PHYSICAL REVIEW E **80**, 031124 (2009)

Dynamic scaling functions and amplitude ratios of stochastic models with energy conservation above T_c

M. Dudka,[1,2,*] R. Folk,[2,†] and G. Moser[3,‡]

[1]Institute for Condensed Matter Physics, National Academy of Sciences of Ukraine, 1 Svientsitskii Str., UA-79011 Lviv, Ukraine
[2]Institute for Theoretical Physics, Johannes Kepler University Linz, Altenbergerstrasse 69, A-4040 Linz, Austria
[3]Department for Material Research and Physics, Paris Lodron University Salzburg, Hellbrunnerstrasse 34, A-5020 Salzburg, Austria

(Received 9 April 2009; revised manuscript received 15 June 2009; published 17 September 2009)

Dynamical scaling functions above T_c for the characteristic frequencies and the dynamical correlation functions of the order parameter and the conserved density of model C are calculated in one loop order. By a proper exponentiation procedure these results can be extended in order to consider the changes in these functions using the fixed point values and exponents in two loop order. The dynamical amplitude ratio R of the characteristic frequencies is generalized to the critical region. Surprisingly the decay of the shape functions at large scaled frequency does not behave as expected from applying scaling arguments. The exponent υ of the decay does not change when going from the critical to the hydrodynamic region although the shape functions change. The value of υ for the order parameter is in agreement with its value in the critical region, whereas for the conserved density it is equal to 2, the value in the hydrodynamic region.

DOI: 10.1103/PhysRevE.80.031124 PACS number(s): 05.70.Jk, 64.60.ae, 64.60.Ht

I. INTRODUCTION

Critical dynamics is described by the slow variables of a system containing the order parameter (OP) due to critical slowing down and the secondary densities of the conserved variables (CD) [1]. If the dynamics of these quantities are coupled the dynamical behavior of the OP is changed. The simplest case is realized when the dynamical coupling is due to a static coupling only of these slow variables, which are described by a relaxation equation for the OP and a diffusion equation for the CD. This model has been set up by Halperin *et al.* [2,3] and later has been called model C [1]. From renormalization-group theory it is known that the static asymmetric coupling between the OP and the CD is irrelevant if the specific heat of the system considered is not diverging. Thus, for a system with the specific-heat exponent $\alpha \equiv 0$ (except logarithmic divergence) in dynamics one recovers model A [2]. If, however, the specific heat is diverging the situation is more complicated. The asymmetric coupling is relevant and the dynamical behavior may be characterized by strong dynamical scaling (both OP and CD scale with the same time scale) or weak dynamical scaling (there are two different time scales, one for the OP and one for the CD) depending on the dimension of space $d=4-\varepsilon$ and the number n of components of the OP (for more details see Refs. [4,5]). For $n=1$ strong scaling is realized, whereas for $n=2$ the asymptotics is described by model A, but non-asymptotic effects might be present due to a small dynamical transient exponent.

The most prominent example where the model C dynamics and strong scaling will be realized is the anisotropic antiferromagnet in an external magnetic field [6,7]. There exists a whole *line* of critical points with the z component of the staggered magnetization as OP and the z component of the magnetization as CD. The advantage of this example is also that both, the OP and the CD, are experimentally accessible. Thus the dynamical correlation function can be measured by neutron scattering and the transport coefficients, the relaxation coefficient for the OP and the diffusion coefficient of the CD are in principle measurable. There are also other physical systems where the critical dynamical behavior is described by model C such as systems with quenched impurities [8]; for more examples see [9].

The paper is organized as follows: in the next section we summarize the results expected by the dynamical scaling theory for the scaling functions to be calculated above T_c. Then the dynamical model is defined and in Sec. IV the one loop results for the correlation functions are presented. Then in Sec. V the scaling functions for the width of the OP and CD are given. These results are then used in the next section to evaluate the shape functions of the correlation functions. Section VII contains the limiting behavior of these shape functions ("shape crossover") going from the critical into the hydrodynamic region above T_c. Section VIII considers the dynamical amplitude ratio of model C and generalizes this expression in order to be valid in the hydrodynamic and critical region above T_c. The calculations are based on the validity of strong scaling. Short remarks if weak scaling is valid are made in Sec. IX followed by the conclusions. In the Appendix details of the one loop integrals are given.

II. DYNAMICAL SCALING

The dynamical scaling hypothesis states that dynamical critical phenomena are described by one time scale. The time scale is defined by the wave vector dependence of the characteristic frequency of the OP at the phase transition. This dependence is a power law defining the dynamical critical exponent z. However this only holds in the case of *strong* dynamical scaling, whereas in the case of *weak* dynamical scaling several dynamical critical exponents for the OP, ϕ,

*maxdudka@icmp.lviv.ua
†folk@tphys.uni-linz.ac.at
‡guenter.moser@sbg.ac.at

and for the CD, m, are involved. In scattering experiments the dynamical correlation functions for the OP, $C_{\phi\phi}(\xi,k,\omega)$, and the CD, $C_{mm}(\xi,k,\omega)$ are accessible, which depend on the correlation length ξ, the wave vector modulus k, and the frequency ω. The dynamical scaling assumptions state that the dynamical correlation functions are homogenous functions of their arguments and may therefore be expressed in the form

$$C_{aa}(\xi,k,\omega)=\frac{C^{(s)}_{aa}(\xi,k)}{\omega_a(\xi,k)}\mathcal{F}_a(y_a,x), \tag{1}$$

with $a=\phi,m$. $C^{(s)}_{aa}(\xi,k)$ denotes the static correlation function and $\omega_a(\xi,k)$ is the characteristic frequency for the OP or the CD depending on a. Note that we have introduced for each dynamical correlation function its times scale y_a by the corresponding characteristic frequency. Introducing the parameters

$$x=k\xi, \quad y_a=\frac{\omega}{\omega_a(\xi,k)}, \tag{2}$$

$\mathcal{F}_a(y_a,x)$ are the dynamical shape functions. The static correlation functions and the characteristic frequencies again obey scaling relations. From static scaling it is known that the correlation functions are of the form

$$C^{(s)}_{\phi\phi}(\xi,k)=k^{-2+\eta}g_\phi(x), \quad C^{(s)}_{mm}(\xi,k)=g_m(x) \tag{3}$$

with $g_a(x)$ as static shape functions. The shape functions adopt finite values in the critical limit $x\to\infty$ leading to the static amplitude ratios when they are considered above and below the critical temperature. The critical exponent η is usually called the anomalous dimension. The characteristic frequencies fulfill also a dynamical scaling relation of the form

$$\omega_a=A_ak^{z_a}f_a(x) \text{ with } a=\phi,m. \tag{4}$$

In the critical region, $x>1$, the universal scaling functions f_a reach finite nonzero limits for $x\to\infty$, whereas in the hydrodynamic region, $x<1$, in the limit $x\to 0$ the scaling function f_ϕ diverges like x^{-z_ϕ}, and f_m like x^{-z_m+2} due to its conservation property. The characteristic frequencies have the form

$$\omega_\phi=\bar{A}_\phi\xi^{-z_\phi}=\Gamma_\phi(\xi) \text{ and} \tag{5}$$

$$\omega_m=\bar{A}_m\xi^{-z_m+2}k^2=D(\xi)k^2. \tag{6}$$

The shape functions at T_c ($x=\infty$) are functions of the scaled frequency alone and behave in the limit of large argument as

$$\lim_{y_a\to\infty}\mathcal{F}_a(y_a,\infty)\sim y_a^{-\upsilon_a} \tag{7}$$

with

$$\upsilon_\phi=\frac{z_\phi+2-\eta}{z_\phi} \text{ and } \upsilon_m=\frac{z_m+2}{z_m}. \tag{8}$$

Ratios of different dynamical quantities may be defined which become universal in the asymptotic limit. Such ratios might involve dynamical quantities above and below the phase transition temperature T_c. In systems described by at least two dynamical variables also the ratio of dynamical quantities defined for each of these variables separately above T_c may be defined. One such quantity is the dynamical amplitude ratio [10]

$$R(k,\xi)=\lim_{k\to 0}\left[\frac{\omega_m(k,\xi)}{\omega_\phi(k,\xi)x^2}\right] \tag{9}$$

given by ratio of the characteristic frequencies for the case of *strong* scaling where $z_\phi=z_m=z$. The limit of $k\to 0$ has to be taken before ξ goes to ∞ (or $T\to T_C$), in order to guarantee the approach to the asymptotics from the hydrodynamic region. Inserting the expressions for the characteristic frequencies, Eqs. (5) and (6), the ratio is expressed by the hydrodynamic transport coefficients

$$R=\lim_{\xi\to\infty}\frac{D(\xi)}{\Gamma_\phi(\xi)\xi^2}=\frac{\lambda(\xi)\chi_\phi(\xi)}{\Gamma(\xi)\chi_m(\xi)\xi^2}. \tag{10}$$

In the last step the kinetic coefficients Γ and λ for the OP and the CD have been introduced by the relations $\Gamma_\phi(\xi)=\Gamma(\xi)/\chi_\phi(\xi)$ and $D(\xi)=\lambda(\xi)/\chi_m(\xi)$. χ_ϕ and χ_m are the corresponding static susceptibilities. When T goes to T_C and the *asymptotic* region is entered, the model parameters (appearing in the kinetic coefficients and the susceptibilities) reach their fixed point values and the amplitude reaches a universal value depending only on the spatial dimension d and the number of components n of the OP.

Such amplitude ratios have been defined for more complicated cases [11] when the dynamics of the OP and the CD contain beside the static couplings mode coupling terms as for the superfluid transition. The most prominent example is the universal amplitude ratio of the thermal conductivity at the superfluid transition in ^{4}He [12]. However, for the superfluid transition it turned out that the asymptotic value of this amplitude ratio is not reached in the experimental accessible region and one had to compare with its nonasymptotic extension [13].

