Wobbling kinks in ϕ^4 theory

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We present a uniform asymptotic expansion of the wobbling kink to any order in the amplitude of the wobbling mode. The long-range behavior of the radiation is described by matching the asymptotic expansions in the far field and near the core of the kink. The complex amplitude of the wobbling mode is shown to obey a simple ordinary differential equation with nonlinear damping. We confirm the $t^{-1/2}$-decay law for the amplitude, which was previously obtained on the basis of energy considerations.

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I. INTRODUCTION

Since the early 1960s, the one-dimensional ϕ^4 theory has been among the most heavily utilized models of statistical mechanics and condensed-matter physics [1]. At the same time, it served as a testing ground for a variety of ideas in topological quantum field theory [2] and cosmology [3]. The equation of motion for the model reads

$$\frac{1}{2}\phi_{tt} - \frac{1}{2}\phi_{xx} - \phi + \phi^3 = 0, \tag{1}$$

and the fundamental role in applications is played by its kink solution,

$$\phi(x,t) = \tanh x. \tag{2}$$

The ϕ^4 kinks describe domain walls in ferromagnets [4] and ferroelectrics [5–7], and represent elementary excitations in the corresponding partition function [7]. They were employed to model proton transport in hydrogen-bonded chains [8] and charge-density waves in polymers and some metals [9,10]. Topological defects described by kinks are generated in large numbers during second-order phase transitions [11]; such transitions occur in a variety of condensed-matter systems and are believed to have been made by different parts of the early universe [3]. In quantum field theory, kinks represent nonperturbative classical solutions that undergo subsequent quantization [12]; one example concerns "bags" containing fermions [13]. (For more recent quantum physics applications, see [14].)

Mathematically, the ϕ^4 kink has a lot in common with its sine-Gordon counterpart; the two kinks are the simplest examples of topological solitons in one dimension. There is an important difference though; the sine-Gordon equation is integrable whereas the ϕ^4 theory is not. Accordingly, the kink-antikink interaction becomes a nontrivial matter in the ϕ^4 case [6,15,16]. Another (not unrelated) difference is that, unlike the kink of the sine-Gordon equation, the ϕ^4 kink has an internal mode—an extra degree of freedom that allows for oscillations in the width of the kink. Although these oscillations are accompanied by the emission of radiation (another manifestation of the nonintegrability of the ϕ^4 model), the radiation is quite weak and the oscillations are sustained over long periods of time. Since the amplitude of the oscillations can be fairly large, this periodically expanding and contracting kink (termed *wobbling* kink in literature, or simply *wobbler*) can be regarded as one of the fundamental nonlinear excitations of the ϕ^4 theory, on a par with the nonoscillatory kinks and breathers. For small oscillation amplitudes and on short-time intervals, the wobbler can be characterized simply as a linear perturbation of stationary kink (2). However in order to determine the lifetime of this particlelike structure (even when its amplitude is small), or characterize it when it is a large-amplitude excitation, one needs a self-consistent fully nonlinear description.

The wobbling kink was discovered in the early numerical experiments of Getmanov [16] who interpreted it as a bound state of three fundamental (i.e., nonoscillatory) kinks. (For a more recent series of numerical simulations, see [17].) Rice and Mele have reobtained this nonlinear excitation within a variational approach employing the width of the kink as a dynamical variable [10,18]. Segur then constructed the quiescent (i.e., nonpropagating) wobbler as a *regular* perturbation expansion in powers of the oscillation amplitude [19]. He calculated the first two orders of the perturbation series and noted the likely occurrence of unbounded terms at the third (ϵ^3-) order, implying the consequent breakdown of the expansion. His construction was extended in Ref. [20] where the effect of the wobbling on the stationary component of the kink was evaluated. It is also appropriate to mention Ref. [21] where its author derived an expression for the radiation wave emitted by an initially nonradiating wobbler, and a series of publications [22] where the interaction of the wobbler with radiation waves was studied in more detail and from a variety of perspectives. From the fact that the energy of the wobbling kink is quadratic in the amplitude of the wobble while the second-harmonic radiation flux is quartic, it is straightforward to conclude that the amplitude decays as $t^{-1/2}$ [21,23,24].

