(\omega t)}{dt} \int_0^t Z(\omega s)C(s-t)dsdt. \quad (3)$$

Since the phase response is a periodic function of the phase, we can expand it as a Fourier series: $Z(\varphi) = \sum_n A_n \cos(n\varphi) + B_n \sin(n\varphi)$. Substituting this expression in Eq. (3) and after some tedious calculus we arrive at an analytical expression for the mean frequency shift

$$\langle \Delta\omega \rangle = -\omega \frac{\sigma^2}{2} \sum_n \frac{n^2 \tau^2 C_n^2}{1 + n^2 \omega^2 \tau^2}, \quad (4)$$

where $C_n^2 = A_n^2 + B_n^2$. Note that the average frequency change is always negative, indicating that colored noise lowers the natural frequency of the oscillator. In the white-noise limit ($\tau \rightarrow 0$), the mean frequency change goes to zero. In the limit of slowly varying inputs ($\tau \rightarrow \infty$), the asymptotic value of the mean frequency change is

$$\langle \Delta\omega \rangle \approx -\frac{\sigma^2}{2\omega} \sum_n C_n^2,$$

which is the maximal decrease in frequency than can be obtained in the approximation of weak perturbations. Note that the asymptotic frequency shift depends on the L^2 -norm of the phase-response curve, $\|Z(\varphi)\|^2 = \sum_n C_n^2$ rather than on any specific shape of the curve. This demonstrates that the frequency shift due to colored noise is a general phenomenon for phase oscillators.

To test our theoretical prediction we first consider a canonical model of a dynamical system displaying sustained oscillations, the Stuart-Landau oscillator. On the complex plane, we thus have: $\dot{z} = (\gamma + i\omega)z - \beta z|z|^2 + \sigma u(t)$, where γ , ω , and β are real parameters. Without loss of generality, we choose $\gamma = 1$ and $\beta = 1$. We then transform to polar coordinates, $z = r \exp(i\varphi)$, thereby obtaining

$$\begin{cases} \frac{dr}{dt} = r - r^3 + \sigma u(t) \cos \varphi \\ \frac{d\varphi}{dt} = \omega - \frac{\sigma}{r} u(t) \sin \varphi. \end{cases} \quad (5)$$

For small σ , the radius of the limit cycle is roughly constant and close to one, which is the nonzero root of $dr/dt = 0$ ignoring the term with σ . Note that in this case, the phase in polar coordinates coincides with the asymptotic phase of this oscillator. By comparing the phase equation in Eq. (5) with the phase equation in Eq. (1), we note that $Z(\varphi) = -\sin \varphi$, and then, according to Eq. (4), the mean frequency shift reads

$$\langle \Delta\omega \rangle = -\frac{\sigma^2}{2} \frac{\omega \tau^2}{1 + \omega^2 \tau^2}.$$

Figure 1 displays the trajectories of the Stuart-Landau oscillator in the state space for large τ , as well as the comparison between the theoretically predicted frequency shift and the shift observed in computer simulations of the stochastic differential equations. Note the good agreement between theory and simulations.

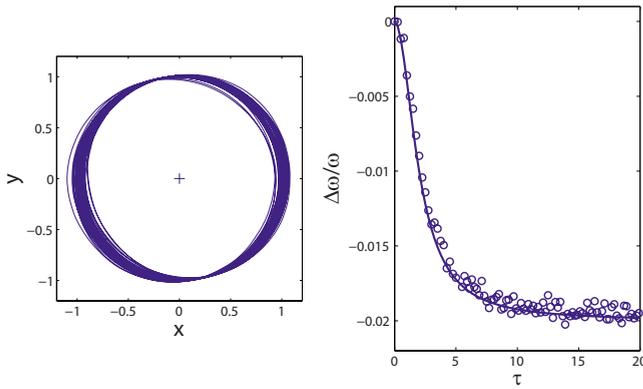


FIG. 1. (Color online) Stuart-Landau oscillator. Left: perturbed trajectories for colored noise with $\tau=20$. The cross indicates the origin (0,0) with respect to which polar coordinates are calculated. Right: relative frequency shift as a function of τ . The data from simulations (circles) match the theoretical prediction (line). Simulation parameters: $\gamma=1$, $\beta=1$, $\omega=0.5$, $\sigma=0.1$, and $dt=0.05$; data points for each value of $\tau: 4 \times 10^6$.