From its definition it is clear that in the case of the *weak* dynamical scaling fixed point the amplitude ratio may be not finite or zero depending on the values of the dynamical exponents. In zeroth loop order the dynamical amplitude ratio is simply given by $w^{\star -1}$ the inverse fixed point value of the time scale ratio $w=\Gamma/\lambda$ (the ratio of the renormalized kinetic coefficients). In the plane of spatial dimension d and number of OP components n qualitative different regions depending on the fixed point value of w are found: (i) the *strong scaling* region where $w^\star$ is nonzero and finite, (ii) the *weak* scaling region with $w^\star=0$, and (iii) the decoupling region where the two equations decouple since the fixed point value of the static coupling is zero (see [4,5] for further details). Here we restrict model C to the case of strong scaling, which also is the case of the most important physical examples.

III. DYNAMICAL EQUATIONS OF MODEL C

Model C has been introduced by Halperin *et al.* in 1974 [2] in order to investigate the influence of energy conservation on dynamical critical phenomena. The system under consideration contains a real n-component order parameter

$\vec{\phi}_0(x,t)$ which is not conserved and a real conserved scalar secondary density $m_0(x,t)$. One might generalize this model to a complex order parameter [5,14] but this is not considered here. The dynamics of the order parameter is purely relaxational, while the dynamics of the secondary density is determined by a diffusion process. This leads to the following dynamical equations:

$$\frac{\partial \vec{\phi}_0}{\partial t} = -\mathring{\Gamma}\frac{\delta H}{\delta \vec{\phi}_0} + \vec{\theta}_\phi, \tag{11}$$

$$\frac{\partial m_0}{\partial t} = \mathring{\lambda}\nabla^2\frac{\delta H}{\delta m_0} + \theta_m. \tag{12}$$

$\mathring{\Gamma}$ is the real kinetic coefficient OP. The stochastic forces θ_{α_i} fulfill Einstein relations

$$\langle \theta_{\phi_i}(x,t)\theta_{\phi_j}(x',t')\rangle = 2\mathring{\Gamma}\delta(x-x')\delta(t-t')\delta_{ij}, \tag{13}$$

$$\langle \theta_m(x,t)\theta_m(x',t')\rangle = -2\mathring{\lambda}\nabla^2\delta(x-x')\delta(t-t'). \tag{14}$$

The critical behavior of the thermodynamic derivatives follows from the static functional

$$H = \int d^dx\left\{\frac{1}{2}\mathring{\tau}\vec{\phi}_0^2 + \frac{1}{2}\sum_{i=1}^{n}(\nabla\phi_{0i})^2 + \frac{\mathring{\tilde{u}}}{4!}(\vec{\phi}_0^2)^2 + \frac{1}{2}m_0^2 + \frac{1}{2}\mathring{\gamma}m_0\vec{\phi}_0^2 - \mathring{h}m_0\right\}. \tag{15}$$

The symbol $\mathring{}$ and $_\circ$ denote unrenormalized quantities from now on. The above static functional may be reduced to the usual Ginzburg-Landau-Wilson functional by integrating over the CD m_0,

$$H_\phi = \int d^dx\left\{\frac{1}{2}\mathring{r}\vec{\phi}_0^2 + \frac{1}{2}\sum_{i=1}^{n}(\nabla\phi_{0i})^2 + \frac{\mathring{u}}{4!}(\vec{\phi}_0^2)^2\right\}. \tag{16}$$

The parameters $\mathring{r}$ and $\mathring{u}$ in Eq. (16) are related to $\mathring{\tau}$, $\mathring{\tilde{u}}$, $\mathring{\gamma}$, and $\mathring{h}$ in Eq. (15) by

$$\mathring{r} = \mathring{\tau} + \mathring{\gamma}\mathring{h}, \quad \mathring{u} = \mathring{\tilde{u}} - 3\mathring{\gamma}^2. \tag{17}$$

The ability to eliminate the secondary density in Eq. (15) also leads to relation between the static correlations of the CD and OP correlations. For the first and second cumulant one obtains

$$\langle m_0\rangle = \mathring{h} - \mathring{\gamma}\left\langle\frac{1}{2}\vec{\phi}_0^2\right\rangle, \tag{18}$$

$$\langle m_0 m_0\rangle_c = 1 + \mathring{\gamma}^2\left\langle\frac{1}{2}\vec{\phi}_0^2\frac{1}{2}\vec{\phi}_0^2\right\rangle_c. \tag{19}$$

Note that the angular brackets in Eqs. (18) and (19) have to be calculated with a probability density $\exp(-H)/\mathcal{N}$ on the left-hand side, and with $\exp(-H_\phi)/\mathcal{N}'$ on the right-hand side, where $\mathcal{N}$ and $\mathcal{N}'$ are appropriate normalization factors. The static and dynamical vertex functions have to be calculated within the usual Feynman graph expansion.

More details of the field theoretic treatment of this model can be found in [5]. There, the renormalization is defined (see Appendix, Sec. I) and the field theoretic ζ functions are given (see Sec. III). The unrenormalized quantities are represented in the following by the corresponding parameters a replacing $\mathring{a}$ or a_0. We restrict our numerical results in the following to the Ising case $n=1$, which corresponds to the physical example of the uniaxial antiferromagnet in an external magnetic field. In such a case, strong dynamic scaling holds.

IV. CORRELATION FUNCTIONS OF MODEL C

The calculation of the various quantities of interest is performed by loopwise expansion in the couplings present in the theoretical model (that is the nonquadratic part of the corresponding dynamical functional). Renormalization removes the poles present in the vertex functions. The finite amplitude functions, however, contain logarithmic terms diverging in the limit of small modulus of the wave vector k, frequency ω, and/or infinite correlation length. A proper exponentiation of these logarithmic terms brings the perturbational result into the scaling form. This exponentiation, however, is ambiguous in the orders exceeding the loop expansion used.

Nevertheless, it is useful to make use of the general scaling laws since one can combine results in different loop order. In dynamics (and this is the case also for model C) the dynamical exponent z might be known exactly and one is left with the task to calculate scaling functions. These functions may be well represented by the lowest loop orders, then exponentiated, the exponents identified and the result in a certain loop order for the exponents replaced by the exact values. The same might then happen with the fixed point values left in the asymptotic expression. They may be replaced by their two loop counterparts. This is important since the FP value of the time scale ratio might be considerably changed in two loop order.

A. Basic relations

Within the current approach, which is based on the work of Bausch *et al.* [15], the dynamical correlation functions $\mathring{C}_{aa}(\xi,k,\omega)$ can be expressed by the vertex functions $\mathring{\Gamma}_{a\tilde{a}}(\xi,k,\omega)$ via

$$\mathring{C}_{aa}(\xi,k,\omega) = -\frac{\mathring{\Gamma}_{\tilde{a}\tilde{a}}(\xi,-k,-\omega)}{|\mathring{\Gamma}_{a\tilde{a}}(\xi,-k,-\omega)|^2}, \tag{20}$$

with $a=\phi,m$, as in the previous sections. The dynamical vertex functions $\mathring{\Gamma}_{a\tilde{a}}(\xi,k,\omega)$ and $\mathring{\Gamma}_{\tilde{a}\tilde{a}}(\xi,k,\omega)$ are obtained by collecting all one particle irreducible graphical contributions within perturbation expansion. Independent from perturbation expansion they have the general structure [16]

$$\mathring{\Gamma}_{a\tilde{a}}(\xi,k,\omega) = -i\omega\mathring{\Omega}_{a\tilde{a}}(\xi,k,\omega) + \mathring{\Gamma}_{aa}(\xi,k)\mathring{\Gamma}_a(k). \tag{21}$$

$\mathring{\Gamma}_{aa}(\xi,k)$ denotes the vertex function calculated within statics. The generalized kinetic coefficient $\mathring{\Gamma}_a(k)$ is equal to $\mathring{\Gamma}$ for the nonconserved OP ($a=\phi$) and equal to $\mathring{\lambda}k^2$ in the case of the conserved secondary density ($a=m$). All dynamical contributions are collected in the function $\mathring{\Omega}_{a\tilde{a}}(\xi,k,\omega)$ which will be calculated loopwise within perturbation expansion. The vertex functions $\mathring{\Gamma}_{\tilde{a}\tilde{a}}(\xi,k,\omega)$ are also determined by an exact relation of the form

$$\mathring{\Gamma}_{\tilde{a}\tilde{a}}(\xi,k,\omega) = -2\Re[\mathring{\Gamma}_a(k)\mathring{\Omega}_{a\tilde{a}}(\xi,k,\omega)]. \tag{22}$$

The static correlation functions $\mathring{C}^{(s)}_{aa}$ are equal to the dynamical correlation functions $\mathring{C}_{aa}$ at the initial time $t=0$. This leads to

$$\mathring{C}^{(s)}_{aa}(\xi,k) = \int_{-\infty}^{\infty} d\omega \mathring{C}_{aa}(\xi,k,\omega) \tag{23}$$

for the Fourier transformed functions. The connection to the static vertex functions is given by

$$\mathring{C}^{(s)}_{aa}(\xi,k) = \frac{1}{\mathring{\Gamma}_{aa}(\xi,k)}. \tag{24}$$

B. Renormalization

Within the current calculations we will use the same renormalization scheme as presented in [5] which is the minimal subtraction scheme (for an overview see [17]). All renormalization factors have been introduced therein in detail. Thus, it is not necessary to present all the definitions here once again. The amplitude functions are calculated in first-order ε expansion.