Moving on to *singular* perturbation expansions, Kiselev [25] studied the perturbed ϕ^4 kink using the Krylov-Bogoliubov-Mitropolskii method. (Later, he extended his

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analysis to the ϕ^4 equation with a conservative autonomous perturbation [26].) His two papers have mathematical rigor and a wealth of useful formulas; however a self-consistent system of equations for the kink's parameters was not derived in [25,26] and the long-term evolution of the wobbling kink has, therefore, remained unexplored. Manton and Merabet [24] used the Lindstedt-Poincaré method [27], where the expansion of the field is supplemented by an expansion of the frequency of the wobbling. Manton-Merabet's theory was successful in reproducing the decay law of the wobbling amplitude (which was previously obtained from the energy considerations [21,23,24]). However, the Lindstedt-Poincaré method, although efficient in finding periodic orbits, may lead to erroneous conclusions about nonperiodic regimes [27]. (One manifestation of this inadequacy in the case at hand is that the nonlinear corrections to the frequency become complex and time dependent [24]; a less obvious difficulty is the infinite speed of the signal propagation, see below.) This motivates the search for new approaches that would be mathematically self-consistent (like the one in [25,26]) on the one hand, and preserve the physical insights of phenomenological expansions [21,24] on the other.

The aim of the present paper is to develop a singular perturbation expansion of this kind. Our approach recognizes the existence of a hierarchy of space and time scales associated with the kink+radiation system, and generates a perturbative expansion that remains uniform to all orders. The consistent treatment of radiation requires also the introduction of an independent expansion of the far field, which is then matched to the expansion near the core of the kink. This produces physically consistent and asymptotically accurate results at all space and time scales. In particular we will obtain a nonlinear ordinary differential equation obeyed by the amplitude of the wobbling mode. In the follow-up paper [28] our multiscale approach will be used for the analysis of the wobbling kink driven by a resonant force.

The basics of our method are outlined in the next section. In Secs. III and IV we evaluate the first- and second-order corrections to the shape of the wobbling kink, and in Sec. VI we derive an equation for the amplitude of the wobbling mode. The asymptotic matching of the radiation on the short and long scales is carried out in Sec. V; here we show, in particular, how to account for finite propagation speed of radiation in a mathematically consistent way. Finally, conclusions of this study are summarized in Sec. VII.

II. METHOD

We consider the kink moving with the velocity v. Making the change of variables $(x,t) \rightarrow (\xi,\tau)$, where

$$\xi = x - \int_0^t v(t') \, dt', \quad \tau = t,$$

we transform Eq. (1) to the comoving frame,

$$\frac{1}{2}\phi_{\tau\tau} - v\phi_{\xi\tau} - \frac{v_\tau}{2}\phi_\xi - \frac{1-v^2}{2}\phi_{\xi\xi} - \phi + \phi^3 = 0. \quad (3)$$

Like the authors of [29], we shall determine the kink's velocity $v(\tau)$ by imposing the condition that the kink be always centered at $\xi=0$ [i.e., at $x=\int_0^t v(t')dt'$].

At first glance, the inclusion of the function $v(t)$ is unnecessary: having constructed a quiescent wobbling kink, we could make it move at any speed simply by a Lorentz boost. The reason we have introduced the velocity explicitly in Eq. (3) is twofold. Firstly, this will allow us to check whether the wobbling kink can drift with a *nonconstant* velocity. The soliton moving with a variable $v(t)$ could obviously not be Lorentz transformed to the rest frame. Secondly, we include the velocity in preparation for the analysis of the damped-driven ϕ^4 equation in the second part of this project [28]. Since the damping and driving terms violate relativistic invariance, the explicit introduction of the velocity becomes essential even when considering the damped-driven wobblers moving at a constant speed.