We now consider the van der Pol oscillator, which can be implemented in electronic circuits to develop sustained oscillations. It has also been widely used to model the physiological mechanisms controlling the heart beat and other excitable tissues, like neurons [11]. In particular, we consider $\ddot{x} - \mu(1-x^2)\dot{x} + \omega_0^2 x = \sigma u(t)$, where μ is a positive real parameter indicating the nonlinearity of the system. After defining $y = \dot{x}$, one obtains a two-dimensional representation of the van der Pol oscillator. Figure 2 shows the results of the simulations for $\mu=1$, $\omega_0=1$, and $\sigma=0.2$. Contrary to the previous case, the asymptotic phase and hence, $d\varphi/dt$ cannot be calculated from the equations explicitly. We thus quantify the frequency of the oscillator, as 2π over the period, and the period is the time elapsed between two successive crossings

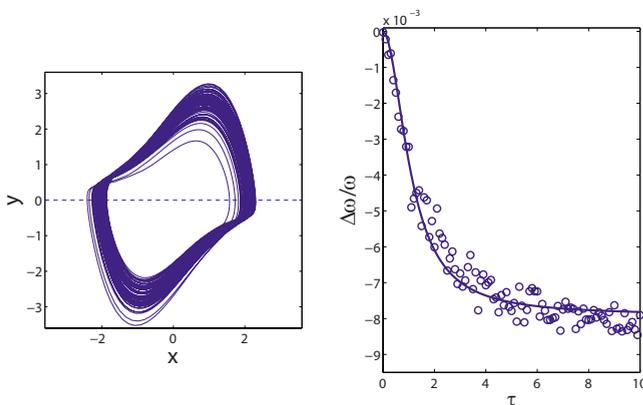


FIG. 2. (Color online) Van der Pol oscillator. Left: perturbed trajectories for colored noise with $\tau=10$. The dashed line indicates the Poincaré section with respect to which the instantaneous phase is calculated. Right: relative frequency shift as a function of the correlation time of the noise, τ . The data from simulations (circles) fit the trend predicted by the theory (line) in Eq. (4). Simulation parameters: $\mu=1$, $\omega_0=1$, $\sigma=0.2$, and $dt=0.05$. Number of integration time steps for each $\tau: 1 \times 10^6$. Fitting coefficients: $C_1=0.5646$, $C_2=0.0048$, and $C_3=0.0760$.

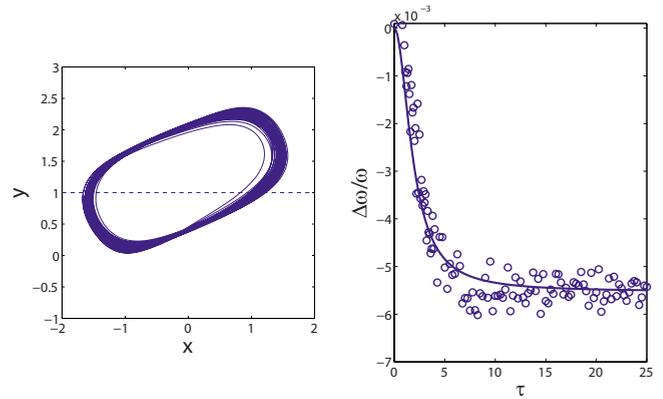


FIG. 3. (Color online) FitzHugh-Nagumo oscillator. Left: perturbed trajectories for colored noise with $\tau=25$. The dashed line indicates the Poincaré section with respect to which the instantaneous phase is calculated. Right: relative frequency shift as a function of τ . The data from simulations (circles) fit the trend predicted by the theory (line) in Eq. (4). Simulation parameters: $a=0.5$, $b=1$, $c=0.8$, $I=1.2$, $\sigma=0.08$, and $dt=0.05$. Number of integration time steps for each $\tau: 2 \times 10^6$. Fitting coefficients: $C_1=0.7002$, $C_2=0.0125$, and $C_3=-0.0007$.

of $y=0$ (Fig. 2, left, dashed line) with positive slope. Then, in order to check the validity of our theoretical prediction, we fit the results of the simulations to expression (4) for $n=1$ to 3, using nonlinear regression techniques. Note the good fit of the data to the theoretically predicted frequency shift.

Finally, we consider another oscillator that has been used in numerous models of physiological oscillations: the FitzHugh-Nagumo oscillator [11]

$$\begin{cases} \frac{dx}{dt} = x - x^3/3 - y + I + \sigma u(t) \\ \frac{dy}{dt} = a(x + b - cy) \end{cases}$$

where a , b , and c are real, positive parameters. Figure 3 displays the results of the simulations for $a=0.5$, $b=1$, $c=0.8$, $I=1.2$, and $\sigma=0.08$. We use the same definition of phase as in the previous case to quantify frequency changes. Note also in this case the good fit of the data to the theoretically predicted frequency shift.

III. DISCUSSION

We have demonstrated that when the limit cycle underlying the oscillations has infinite attraction, the average frequency shift in oscillators driven by colored noise is always negative and proportional to the variance of the noise. Moreover, the absolute value of the frequency shift monotonically increases as τ increases. For large τ , the mean frequency shift approaches the asymptotic value $-k\sigma^2/(2\omega)$, where k is the L^2 -norm of the phase sensitivity of the oscillator, $Z(\varphi)$.