The mentioned renormalization scheme leads to the following relations for the correlation and vertex functions. The correlation functions renormalize as

$$C_{\phi\phi} = Z_\phi^{-1}\mathring{C}_{\phi\phi}, \quad C_{mm} = Z_m^{-2}\mathring{C}_{mm} \tag{25}$$

independent if they are the dynamical or static correlation functions. The static vertex functions renormalize as

$$\Gamma_{\phi\phi} = Z_\phi \mathring{\Gamma}_{\phi\phi}, \quad \Gamma_{mm} = Z_m^2 \mathring{\Gamma}_{mm} \tag{26}$$

while the dynamical vertex functions fulfill

$$\Gamma_{\phi\tilde{\phi}} = Z_\phi^{1/2} Z_{\tilde{\phi}}^{1/2} \mathring{\Gamma}_{\phi\tilde{\phi}}, \quad \Gamma_{\tilde{\phi}\tilde{\phi}} = Z_{\tilde{\phi}} \mathring{\Gamma}_{\tilde{\phi}\tilde{\phi}}, \tag{27}$$

$$\Gamma_{m\tilde{m}} = Z_m Z_{\tilde{m}} \mathring{\Gamma}_{m\tilde{m}}, \quad \Gamma_{\tilde{m}\tilde{m}} = Z_{\tilde{m}}^2 \mathring{\Gamma}_{\tilde{m}\tilde{m}}. \tag{28}$$

Replacing in the basic relations (20), (23), and (24) the functions with their renormalized counterparts by using Eqs. (25)–(28) one can see that they remain valid for the renormalized functions. This has to be the case also for relations (21) and (22) leading to the renormalizations

$$\Omega_{\phi\tilde{\phi}} = Z_\phi^{1/2} Z_{\tilde{\phi}}^{1/2} \mathring{\Omega}_{\phi\tilde{\phi}}, \quad \Omega_{m\tilde{m}} = Z_m Z_{\tilde{m}} \mathring{\Omega}_{m\tilde{m}} \tag{29}$$

for the dynamical functions and

$$\Gamma = Z_\phi^{-1/2} Z_{\tilde{\phi}}^{1/2} \mathring{\Gamma} \equiv Z_\Gamma^{-1}\mathring{\Gamma}, \tag{30}$$

$$\lambda = Z_m^{-1} Z_{\tilde{m}} \mathring{\lambda} \equiv Z_\lambda^{-1}\mathring{\lambda} \tag{31}$$

for the kinetic coefficients. One has to keep in mind that for the conserved secondary density the relation $Z_{\tilde{m}}=Z_m^{-1}$ holds, which may be inserted into the above equations. Thus, with the presented renormalization scheme in all relations of the previous subsections the unrenormalized functions may be replaced directly by their renormalized counterparts.

C. Vertex functions in one loop order

When the correlation length ξ carefully has been summed up the static vertex function of the OP in one loop order reduces to

$$\mathring{\Gamma}_{\phi\phi}(\xi,k) = \xi^{-2} + k^2, \tag{32}$$

while the corresponding function for the secondary density reads

$$\mathring{\Gamma}_{mm}(\xi,k) = 1 - \frac{n}{2}\mathring{\gamma}^2 \mathring{I}^{(s)}_m(\xi,k) \tag{33}$$

with the static one loop integral

$$\mathring{I}^{(s)}_m(\xi,k) = \int_{k'} \frac{1}{(\xi^{-2}+k'^2)[\xi^{-2}+(k'+k)^2]}. \tag{34}$$

Within dynamics the functions $\mathring{\Omega}_{\phi\tilde{\phi}}(\xi,k,\omega)$ and $\mathring{\Omega}_{m\tilde{m}}(\xi,k,\omega)$ have to be calculated. From relations (20)–(22) in the previous subsection it is clear that all other functions are then determined. In one loop order we obtain

$$\mathring{\Omega}_{\phi\tilde{\phi}}(\xi,k,\omega) = 1 + \mathring{w}\mathring{\gamma}^2 \mathring{I}^{(d)}_\phi(\xi,k,\omega), \tag{35}$$

$$\mathring{\Omega}_{m\tilde{m}}(\xi,k,\omega) = 1 + k^2 \frac{n}{2}\frac{\mathring{\gamma}^2}{\mathring{w}} \mathring{I}^{(d)}_m(\xi,k,\omega), \tag{36}$$

where the time scale ratio

$$\mathring{w} = \frac{\mathring{\Gamma}}{\mathring{\lambda}} \tag{37}$$

has been introduced. The dynamical one loop integrals are

$$\mathring{I}^{(d)}_\phi(\xi,k,\omega) = \int_{k'} \frac{1}{(\xi^{-2}+k'^2)[\frac{-i\omega}{\mathring{\lambda}} + \mathring{w}(\xi^{-2}+k'^2) + (k'+k)^2]}, \tag{38}$$

and

$$\mathring{I}^{(d)}_m(\xi,k,\omega) = \int_{k'} \frac{1}{(\xi^{-2}+k'^2)(\xi^{-2}+(k+k')^2)} \times \frac{1}{[\frac{-i\omega}{\mathring{\Gamma}} + 2\xi^{-2} + k'^2 + (k'+k)^2]}. \tag{39}$$

Applying the renormalization scheme to Eqs. (33), (35), and (36) the renormalized functions read

$$\Gamma_{mm}(\xi,k) = 1 - \frac{n}{2}\gamma^2 I_m^{(s)}\left(\kappa\xi,\frac{k}{\kappa}\right), \tag{40}$$

$$\Omega_{\phi\tilde{\phi}}(\xi,k,\omega) = 1 + w\gamma^2 I_\phi^{(d)}\left(\kappa\xi,\frac{k}{\kappa},\frac{\omega}{\kappa^2}\right), \tag{41}$$

$$\Omega_{m\tilde{m}}(\xi,k,\omega) = 1 + \frac{n}{2}\left(\frac{k}{\kappa}\right)^2\frac{\gamma^2}{w} I_m^{(d)}\left(\kappa\xi,\frac{k}{\kappa},\frac{\omega}{\kappa^2}\right). \tag{42}$$

κ represents an arbitrary wave number scale. The renormalized counterparts of the integrals in Eqs. (41) and (42) are defined as

$$I_m^{(s)}\left(\kappa\xi,\frac{k}{\kappa}\right) = \kappa^\varepsilon A_d^{-1}(\mathring{I}_m^{(s)}(\xi,k) - [\mathring{I}_m^{(s)}(\xi,0)]_S), \tag{43}$$

$$I_\phi^{(d)}\left(\kappa\xi,\frac{k}{\kappa},\frac{\omega}{\kappa^2}\right) = \kappa^\varepsilon A_d^{-1}(\mathring{I}_\phi^{(d)}(\xi,k,\omega) - [\mathring{I}_\phi^{(d)}(\xi,0,0)]_S), \tag{44}$$

$$I_m^{(d)}\left(\kappa\xi,\frac{k}{\kappa},\frac{\omega}{\kappa^2}\right) = \kappa^{2+\varepsilon} A_d^{-1}\mathring{I}_m^{(d)}(\xi,k,\omega). \tag{45}$$

$[.]_S$ denotes the singular part of an integral which only contains the ε poles. The explicit expressions are presented in the Appendix. Integral (45) does not have poles at $k=0$, $\omega=0$ therefore nothing has to be subtracted;

$$A_d = \Gamma\left(1-\frac{\varepsilon}{2}\right)\Gamma\left(1+\frac{\varepsilon}{2}\right)\frac{\Omega_d}{(2\pi)^d} \tag{46}$$

is a suitable geometric factor with Ω_d the surface of the d-dimensional unit sphere and $\Gamma(x)$ the Euler Γ function.

V. CHARACTERISTIC FREQUENCIES

There are several possibilities to define the characteristic frequency of a frequency-dependent function. For a monotonic decreasing function one may use the half width at half height. Of more general use is a definition via the linear-response function and the wave-vector-dependent kinetic coefficient (see [1]). Thus, the characteristic frequency is defined here by the dynamic correlation function at zero frequency, which is directly calculable by the corresponding vertex function

$$\frac{1}{\omega_a(\xi,k)} = \frac{\mathring{C}_{aa}(\xi,k,\omega=0)}{2\mathring{C}_{aa}^{(s)}(\xi,k)} = \frac{C_{aa}(\xi,k,\omega=0)}{2C_{aa}^{(s)}(\xi,k)} \tag{47}$$

for the OP and the CD. Note that like the correlation length the characteristic frequency does not renormalize according to Eq. (25).

A. Characteristic frequency of the OP

Putting Eqs. (20)–(22) and (41) together and inserting the result into Eq. (47) lead to the characteristic frequency of the order parameter

$$\omega_\phi(\xi,k) = \Gamma(\xi^{-2}+k^2)\left\{1 - w\gamma^2 I_\phi^{(d)}\left(\kappa\xi,\frac{k}{\kappa},0\right)\right\}. \tag{48}$$

Inserting the explicit expression (A5) of the one loop integral of the order parameter at $\omega=0$ into Eq. (48) one obtains

$$\omega_\phi(\xi,k) = \Gamma k^2\left(1+\frac{1}{x^2}\right)\left\{1 - \frac{\zeta_\Gamma}{2}\left[1 - 2\ln\frac{k}{\kappa} + w\ln\frac{w}{1+w} - \ln\left(\frac{1}{1+w}+\frac{1}{x^2}\right) - \frac{1+w}{x^2}\ln\left(1+\frac{x^2}{1+w}\right)\right]\right\}, \tag{49}$$

with x defined in Eq. (2). In the above equation the one loop expression

$$\zeta_\Gamma = \frac{w}{1+w}\gamma^2 \tag{50}$$

of the dynamic order parameter ζ function already has been identified. One should note that there are two different types of logarithmic terms: (i) logarithms in k leading to the power law for the characteristic frequency at T_c for small wave vector modulus k in the so-called critical region; (ii) logarithms in x leading to a power-law behavior of the scaling function for small values of the scaling variable x to accomplish the asymptotic crossover from the critical to the hydrodynamic region. Correspondingly, two exponentiations will be performed in the following.

(i) In order to obtain the asymptotic critical exponent of the wave vector modulus, the term proportional to $\ln k$ can be exponentiated to k^{ζ_Γ}.

(ii) In order to obtain a finite amplitude function in the limits $x\to 0$ and $x\to\infty$, the prefactor $1+(1/x^2)$ is resummed to $[1+(1/x^2)]^{1+\zeta_\Gamma/2}$ leading to logarithmic terms in the amplitude function which make it finite in the mentioned limits.