We expand the field about the kink $\phi_0 \equiv \tanh \xi$,

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots. \quad (4)$$

Here ϵ is a (formal) small parameter; it will drop out of the final expression for the solution [see Eq. (48) below]. Substituting Eq. (4) in Eq. (3) and setting to zero coefficients of like powers of ϵ would constitute Segur's approach, which is expected to produce secular terms in the expansion [19]. To avoid these, we introduce a sequence of stretched space and time variables

$$X_n \equiv \epsilon^n \xi, \quad T_n \equiv \epsilon^n \tau, \quad n = 0, 1, 2, \ldots, \quad (5)$$

which describe slower times and longer distances. In the limit $\epsilon \rightarrow 0$, the different scales become uncoupled and may be treated as independent variables. We expand the ξ and τ derivatives in terms of the scaled variables by using the chain rule,

$$\frac{\partial}{\partial \xi} = \partial_0 + \epsilon\partial_1 + \epsilon^2\partial_2 + \cdots,$$

$$\frac{\partial}{\partial \tau} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots, \quad (6)$$

where we have used the standard short-hand notation

$$\partial_n \equiv \frac{\partial}{\partial X_n}, \quad D_n \equiv \frac{\partial}{\partial T_n}.$$

Substituting these expansions into ϕ^4 equation (3), along with series (4), and equating coefficients of like powers of ϵ, we obtain a hierarchy of equations. We assume that the velocity of the kink is slowly varying and that, for simplicity, it is small, i.e., $v = \epsilon V$, where $V = V(T_1, T_2, \ldots)$ is of order one.

III. LINEAR PERTURBATIONS

At $O(\epsilon^1)$, we obtain the linearization of Eq. (1) about the kink $\phi_0 = \tanh X_0$,

$$\frac{1}{2}D_0^2\phi_1 + \mathcal{L}\phi_1 = 0, \quad (7)$$

where we have introduced the Schrödinger operator

$$\mathcal{L} = -\frac{1}{2}\partial_0^2 - 1 + 3\phi_0^2 = -\frac{1}{2}\partial_0^2 + 2 - 3 \operatorname{sech}^2 X_0. \quad (8)$$

The general solution of the variable-coefficient Klein-Gordon equation (7) can be written as

$$\phi_1 = C\,y_T(X_0) + A e^{i\omega_0 T_0}\,y_w(X_0) + \text{c.c.} + \phi_R(X_0, T_0), \quad (9)$$

with

$$\phi_R = \int_{-\infty}^{\infty} [\mathcal{R}(p) e^{i\omega(p)T_0} + \mathcal{R}^*(-p) e^{-i\omega(p)T_0}]\,y_p(X_0)\,dp. \quad (10)$$

Here y_T and y_w are eigenfunctions of the operator $\mathcal{L}$ associated with eigenvalues 0 and $\frac{3}{2}$, respectively,

$$y_T(X_0) = \text{sech}^2 X_0, \quad (11)$$

$$y_w(X_0) = \text{sech}\,X_0 \tanh X_0. \quad (12)$$

The eigenfunction y_w gives the spatial profile of the so-called internal mode, also known as the wobbling mode in the current context. The internal mode frequency $\omega_0 = \sqrt{3}$. The functions $y_p(X_0)$ are solutions pertaining to the continuous spectrum of $\mathcal{L}$,

$$\mathcal{L}y_p = \left(2 + \frac{p^2}{2}\right)y_p, \quad -\infty < p < \infty. \quad (13)$$

These were constructed by Segur [19],

$$y_p(X_0) = e^{ipX_0}\left[1 + \frac{3(1-ip)}{1+p^2}\tanh X_0(1 + \tanh X_0) - \frac{3(2-ip)}{4+p^2}(1 + \tanh X_0)^2\right]. \quad (14a)$$

We also mention an equivalent form for these solutions,

$$y_p(X_0) = \frac{1}{(1+ip)(2+ip)}e^{ipX_0} \times (2 - p^2 - 3ip\tanh X_0 - 3\,\text{sech}^2 X_0). \quad (14b)$$

The continuous spectrum solutions are usually referred to as phonon modes; the phonon frequencies $\omega(p)$ are given by $\omega(p) = \sqrt{4 + p^2} > 0$. Finally, the coefficients $\mathcal{R}(p)$ and A are complex, C_0 is real, and c.c. in Eq. (9) stands for the complex conjugate of the immediately preceding term.