Teramae *et al.* previously demonstrated that the mean frequency of a generic noise-driven oscillator is affected by the correlation time of the noise, τ as well as by the relaxation time of the oscillator, τ_p [2]. The perturbation expansion of

the Fokker-Planck equation used in their paper provides analytical results for relatively small τ . Here, we have investigated the case in which τ can be arbitrarily large but the relaxation time is negligible, i.e., we consider a limit cycle with infinite attraction and hence, $\tau_p=0$. Thus, their results and ours are different but complementary in the study of nonlinear oscillators driven by colored noise. Indeed, by changing parameters appropriately, the three oscillator types considered here can be set into regimes in which moderate perturbations may provoke relatively large phase and amplitude changes, leading to positive rather than negative mean frequency shifts (data not shown), as reported by Teramae *et al.* for the Stuart-Landau oscillator [2].

In this paper, we have also considered two important contexts in which the infinite-attraction approximation is applicable and where our results for phase oscillators are meaningful. One is the design of radiofrequency oscillators, where thermal noise is known to induce a small but appreciable frequency shift [5,7]. In effect, it has been shown with computer simulations of phase oscillator models for electronic circuits that the frequency shift is negative and grows quadratically with the standard deviation, σ of the noise [7]. Here, for the first time we have provided an analytical expression that makes this dependence explicit. In addition, we have mathematically shown the dependence of the mean frequency shift on τ and ω . The fact that our theory explains the results from simulations of electronic circuits that in turn reproduce experimental observations, suggests that the relaxation time of amplitude perturbations in these systems is negligible with respect to both, the period of the oscillations and the correlation time of the noise.

Another context in which our results are useful is neuroscience. Neural network oscillations are thought to provide a clock signal with respect to which individual neurons can fire at certain phases in a stimulus dependent fashion. This way, a sequence of action potentials across the network can en-

code a stimulus consisting of which neuron has fired and when [8]. This form of spatiotemporal code is based on at least two assumptions: (1) neurons are precise enough to detect the phase of the network oscillation at which they are supposed to fire; (2) the global oscillation, i.e., the clock signal is robust enough with respect to background perturbations resulting from both extrinsic and intrinsic sources: the extrinsic sources are varying conditions in the presentation of stimuli, whereas the intrinsic sources include spontaneous release of neurotransmitters and thermal fluctuations of ion channel conductances. The first assumption is known to be true, as numerous studies have reported the ability of neurons to fire in response to stimuli with millisecond precision (see, e.g., [12]). The second assumption is supported by the results reported here. Note that the relative frequency shift in the cases studied here is 2% or less. Whether this small change is negligible or not in the brain depends on the tolerance to error that neural codes based on spike timing allow. In any case, the frequency shift can be reduced below any given tolerance by increasing the frequency of the network oscillations. Indeed, we have shown that the largest frequency shift induced by colored noise is inversely proportional to ω and therefore, the relative frequency shift $\langle\Delta\omega\rangle/\omega$ is inversely proportional to ω^2 . Thus, if for a network oscillation of 1 rad/s the relative frequency shift due to background noise is 5%, the relative frequency shift for the same network oscillating at 10 rad/s will be 0.05%. Taking into account that typical network oscillations are in the theta-beta range (5–20 Hz \approx 30–120 rad/s), we conclude that they can provide a robust clock signal for the timing of action potentials even in the presence of background noise.

ACKNOWLEDGMENTS

This work has been supported by The Mount Sinai Health Care Foundation and The Alfred P. Sloan Foundation.

-
- [1] K. Yoshimura and K. Arai, *Phys. Rev. Lett.* **101**, 154101 (2008).
 - [2] J. N. Teramae, H. Nakao, and G. B. Ermentrout, *Phys. Rev. Lett.* **102**, 194102 (2009).
 - [3] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences* (Springer, Berlin, 2004).
 - [4] A. Demir, *IEEE Trans. Circuits Syst., I: Fundam. Theory Appl.* **49**, 1782 (2002).
 - [5] J. P. Gleeson, *IEEE Trans. Circuits Syst., II: Express Briefs* **53**, 183 (2006).
 - [6] F. O’Doherty and J. P. Gleeson, *IEEE Trans. Circuits Syst., II: Express Briefs* **54**, 435 (2007).
 - [7] P. Maffezzoni, *IEEE Trans. Circuits Syst., II: Express Briefs* **54**, 887 (2007).
 - [8] G. Buzsáki, *Rhythms of the Brain* (Oxford University Press, New York, 2006).
 - [9] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer-Verlag, Berlin, 1984).
 - [10] G. B. Ermentrout, *Simulating, Analyzing, and Animating Dynamical Systems: A Guide to XPPAUT for Researchers and Students* (SIAM, Philadelphia, 2002).
 - [11] J. Keener and J. Sneyd, *Mathematical Physiology* (Springer, New York, 2004).
 - [12] R. F. Galán, G. B. Ermentrout, and N. N. Urban, *J. Neurophysiol.* **99**, 277 (2008).