In both cases the difference between expanded and exponentiated expressions is of order two loop. The two steps lead to the expression

$$\omega_\phi(\xi,k) = \kappa^2\Gamma\left(\frac{k}{\kappa}\right)^{2+\zeta_\Gamma}\left(1+\frac{1}{x^2}\right)^{2+\zeta_\Gamma/2}\left\{1 - \frac{\zeta_\Gamma}{2}\left[1 + w\ln\frac{w}{1+w} - \ln\left(\frac{\frac{1}{1+w}+\frac{1}{x^2}}{1+\frac{1}{x^2}}\right) - \frac{1+w}{x^2}\ln\left(1+\frac{x^2}{1+w}\right)\right]\right\}. \tag{51}$$

The scaling function $f_\phi(x)$ may now be identified as

$$f_\phi(x)=\left(1+\frac{1}{x^2}\right)^{2+\zeta_\Gamma/2}\left\{1-\frac{\zeta_\Gamma}{2}\left[1+w\ln\frac{w}{1+w}-\ln\left(\frac{\frac{x^2}{1+w}+1}{1+x^2}\right)-\frac{1+w}{x^2}\ln\left(1+\frac{x^2}{1+w}\right)\right]\right\}. \tag{52}$$

In the asymptotic region the fixed point values $\gamma^\star$ and $w^\star$ may be inserted. The ζ function $\zeta_\Gamma^\star\equiv\zeta_\Gamma(\gamma^\star,w^\star)$ can then be replaced by the dynamic critical exponent z_ϕ via the relation $z_\phi=2+\zeta_\Gamma^\star$. In this case the scaling function is a function of x only with the appropriate behavior in the limits $x\to 0$ and $x\to\infty$. Otherwise, Eq. (52) would be a nonasymptotic expression dependent on the flows $\gamma(l)$ and $w(l)$. At last the characteristic frequency can be written as

$$\omega_\phi(\xi,k)=\kappa^2\Gamma\left(\frac{k}{\kappa}\right)^{2+\zeta_\Gamma}f_\phi(x), \tag{53}$$

which is consistent to Eq. (4). The advantage of the presented approach is that one can insert the fixed point values for γ and w obtained in any loop order or by a summation procedure without destroying the scaling properties corresponding to Eqs. (5) and (6) since $z_\phi=2+\zeta_\Gamma^\star$.

We now have the scaling function in a form where we have separated its singular behavior from the part containing only perturbational contributions finite in all limits. Therefore, we can extend this one loop expression for the scaling function using knowledge of higher loop order results. E.g., one may insert the exact value for the dynamical critical exponent $z_\phi=z=2+\alpha/\nu$ of model C for the scaling fixed point or even experimental values if at hand. One also may use higher loop order results for the fixed point value of the dynamic scale ratio w. The error made in this way is only in perturbational terms contributing to the amplitudes of the singular behavior of the scaling function in the different limits. This concept will be used in the following for all scaling functions calculated.

In Fig. 1 the scaling function Eq. (52) is plotted using the fixed point values of the coupling γ and the time scale ratio w for the three cases (strong scaling) presented in Table I.

B. Characteristic frequency of the CD

The same procedure as for the characteristic frequency of the OP is now accomplished for the characteristic frequency of the CD. Equations (20)–(22), (40), and (42) lead together with Eq. (47) to the expression

$$\omega_m(\xi,k)=\lambda k^2\left\{1-\frac{n}{2}\gamma^2\left[I_m^{(s)}\left(\kappa\xi,\frac{k}{\kappa}\right)+\frac{k^2}{\kappa^2 w}I_m^{(d)}\left(\kappa\xi,\frac{k}{\kappa},0\right)\right]\right\}. \tag{54}$$

Taking Eq. (A12) at $\omega=0$ the characteristic frequency of the CD in order ε reads

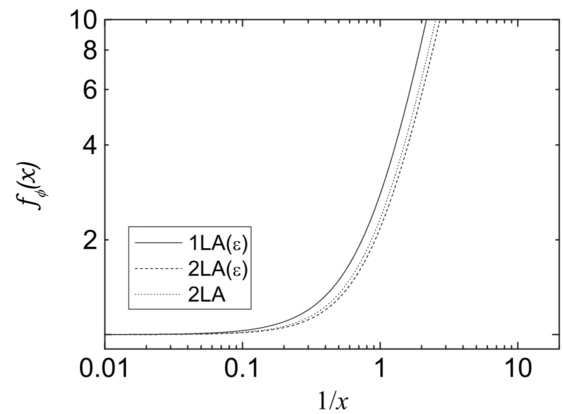

FIG. 1. Scaling function of the width of the dynamical correlations of the OP according to Eq. (52) as function of $x=k\xi$, where the following different approximations for the fixed point values are used. Solid line: one loop order ε expanded; dashed line: two loop order ε expanded; dotted line: two loop order fixed dimension $d=3$.

$$\omega_m(\xi,k)=\lambda k^2\left\{1-\frac{\zeta_\lambda}{2}\left[1-\ln\left(\frac{sk}{2\kappa}\right)^2-a(w,x)\right]\right\}, \tag{55}$$

where the parameter

$$s=\sqrt{1+\frac{4}{x^2}} \tag{56}$$

has been introduced. The function $a(w,x)$ is defined as

$$a(w,x)=\left[1+s-\frac{1}{w}\left(1+\frac{1}{s}\right)\right]\ln\left(1+\frac{1}{s}\right)+\left[1-s-\frac{1}{w}\left(1-\frac{1}{s}\right)\right]\ln\left(1-\frac{1}{s}\right). \tag{57}$$

In Eq. (55) the one loop ζ function

$$\zeta_\lambda=\frac{n}{2}\gamma^2 \tag{58}$$

already has been identified. For the same reasons as discussed in the previous subsection the term $\ln(\frac{sk}{2\kappa})^2=2\ln\frac{k}{\kappa}+\ln\frac{s^2}{4}$ will be resummed in powers of k. This leads to

$$\omega_m(\xi,k)=\kappa^2\lambda\left(\frac{k}{\kappa}\right)^{2+\zeta_\lambda}\left\{1-\frac{\zeta_\lambda}{2}\left[1-\ln\frac{s^2}{4}-a(w,x)\right]\right\}. \tag{59}$$

For convenience we introduce the power of $1+1/x^2$, as it appears in the OP characteristic frequency, instead of the power of s. At least the scaling function is identified as

$$f_m(x)=\left(1+\frac{1}{x^2}\right)^{\zeta_\lambda/2}\left\{1-\frac{\zeta_\lambda}{2}\left[1-a(w,x)-\ln\left(\frac{s^2}{4(1+x^{-2})}\right)\right]\right\}. \tag{60}$$

Quite analogous as discussed in the previous subsection the fixed point values $\gamma^\star$ and $w^\star$ may be inserted in order to obtain the asymptotic expression. As a consequence the ζ function $\zeta_\lambda^\star\equiv\zeta_\lambda(\gamma^\star)$ can be replaced by the dynamic expo-

TABLE I. Fixed point values of model C for $n=1$ and $d=3$ for the static coupling $\gamma^{\star 2}$, the time scale ratio $w^\star$, the dynamic critical exponent z, the dynamical exponent υ_ϕ [see Eq. (8)] and the stability borderline value n_c of *strong* dynamic scaling in one-loop ε-expansion, two-loop ε-expansion, and two-loop fixed dimension schemes (these values are taken from [5]). R and $R(\infty)$ are calculated according to Eqs. (111) and (114), respectively, at the corresponding fixed points.

Loop	$\gamma^{\star 2}$	$w^\star$	z	υ_ϕ	n_c	R	$R(\infty)$
1, ε-exp.	2/3	1	2.33	13/7	2	1.05	0.72
2, ε-exp.	0.2	0.56	2.01	1.995	2	1.82	1.55
2, $d=3$	0.35	0.49	2.18	1.92	1.3	2.12	1.45

nent z_m via the relation $z_m=2+\zeta_\lambda^\star$. Scaling function (59) is then a function of x alone. According to Eq. (4) the characteristic frequency is

$$\omega_m(\xi,k) = \kappa^2\lambda\left(\frac{k}{\kappa}\right)^{2+\zeta_\lambda} f_m(x). \quad (61)$$

Note that at this stage we did not relate the characteristic width to the scale set by the OP. This will be done in the later section on the dynamical amplitude ratio R.

In Fig. 2 the scaling function Eq. (59) is plotted using the fixed point values of the coupling γ and the time scale ratio w for the three cases (strong scaling) presented in Table I.

VI. DYNAMIC SHAPE FUNCTIONS OF MODEL C

After calculating the characteristic width of the OP and the CD we are able to extract the shape functions $\mathcal{F}_a$ introduced in Eq. (1) from the corresponding dynamical correlation function (Figs. 3 and 4). The program follows similar steps as for the characteristic frequency. Starting from the perturbational result in one loop order we exponentiate logarithmic terms in order to get the singular behavior of the shape function correct in all possible limits. This result is then extended using higher loop values for exponents and/or fixed point values.

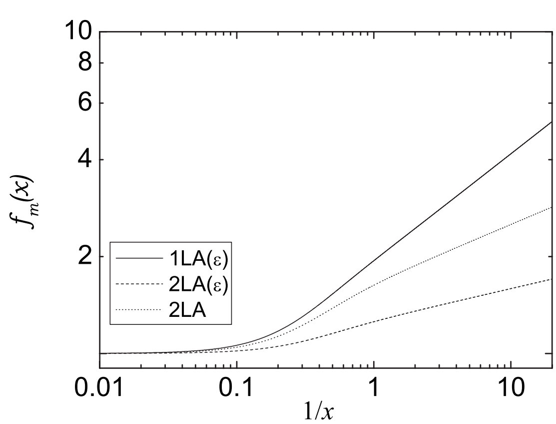

FIG. 2. Scaling function of the width of the dynamical correlations of the CD according to Eq. (60) as function of $x=k\xi$, where the following different approximations for the fixed point values are used. Solid line: one loop order ε expanded; dashed line: two loop order ε expanded; dotted line: two loop order fixed dimension $d=3$.

A. Dynamic OP shape function

Substituting the expressions of dynamic vertex functions for OP [Eqs. (21) and (22)] into correlation function (20) we may write

$$C_{\phi\phi}(\xi,k,\omega) = \frac{C^{(s)}_{\phi\phi}(\xi,k)}{\omega_\phi(\xi,k)} \frac{2\Re F_\phi(x,y_\phi)}{|iy_\phi F_\phi(x,y_\phi)+1|^2}, \quad (62)$$

where the complex function $F_\phi(x,y_\phi)=\Re F_\phi(x,y_\phi)+i\Im F_\phi(x,y_\phi)$ has been introduced. Comparing Eq. (62) with scaling form of correlation function introduced in Eq. (1), one may identify the dynamic shape function $\mathcal{F}_\phi$ as function of the scaled frequency y_ϕ and the scaled wave vector x as

$$\mathcal{F}_\phi(y_\phi,x) = \frac{2\Re F_\phi(x,y_\phi)}{[y_\phi\Re F_\phi(x,y_\phi)]^2+[1-y_\phi\Im F_\phi(x,y_\phi)]^2}, \quad (63)$$

where $F_\phi(x,y_\phi)$ has been separated into its real and imaginary part. The explicit expression of the function $F_\phi(x,y_\phi)$ is

$$F_\phi(x,y_\phi) \equiv \Gamma^{-1}\Omega_{\phi\tilde{\phi}}(\xi,-k,-\omega)C^{(s)}_{\phi\phi}(\xi,k)\omega_\phi(\xi,k). \quad (64)$$

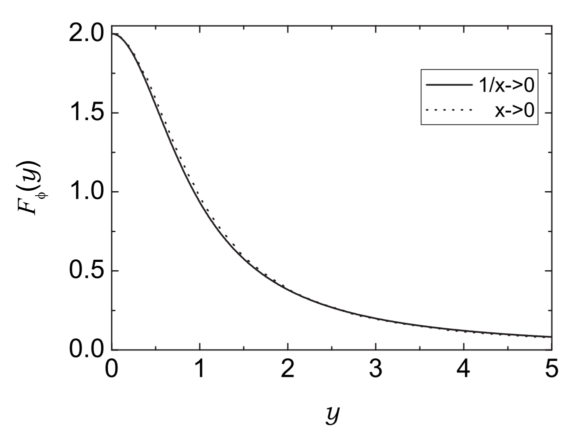

FIG. 3. Shape function of the dynamical correlations of the OP as function of $y=\omega/\omega_\phi$ in the critical limit $x\to\infty$ (solid line) and hydrodynamic limit $x\to 0$ (dotted line) calculated from Eq. (73) with $x=k\xi$.

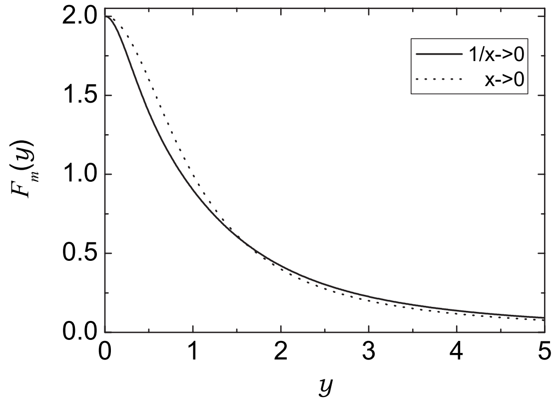

FIG. 4. Shape function of the dynamical correlations of the CD as function of $y=\omega/\omega_m$ in the critical limit $x\to\infty$ (solid line) and hydrodynamic limit $x\to 0$ (dotted line) calculated from Eq. (85) with $x=k\xi$.

Inserting Eqs. (24), (32), (41), and (48) into the above equation one obtains

$$F_\phi(x,y_\phi)=1+w\gamma^2\left[I_\phi^{(d)}\left(\kappa\xi,\frac{-k}{\kappa},\frac{-\omega}{\kappa^2}\right)-I_\phi^{(d)}\left(\kappa\xi,\frac{k}{\kappa},0\right)\right] \tag{65}$$

in one loop order. Using the explicit expression (A5) for the one loop integral (44) given in the Appendix, the function F_ϕ reads

$$F_\phi(x,y_\phi)=1+\frac{\zeta_\Gamma}{2}\Bigg[-(1+w)\ln\frac{w}{1+w}+\frac{1+w}{x^2}\ln\left(1+\frac{x^2}{1+w}\right)-\ln\left(1+\frac{i\omega}{\lambda k^2 w}\frac{1}{\frac{1}{1+w}+\frac{1}{x^2}}\right)+(1+w)\times\left(\bar{y}_+\ln\frac{(1+w)\bar{y}_+-1}{(1+w)\bar{y}_+}+\bar{y}_-\ln\frac{(1+w)\bar{y}_--1}{(1+w)\bar{y}_-}\right)\Bigg], \tag{66}$$

where

$$\bar{y}_\pm=y_\pm(-\omega)=\frac{1}{2}\left[1-\frac{1}{x^2}+\frac{i\omega}{\lambda k^2}\pm\sqrt{\left(1-\frac{1}{x^2}+\frac{i\omega}{\lambda k^2}\right)^2+\frac{4}{x^2}}\right]. \tag{67}$$

The parameter $y_\pm$ is given in Eq. (A6). In order to obtain a function of the scaling variable y_ϕ we have to introduce the characteristic frequency ω_ϕ consistently. In Eq. (66) the scaling variable has only to be introduced in one loop terms. Therefore, it is sufficient to use the lowest order expression for the characteristic frequency. From Eq. (49) follows that $\omega_\phi=\Gamma k^2(1+x^{-2})$ in lowest order. Together with the definition of the time scale ratio $w=\Gamma/\lambda$ we may introduce the scaling variable by

$$\frac{\omega}{\lambda k^2}=y_\phi w\left(1+\frac{1}{x^2}\right). \tag{68}$$

The function (66) can then be rewritten as

$$F_\phi(x,y_\phi)=1+\frac{\zeta_\Gamma}{2}\left[-\ln\left(1+iy_\phi\frac{1+\frac{1}{x^2}}{\frac{1}{1+w}+\frac{1}{x^2}}\right)+Y_\phi(x,y_\phi)\right], \tag{69}$$

where we have introduced the function

$$Y_\phi(x,y_\phi)\equiv-(1+w)\ln\frac{w}{1+w}+\frac{1+w}{x^2}\ln\left(1+\frac{x^2}{1+w}\right)+(1+w)\left(\bar{y}_+\ln\frac{(1+w)\bar{y}_+-1}{(1+w)\bar{y}_+}+\bar{y}_-\ln\frac{(1+w)\bar{y}_--1}{(1+w)\bar{y}_-}\right). \tag{70}$$

The parameter $\bar{y}_\pm$ from Eq. (67) then reads

$$\bar{y}_\pm=\frac{1}{2}\left(1+\frac{1}{x^2}\right)\left[\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}+iy_\phi w\pm\sqrt{\left(\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}+iy_\phi w\right)^2+\frac{4}{x^2\left(1+\frac{1}{x^2}\right)^2}}\right]. \tag{71}$$

The function F_ϕ has several limiting properties which have to be fulfilled also by expression (69). First of all one can see immediately from Eq. (65) that $F_\phi=1$ at $\omega=0$. Setting $y_\phi=0$ in Eq. (69), the property is of course fulfilled because $Y_\phi(x,y_\phi=0)=0$. Further properties of the function F_ϕ are that it stays finite in the critical limit ($x\to\infty$), as well as in the hydrodynamic limit ($x\to 0$). However, a singular behavior may remain in expression (69) in the limit to large frequencies $y_\phi\to\infty$. In order to deal with this limit appropriate exponentiations in the scaling function $\mathcal{F}_\phi$ have to be performed.

Substituting Eq. (69) into shape function (63) and expanding the resulting expression into powers of the static coupling contained in ζ_Γ, we obtain

$$\mathcal{F}_{\phi}(x,y_{\phi}) = \frac{2}{y_{\phi}^2+1}\left[1-\frac{\frac{\zeta_\Gamma}{2}}{y_{\phi}^2+1}\left((y_{\phi}^2-1)\left\{\Re Y_{\phi}(x,y_{\phi})-\frac{1}{2}\ln\left[1+y_{\phi}^2\left(\frac{1+\frac{1}{x^2}}{\frac{1}{1+w}+\frac{1}{x^2}}\right)^2\right]\right\}\right.\right.$$
$$\left.\left.-2y_{\phi}\left[\Im Y_{\phi}(x,y_{\phi})-\arctan\left(y_{\phi}\frac{1+\frac{1}{x^2}}{\frac{1}{1+w}+\frac{1}{x^2}}\right)\right]\right)\right]. \tag{72}$$

The above expression for the shape function contains a term which diverges proportional to $\ln y_{\phi}$ for large frequencies. In order to obtain a finite amplitude in the curly bracket the corresponding logarithmic term may be exponentiated. This leads to the scaling form

$$\mathcal{F}_{\phi}(x,y_{\phi}) = \frac{2\left[1+y_{\phi}^2\frac{\left(1+\frac{1}{x^2}\right)^2}{\left(\frac{1}{1+w}+\frac{1}{x^2}\right)^2}\right]^{\zeta_\Gamma/4}}{y_{\phi}^2+1}\left(1-\frac{\frac{\zeta_\Gamma}{2}}{y_{\phi}^2+1}\left\{\ln\left[1+y_{\phi}^2\frac{\left(1+\frac{1}{x^2}\right)^2}{\left(\frac{1}{1+w}+\frac{1}{x^2}\right)^2}\right]\right.\right.$$
$$\left.\left.+(y_{\phi}^2-1)\Re Y_{\phi}(x,y_{\phi})-2y_{\phi}\left[\Im Y_{\phi}(x,y_{\phi})-\arctan\left(y_{\phi}\frac{1+\frac{1}{x^2}}{\frac{1}{1+w}+\frac{1}{x^2}}\right)\right]\right\}\right). \tag{73}$$

B. Dynamic CD shape function

Quite analogous to Eq. (62) in case of the OP we may write the dynamic CD correlation function as

$$C_{mm}(\xi,k,\omega) = \frac{C_{mm}^{(s)}(\xi,k)}{\omega_m(\xi,k)}\frac{2\Re F_m(x,y_m)}{|iy_mF_m(x,y_m)+1|^2}. \tag{74}$$

Comparing Eq. (74) with the scaling form of correlation function (1) and separating the function $F_m(x,y_m) = \Re F_m(x,y_m)+i\Im F_m(x,y_m)$ into its real and imaginary part we obtain an expression

$$\mathcal{F}_m(y_m,x) = \frac{2\Re F_m(x,y_m)}{[y_m\Re F_m(x,y_m)]^2+[1-y_m\Im F_m(x,y_m)]^2}, \tag{75}$$

which is the counterpart to Eq. (63) now for the CD. It is a function of the scaled frequency

$$y_m \equiv \frac{\omega}{\omega_m(\xi,k)}. \tag{76}$$

The explicit expression of the function $F_m(x,y_m)$ is

$$F_m(x,y_m) \equiv \frac{1}{\lambda k^2}\Omega_{\tilde{m}m}(\xi,-k,-\omega)C_{mm}^{(s)}(\xi,k)\omega_m(\xi,k). \tag{77}$$

Using Eqs. (24), (33), (42), (45), and (54) the function (77) can be written as

$$F_m(x,y_m) = 1+\frac{n}{2}\frac{\gamma^2}{w}\left(\frac{k}{\kappa}\right)^2\left[I_m^{(d)}\left(\kappa\xi,\frac{-k}{\kappa},\frac{-\omega}{\kappa^2}\right)-I_m^{(d)}\left(\kappa\xi,\frac{k}{\kappa},0\right)\right]. \tag{78}$$

Inserting the result for the one loop integral (A12) given in the Appendix and using the one loop expression (58) for ζ_λ, the function F_m reads

$$F_m(x,y_m) = 1+\frac{1}{w}\frac{\zeta_\lambda}{2}\left[\left(i\frac{s}{\frac{\omega}{\Gamma k^2}}-\frac{1}{s}\right)\ln\left(\frac{s+1}{s-1}\right)+\ln\frac{s^2}{2i\frac{\omega}{\Gamma k^2}+s^2}\right.$$
$$\left.+i\frac{\bar{W}'}{\frac{\omega}{\Gamma k^2}}\ln\frac{i\frac{\omega}{\Gamma k^2}+s^2-\bar{W}'}{i\frac{\omega}{\Gamma k^2}+s^2+\bar{W}'}\right], \tag{79}$$

where $\bar{W}'$ is given by

$$\bar{W}' = W'(-\omega) = \sqrt{\left(1+\frac{i\omega}{\Gamma k^2}\right)^2+\frac{4}{x^2}} \tag{80}$$

with W' defined in Eq. (A13). In order to introduce the scaling variable y_m defined in Eq. (77) in the one loop contributions, it is sufficient to take $\omega_m(\xi,k)$ in lowest order. From

Eq. (59) follows that this is $\omega_m(\xi,k)=\lambda k^2$. Thus, we may insert

$$\frac{\omega}{\Gamma k^2}=\frac{y_m}{w} \tag{81}$$

into Eq. (79) which is correct within the one loop approximation. Equation (79) can then be rewritten as

$$F_m(x,y_m)=1+\frac{1}{w}\frac{\zeta_\lambda}{2}\left[\ln\frac{s^2}{2i\frac{y_m}{w}+s^2}+Y_m(x,y_\phi)\right]. \tag{82}$$

Quite analogous to the OP we have introduced the function

$$Y_m(x,y_\phi)=\left(i\frac{ws}{y_m}-\frac{1}{s}\right)\ln\left(\frac{s+1}{s-1}\right)+i\frac{w\bar{W}'}{y_m}\ln\frac{i\frac{y_m}{w}+s^2-\bar{W}'}{i\frac{y_m}{w}+s^2+\bar{W}'}. \tag{83}$$

The parameter $\bar{W}'$ given in Eq. (80) becomes

$$\bar{W}'=\sqrt{\left(1+i\frac{y_m}{w}\right)^2+\frac{4}{x^2}}. \tag{84}$$

From the general expression (78) one can immediately see that $F_m(x,y_m)=1$ at $\omega=0$. The corresponding scaling function $\mathcal{F}_m(x,y_m)$ has to stay finite in the critical limit ($x\to\infty$) as well as in the hydrodynamic limit ($x\to 0$). Function (82) may be inserted into shape function (75). Expanding the resulting expression into powers of the couplings one obtains

$$\mathcal{F}_m(x,y_m)=\frac{2}{1+y_m^2}\left(1-\frac{\frac{\zeta_\lambda}{2w}}{1+y_m^2}\left\{(y_m^2-1)(\Re Y_m(x,y_m)-\frac{1}{2}\ln\left(1+\frac{4y_m^2}{w^2s^4}\right)-2y_m\left[\Im Y_m(x,y_m)-\arctan\left(\frac{2y_m}{ws^2}\right)\right]\right\}\right), \tag{85}$$

which is the analog expression to Eq. (72) now for the CD. A closer examination of Eq. (85) reveals that in contrast to the OP shape function no further exponentiation is necessary because the expression remains finite also in the high frequency limit $y_m\to\infty$.

VII. LIMITING BEHAVIOR OF THE SHAPE FUNCTIONS

In the limiting cases the general expressions of the shape functions can be given in analytic form. It is also demonstrated in this section that the shape function of the OP remains non-Lorentzian in the hydrodynamic limit. The shape function of the CD crosses over from a non-Lorentzian shape to a Lorentzian in the hydrodynamic region.

A. OP shape function in different limits

1. Shape function at T_c

In the critical limit $T\to T_c$ the correlation length ξ diverges faster than the wave vector modulus k goes to zero. This means that the scaling variable x grows to infinity. In this case the parameters $\bar{y}_\pm$ in Eq. (71) simplify to

$$\lim_{x\to\infty}\bar{y}_+=1+iy_\phi w,\quad \lim_{x\to\infty}\bar{y}_-=0. \tag{86}$$

Inserting into Eqs. (69) and (70) leads to

$$F_\phi(\infty,y_\phi)=1+\frac{\zeta_\Gamma}{2}\{-\ln[1+iy_\phi(1+w)]+Y_\phi(\infty,y_\phi)\}, \tag{87}$$

with

$$Y_\phi(\infty,y_\phi)=(1+w)\left[iy_\phi w\ln\frac{w}{1+w}+(1+iy_\phi w)\ln\frac{1+iy_\phi(1+w)}{1+iy_\phi w}\right]. \tag{88}$$

Using these expressions in the dynamic shape function (73) and expanding the whole expression in the coupling γ (entering via ζ function) one obtains

$$\mathcal{F}_\phi(\infty,y_\phi)=\frac{2[1+y_\phi^2(1+w)^2]^{\zeta_\Gamma/4}}{y_\phi^2+1}\left[1-\frac{\frac{\zeta_\Gamma}{2}}{y_\phi^2+1}(\ln[1+y_\phi^2(1+w)^2]+(y_\phi^2-1)\Re Y_\phi(\infty,y_\phi)-2y_\phi\{\Im Y_\phi(\infty,y_\phi)-\arctan[y_\phi(1+w)]\})\right]. \tag{89}$$

The real and the imaginary parts of the function $Y_\phi(\infty,y_\phi)$ are given as

$$\Re Y_\phi(\infty,y_\phi)=(1+w)\left\{\ln\frac{\sqrt{[1+y_\phi^2 w(1+w)]^2+y_\phi^2}}{1+y_\phi^2w^2}-y_\phi w\arctan\frac{y_\phi}{1+y_\phi^2w(1+w)}\right\}, \tag{90}$$

$$\Im Y_\phi(\infty,y_\phi)=(1+w)\left\{y_\phi w\ln\frac{w}{1+w}+y_\phi w\ln\frac{\sqrt{[1+y_\phi^2 w(1+w)]^2+y_\phi^2}}{1+y_\phi^2w^2}+\arctan\frac{y_\phi}{1+y_\phi^2w(1+w)}\right\}. \tag{91}$$

Considering now scaling function (89) at large frequencies one has to perform the limit $y_\phi\to\infty$ in expressions (89)–(91). One can immediately see that in the curly bracket only the parts proportional to $\Re Y$ and $\Im Y$ may give finite contributions. All other terms vanish in this limit. Performing a careful calculation of the limit reveals that

$$\lim_{y_\phi\to\infty}\frac{y_\phi^2-1}{y_\phi^2+1}\Re Y_\phi(\infty,y_\phi)=(1+w)\ln\frac{1+w}{w}-1 \tag{92}$$

and

$$\lim_{y_\phi\to\infty}\frac{2y_\phi}{y_\phi^2+1}\Im Y_\phi(\infty,y_\phi)=0. \tag{93}$$

Thus, the shape function reduces to

$$\lim_{y_\phi\to\infty}\mathcal{F}_\phi(\infty,y_\phi)\sim 2y_\phi^{-2+\zeta_\Gamma/2}(1+w)^{\zeta_\Gamma/2}\times\left\{1-\frac{\zeta_\Gamma}{2}\left[(1+w)\ln\frac{1+w}{w}-1\right]\right\} \tag{94}$$

behavior with the dynamical exponent $-v_\phi=-2+\zeta_\Gamma^\star/2=(z_\phi+2-\eta)/z_\phi$.

We have exponentiated the logarithmic term and in the last step of Eq. (94) in the power of y the one loop expression of the ζ_Γ function has been used. This agrees in this order with the exact scaling result (8).

The singular behavior of the shape function for large arguments can be included into a compact one loop expression by proper exponentiation. This expression then allows us to improve the shape by using two loop fixed point values consistently. Since the proper ζ functions appearing in the exponents have been identified only the finite perturbational contribution to the shape depends on the coupling and the time scale ratio. Since also the coupling can be expressed by the ζ function the remaining parameter of the shape function is the time scale ratio. The asymptotic value of the ζ function is related to static exponents and one may take the most precise values known from theory or experiment. It remains an uncertainty due to the low order perturbational calculation and the uncertainty in the fixed point value of the time scale ratio.

2. Shape function in the hydrodynamic limit

In the case that the wave vector modulus k is vanishing faster than the correlation length ξ goes to infinity, one approaches the hydrodynamic limit. As a consequence the scaling variable x tends to zero. The parameters $\bar{y}_\pm$ in Eq. (71) reduce to

$$\lim_{x\to 0}\bar{y}_+\sim-\frac{1-iy_\phi w}{x^2},\quad \lim_{x\to 0}\bar{y}_-=\frac{1}{1-iy_\phi w}. \tag{95}$$

Inserting this into Eqs. (69) and (70) leads to

$$F_\phi(0,y_\phi)=1+\frac{\zeta_\Gamma}{2}[-\ln(1+iy_\phi)+Y_\phi(0,y_\phi)], \tag{96}$$

$$Y_\phi(0,y_\phi)=(1+w)\left[-\ln\frac{w}{1+w}+\frac{1}{1-iy_\phi w}\ln\frac{w(1+iy_\phi)}{1+w}\right]. \tag{97}$$

Inserting Eq. (96) into the exponentiated shape function (73) one obtains

$$\mathcal{F}_\phi(0,y_\phi)=2(1+y_\phi^2)^{-1+\zeta_\Gamma/4}\left(1-\frac{\frac{\zeta_\Gamma}{2}}{1+y_\phi^2}\{\ln(1+y_\phi^2)+(y_\phi^2-1)\Re Y_\phi(0,y_\phi)-2y_\phi[\Im Y_\phi(0,y_\phi)-\arctan(y_\phi)]\}\right). \tag{98}$$

The real part and imaginary part of the function $Y_\phi(0,y_\phi)$ in Eq. (97) are given by

$$\Re Y_\phi(0,y_\phi)=\frac{1+w}{1+y_\phi^2w^2}\left[\ln\sqrt{1+y_\phi^2}-y_\phi^2w^2\ln\frac{w}{1+w}-y_\phi w\arctan(y_\phi)\right], \tag{99}$$

$$\Re Y_\phi(0,y_\phi)=\frac{1+w}{1+y_\phi^2w^2}\left[y_\phi w\ln\frac{w\sqrt{1+y_\phi^2}}{1+w}+\arctan(y_\phi)\right]. \tag{100}$$

Analogous to the previous subsection the shape function will be now considered for large frequencies i.e., large values of y_ϕ. From Eqs. (98) and (100) the contributions from the imaginary part of $Y_\phi(0,y_\phi)$ vanish in this limit. The only finite term comes from the real part of $Y_\phi(0,y_\phi)$. A closer examination reveals that one has in the limit $y\to\infty$ the leading term

$$\lim_{y_\phi\to\infty}\frac{y_\phi^2-1}{y_\phi^2+1}\Re Y_\phi(0,y_\phi)=-(1+w)\ln\frac{w}{1+w} \tag{101}$$

leading to a shape function

$$\lim_{y_\phi\to\infty}\mathcal{F}_\phi(0,y_\phi)\sim 2y_\phi^{-2+\zeta_\Gamma/2}\left[1+\frac{\zeta_\Gamma}{2}(1+w)\ln\frac{w}{1+w}\right]. \tag{102}$$

Although the shape depends on x and remains non-Lorentzian in the whole region one sees that the large frequency behavior in both limiting cases is the same. A similar situation was obtained in the antiferromagnet described by the dynamical model of Refs. [18,19] and used for comparison with neutron-scattering experiments in [20]. Correctly performing the limit $x\to 0$ in Eq. (5) of Ref. [18], substitution into formulas (4) and (3) there and considering the case of large frequencies leads to the same behavior of correlation function described by power law (102) with the corresponding dynamical critical exponent v_ϕ. Thus, contrary to the statement in the Appendix of [20], this behavior holds for other dynamic models with a nonconserved OP too (for calculations in the SSS model see [21]).

In the case of a conserved OP the perturbational contributions to frequency terms in $\mathring{\Omega}_{\phi\tilde{\phi}}$ [Eq. (21)] are at least of the order ωk^2. All these contributions vanish in the limit $k\xi\to 0$. For a nonconserved OP pure frequency-dependent terms remain in this function and lead to the non-Lorentzian behavior at large frequencies.

B. CD shape function in different limits

1. Shape function at T_c

In the critical limit $x\to\infty$ the parameters s and $\bar{W}'$ according to Eqs. (56) and (84) have to be expanded in powers of $1/x^2$ in order to perform the limit correctly. Taking into account the corrections in first order one obtains

$$s = 1 + \frac{2}{x^2} + \mathcal{O}\left(\frac{1}{x^2}\right)^2 \qquad (103)$$

and

$$\bar{W}' = \left(1 + i\frac{y_m}{w}\right)\left[1 + \frac{2}{\left(1 + i\frac{y_m}{w}\right)^2}\frac{1}{x^2} + \mathcal{O}\left(\frac{1}{x^2}\right)^2\right]. \qquad (104)$$

Inserting Eqs. (103) and (104) into Eq. (82) one obtains

$$F_m(\infty,y_m) = 1 + \frac{1}{w}\frac{\zeta_\lambda}{2}\left[-\ln\left(1 + 2i\frac{y_m}{w}\right) + Y_m(\infty,y_\phi)\right] \qquad (105)$$

with

$$Y_m(\infty,y_\phi) = \left(\frac{iw}{y_m} - 1\right)\ln\frac{1 + 2i\frac{y_m}{w}}{\left(1 + i\frac{y_m}{w}\right)^2}. \qquad (106)$$

The real part and imaginary part of the function above are

$$\Re Y_m(\infty,y_m) = \left[-\ln\frac{\sqrt{\left(1 + 3\frac{y_m^2}{w^2}\right)^2 + 4\frac{y_m^6}{w^6}}}{\left(1 + \frac{y_m^2}{w^2}\right)^2} + \frac{w}{y_m}\arctan\frac{2\frac{y_m^3}{w^3}}{1 + 3\frac{y_m^2}{w^2}}\right], \qquad (107)$$

$$\Im Y_m(\infty,y_m) = \left[\frac{w}{y_m}\ln\frac{\sqrt{\left(1 + 3\frac{y_m^2}{w^2}\right)^2 + 4\frac{y_m^6}{w^6}}}{\left(1 + \frac{y_m^2}{w^2}\right)^2} + \arctan\frac{2\frac{y_m^3}{w^3}}{1 + 3\frac{y_m^2}{w^2}}\right]. \qquad (108)$$

Inserting this limiting form (107) and (108) into the dynamic shape function (85) reveals that in the high frequency limit $y_m\to\infty$ all logarithmic terms in y_m cancel exactly. The shape function reduces to

$$\lim_{y_m\to\infty} \mathcal{F}_m(\infty,y_m) = \frac{2}{y_m^2}\left\{1 + \frac{\zeta_\lambda}{w}\ln 2 + \mathcal{O}(y_m^{-1})\right\} \sim y_m^{-2} \qquad (109)$$

proofing that a Lorentzian-type decay remains also in the critical limit.

2. Shape function in hydrodynamic case

The hydrodynamic limit $x\to 0$ of Eq. (82) is simply $F_m(0,y_m)=1$ and therefore one ends up with a Lorentzian shape

$$\mathcal{F}_m(0,y_m) = \frac{2}{1 + y_m^2}. \qquad (110)$$

Thus, the shape function of the CD shows the expected crossover from a non-Lorentzian to a Lorentzian shape if one goes from the critical to the hydrodynamic region.

VIII. DYNAMICAL AMPLITUDE RATIOS

Inserting the characteristic frequencies (61) and (53) into the general definition (9) of the amplitude ratio [3,10] one obtains

$$R = \lim_{x\to 0}\frac{1}{w}\frac{f_m(x)}{f_\phi(x)x^2}. \qquad (111)$$

Note that in this expression the structure of the definition has been kept and the characteristic frequencies have been calculated separately. Inserting the amplitude functions (52) and (60) and performing the limit $x\to 0$ one obtains

$$R = \frac{1}{w}\left[1 + \frac{\gamma^2}{2}\left(\frac{n}{2} + \frac{w^2}{1+w}\ln\frac{w}{1+w}\right)\right]. \qquad (112)$$

In the above expression the ratio of the two amplitude functions already has been expanded. This expression differs from the result found in [3] Eq. (6.49). There, the value n instead of $n/2$ appears in the γ^2 term; the difference can be traced back to the procedure chosen in order to remove the poles appearing in the dynamic correlation and response functions. Our method follows the systematic field theoretic procedure of the minimal subtraction scheme.

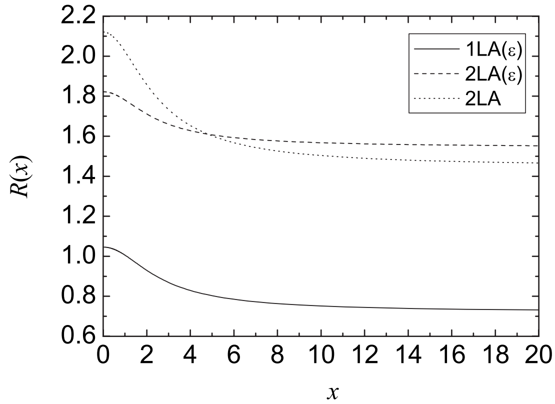

FIG. 5. Generalized dynamic amplitude ratio as function of the scaling variable $x=k\xi$ [see Eq. (114)]. In the hydrodynamic limit $x\to 0$ the former defined asymptotic value is obtained (see Table I). The following different approximations for the fixed point values are used. Solid line: one loop order ε expanded; dashed line: two loop order ε expanded; dotted line: two loop order fixed dimension $d=3$.

In the asymptotic limit the parameters γ and w assume their fixed point values. Inserting the one loop fixed point values one gets for $n=1$

$$R=\left[1+\frac{\varepsilon}{6}(1-\ln 2)\right]=1.05 \quad \text{at} \quad \varepsilon=1 \tag{113}$$

agreeing with the value given in [22], but disagreeing with the value of $R=1.5$ in [3].

One can generalize the asymptotic dynamical amplitude ratio approaching the critical point from any direction in the k-ξ^{-1} plane by

$$R(x)=\lim_{k\to 0,\xi\to\infty,x}\left[\frac{\omega_m(k,\xi)}{\omega_\phi(k,\xi)}\left(1+\frac{1}{x^2}\right)\right]=\frac{1}{w^\star}\frac{f_\phi(x)}{f_m(x)}\left(1+\frac{1}{x^2}\right). \tag{114}$$

Thus, for $x\to\infty$ measurements of the characteristic frequency at T_c but with finite wave vector modulus going to zero can be used to check the theoretical result for $R(x\to\infty)$. Both frequencies can be found from scattering experiments. The dependence of $R(x)$ on x is shown in Fig. 5. For $x\to\infty$ one obtains the values for $R(\infty)$ (see Table I).

IX. DISCUSSION OF THE CASE OF *WEAK* DYNAMIC SCALING

Although in $d=3$ both weak scaling fixed points $w^\star=0$ and $1/w^\star=0$ are not stable we consider these cases as representatives for very small, respectively, very large values of the time scale ratio.

(1) $w^\star=0$: then the OP is slower than the CD and since the static coupling in dynamic quantities and in the corresponding ζ functions always appears in the combination $[w\gamma^2/(1+w)]$ the model reduces to model A with $z_{OP}=z_{modelA}$ and $z_{CD}=2$; the scaling functions of the CD are given by the Van Hove expressions.

(2) $1/w^\star=0$: in this case the CD is slower than the OP, a situation which has been found to be unstable in two loop order for all spatial dimensions [4]. In such a case perturbational contributions from the static coupling remain and a logarithmic diverging contribution appears in the ζ_Γ function.

X. CONCLUSION

We have calculated the dynamic correlation functions for model C. There is a crossover in the dynamic shape functions from the critical ($x\to\infty$) to the hydrodynamic ($x\to 0$) region. However, it turns out that the dynamic shape function of the OP remains non-Lorentzian in the whole region with the same power law for frequency dependence at large frequencies. Thus, in the hydrodynamic region the correlation function of the OP does not cross over to a Lorentzian as expected from general argument used in dynamical scaling theory. This can also be found in the model G but has not been recognized so far [18–21]. For the CD shape function in the hydrodynamic region a Lorentzian shape function is found with a decay exponent $\upsilon=2$ at large scaled frequencies. In the critical region the CD shape function crosses over to a non-Lorentzian but the decay exponent remains the same. This is different from other dynamical models, e.g., model E, where in the critical region the CD shape decays faster than a Lorentzian [21,23] for large frequencies. It would be interesting to check this behavior by computer simulations of model C. Concerning the dynamical critical exponent z the results obtained so far are contradicting [24].

APPENDIX: GENERAL SOLUTION OF THE ONE LOOP INTEGRALS

In order to obtain an explicit expression for the dynamic correlation functions of the OP and the CD, the two integrals in Eq. (35) and in Eq. (36) have to be calculated for finite frequency, wave vector modulus, and correlation length.

1. Order parameter

The generalized form of the one loop integral in the dynamic OP correlation function (35) is

$$I_{\phi\phi}^{(1L)}=\int_{k'}\frac{1}{(a+k'^2)[b+\beta k'^2+(k+k')^2]}. \tag{A1}$$

Folding the denominators with Feynman's method and performing the k' integration one obtains

$$I_{\phi\phi}^{(1L)}=\frac{A_d}{\varepsilon}\frac{1-\frac{\varepsilon}{2}}{1+\beta}\int_0^1 dx\left[(1-x)a+\frac{xb}{1+\beta}+\frac{x}{1+\beta}\left(1-\frac{x}{1+\beta}\right)k^2\right]^{-\varepsilon/2}. \tag{A2}$$

Solving the parameter integrals in ε expansion, the result up to order ε^0 is

$$I_{\phi\phi}^{(1L)} = \frac{A_d}{\varepsilon}\frac{1}{1+\beta}\left\{1+\frac{\varepsilon}{2}\left[1-\ln\frac{b+\frac{\beta k^2}{1+\beta}}{1+\beta}+(1+\beta)\times\left(y_+\ln\frac{(1+\beta)y_+-1}{(1+\beta)y_+}+y_-\ln\frac{(1+\beta)y_--1}{(1+\beta)y_-}\right)\right]\right\}, \quad \text{(A3)}$$

where $y_\pm$ is defined as

$$y_\pm = \frac{1}{2k^2}\{b-(1+\beta)a+k^2 \pm \sqrt{[b-(1+\beta)a+k^2]^2+4ak^2}\}. \quad \text{(A4)}$$

The parameters a, b, and β, appearing in Eq. (A1), may be identified by comparison with Eq. (38). Subtracting the ε pole from Eq. (A3) the renormalized form of integral (38) is

$$I_\phi^{(d)} = \frac{1}{2}\frac{1}{1+w}\left\{1-\ln k^2-\ln\frac{w}{1+w}-\ln\left(-\frac{i\omega}{\lambda k^2 w}+\frac{1}{x^2}+\frac{1}{1+w}\right)+(1+w)\left(y_+\ln\frac{(1+w)y_+-1}{(1+w)y_+}+y_-\ln\frac{(1+w)y_--1}{(1+w)y_-}\right)\right\}, \quad \text{(A5)}$$

where in this case $y_\pm$ is the following expression:

$$y_\pm = \frac{1}{2}\left[1-\frac{1}{x^2}-\frac{i\omega}{\lambda k^2}\pm\sqrt{\left(1-\frac{1}{x^2}-\frac{i\omega}{\lambda k^2}\right)^2+\frac{4}{x^2}}\right]. \quad \text{(A6)}$$

2. Secondary density

The generalized form of the one loop integral in the dynamic SD correlation function (36) is

$$I_{mm}^{(1L)} = \int_{k'}\frac{1}{(a+k'^2)[a+(k+k')^2][b+k'^2+(k+k')^2]}. \quad \text{(A7)}$$

Folding the denominators with Feynman's method two times and performing the k' integration one obtains

$$I_{mm}^{(1L)} = \frac{A_d}{4}\left(1-\frac{\varepsilon}{2}\right)\int_0^1 dx\int_0^1 dy\frac{y}{\left[ya+(1-y)\frac{b}{2}+\frac{1+y(2x-1)}{2}\left(1-\frac{1+y(2x-1)}{2}\right)k^2\right]^{1+\varepsilon/2}}. \quad \text{(A8)}$$

The above integral contains no ε pole, thus it is sufficient in one loop to calculate it at $\varepsilon=0$. Performing the x and y integrations leads to the expression

$$I_{mm}^{(1L)} = \frac{A_d}{8}\frac{1}{a-\frac{b}{2}}\left\{S\ln\left(\frac{1+S}{1-S}\right)^2+\frac{4}{k^2}\left(a-\frac{b}{2}\right)\ln\frac{4a}{2b+k^2}+2\frac{W}{k^2}\ln\frac{k^2+2\left(a+\frac{b}{2}\right)-W}{k^2+2\left(a+\frac{b}{2}\right)+W}\right\}. \quad \text{(A9)}$$

In the above expression the following parameters have been introduced:

$$S = \sqrt{1+\frac{4a}{k^2}} \quad \text{(A10)}$$

and

$$W \equiv \sqrt{4\left(a-\frac{b}{2}\right)^2+k^2(k^2+2b)}. \quad \text{(A11)}$$

The parameters a and b in Eq. (A7) may be identified by comparing with Eq. (39). The renormalized integral (39) obtained by using expression (A9) and definition (45) is

$$I_m^{(d)} = \frac{\Gamma}{4i\omega}\left\{s\ln\left(\frac{1+s}{1-s}\right)^2+\frac{2i\omega}{\Gamma k^2}\ln\frac{4x^{-2}}{2\left(-\frac{i\omega}{\Gamma k^2}+\frac{2}{x^2}\right)+1}+2W'\ln\frac{s^2-\frac{i\omega}{\Gamma k^2}-W'}{s^2-\frac{i\omega}{\Gamma k^2}+W'}\right\} \quad \text{(A12)}$$

with s defined in Eq. (56) and

$$W' \equiv \sqrt{\left(1-\frac{i\omega}{\Gamma k^2}\right)^2+\frac{4}{x^2}}. \quad \text{(A13)}$$

[1] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[2] B. I. Halperin, P. C. Hohenberg, and Shang-keng Ma, Phys. Rev. B **10**, 139 (1974).
[3] B. I. Halperin, P. C. Hohenberg, and Shang-keng Ma, Phys. Rev. B **13**, 4119 (1976).
[4] R. Folk and G. Moser, Phys. Rev. Lett. **91**, 030601 (2003).
[5] R. Folk and G. Moser, Phys. Rev. E **69**, 036101 (2004).
[6] D. R. Nelson, J. M. Kosterlitz, and M. E. Fisher, Phys. Rev. Lett. **33**, 813 (1974).
[7] R. Folk, Y. Holovatch, and G. Moser, Phys. Rev. E **78**, 041125 (2008).
[8] G. Grinstein, Shang-keng Ma, and G. F. Mazenko, Phys. Rev. B **15**, 258 (1977).
[9] P. Calabrese and A. Gambassi, J. Phys. A **38**, R133 (2005).
[10] V. Privman, P. C. Hohenberg, and A. Aharony, *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, UK, 1991).
[11] One of the viewpoints for defining such amplitudes is the observability of the quantities involved. In the definition Eq. (9) the wave-vector-dependent characteristic frequency of the OP is used, which is an unobservable quantity at the superfluid transition. Therefore, there one replaces $\omega_\phi(q)$ by the wave-vector-independent frequency ω_C (see Eq. (6.133) in [10]). This frequency is set by the mode coupling which is related to the Larmor precession term (see B. I. Halperin, in *Statistical Physics*, Proceedings of the International Conference, Budapest, 1975, edited by L. Pál and P. Szépfalusy (North-Holland, Amsterdam, 1976), p. 163). Such terms are not present in model C.
[12] B. I. Halperin, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. B **13**, 1299 (1976); **21**, 2044 (1980).
[13] V. Dohm and R. Folk, Phys. Rev. Lett. **46**, 349 (1981).
[14] E. Brézin and C. De Dominicis, Phys. Rev. B **12**, 4954 (1975).
[15] R. Bausch, H.-K. Janssen, and H. Wagner, Z. Phys. B **24**, 113 (1976).
[16] R. Folk and G. Moser, J. Phys. A **39**, R207 (2006).
[17] D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena*, 2nd ed. (World Scientific, Singapore, 1984).
[18] R. Freedman and G. F. Mazenko, Phys. Rev. Lett. **34**, 1575 (1975).
[19] R. Freedman and G. F. Mazenko, Phys. Rev. B **13**, 4967 (1976).
[20] R. Coldea, R. A. Cowley, T. G. Perring, D. F. McMorrow, and B. Roessli, Phys. Rev. B **57**, 5281 (1998).
[21] M. Weiretmayr, R. Folk, and G. Moser, so far unpublished calculation of the dynamical correlation functions of the SSS model including models G and E.
[22] K. K. Murata, Phys. Rev. B **13**, 2028 (1976).
[23] V. Dohm, Z. Phys. B **33**, 79 (1979).
[24] P. Sen, S. Dasgupta, and D. Stauffer, Eur. Phys. J. B **1**, 107 (1998); D. Stauffer, Int. J. Mod. Phys. C **8**, 1263 (1997).