The y_T eigenfunction is the translation mode; since the kink is assumed to be centered at $X_0 = 0$, we let $C = 0$. Next, we will consider only localized perturbations of the kink, i.e., we assume that $\phi_1 \to 0$ as $|X_0| \to \infty$. This means that the Fourier coefficient $\mathcal{R}(p)$ in integral (10) can be regarded as an ordinary function, i.e., it does not include any δ-function contributions. Sending $T_0 \to \infty$ for the fixed finite X_0, Kelvin's formula of the method of stationary phase gives

$$\phi_R(X_0, T_0) \to \left(\frac{4\pi}{T_0}\right)^{1/2}\mathcal{R}(0)\,y_0(X_0)\,e^{2iT_0 + i\pi/4} + \text{c.c.}$$

Therefore, ϕ_R is a slowly decaying wave packet, which will be dominated by the zero-wavenumber radiation after other harmonics have dispersed away.

Since we are interested in the evolution of the wobbling mode and not that of a general localized initial condition, we set $\mathcal{R}(p) = 0$. Therefore, the first-order perturbation is taken in the form

$$\phi_1 = A(X_1, \dots; T_1, \dots)\,\text{sech}\,X_0 \tanh X_0\,e^{i\omega_0 T_0} + \text{c.c.} \quad (15)$$

The amplitude of the wobbling mode A is constant with respect to X_0 and T_0 but may depend on slower times and longer distances.

IV. QUADRATIC CORRECTIONS

At the second order in the perturbation expansion, we arrive at a nonhomogeneous variable-coefficient Klein-Gordon equation

$$\frac{1}{2}D_0^2\phi_2 + \mathcal{L}\phi_2 = F_2(X_0, \dots; T_0, \dots), \quad (16)$$

where the forcing term is

$$F_2 = (\partial_0\partial_1 - D_0 D_1)\phi_1 - 3\phi_0\phi_1^2 + V D_0\partial_0\phi_1 + \frac{1}{2}D_1 V\partial_0\phi_0 - \frac{1}{2}V^2\partial_0^2\phi_0. \quad (17a)$$

Substituting for ϕ_0 and ϕ_1, this becomes

$$F_2 = -6|A|^2\text{sech}^2 X_0 \tanh^3 X_0 + \frac{1}{2}D_1 V\,\text{sech}^2 X_0$$
$$+ V^2\,\text{sech}^2 X_0 \tanh X_0 + [\partial_1 A(2\,\text{sech}^3 X_0$$
$$- \text{sech}\,X_0) - i\omega_0 D_1 A\,\text{sech}\,X_0 \tanh X_0$$
$$+ i\omega_0 V A(2\,\text{sech}^3 X_0 - \text{sech}\,X_0)]\,e^{i\omega_0 T_0} + \text{c.c.}$$
$$- 3A^2\,\text{sech}^2 X_0 \tanh^3 X_0\,e^{2i\omega_0 T_0} + \text{c.c.} \quad (17b)$$

The T_0-independent term in Eq. (17b) and the term proportional to $e^{i\omega_0 T_0}$ are resonant with the two discrete eigenmodes of the operator in the left-hand side of Eq. (16) while the term proportional to $e^{2i\omega_0 T_0}$ is resonant with its continuous spectrum. The latter part of the forcing is localized in the region near the origin and acts as a source of radiation, which spreads outward from there.

We discard the homogeneous solution of Eq. (16) for the same reason as we have discarded most terms in the solution of Eq. (7); namely, we do not want the evolution of the wobbling mode to be shaded by dispersive transients. Hence the solution that is of interest to us will consist only of the harmonics present in Eqs. (17),

$$\phi_2 = \varphi_2^{(0)} + \varphi_2^{(1)}e^{i\omega_0 T_0} + \text{c.c.} + \varphi_2^{(2)}e^{2i\omega_0 T_0} + \text{c.c.}, \quad (18)$$

where $\varphi_2^{(0)}$, $\varphi_2^{(1)}$, and $\varphi_2^{(2)}$ are functions of X_0 that satisfy the three linear nonhomogeneous equations,

$$\mathcal{L}\varphi_2^{(0)} = -6|A|^2 \text{sech}^2 X_0 \tanh^3 X_0$$
$$+ \frac{1}{2} D_1 V \text{sech}^2 X_0 + V^2 \text{sech}^2 X_0 \tanh X_0, \quad (19)$$

$$\left(\mathcal{L} - \frac{3}{2}\right)\varphi_2^{(1)} = \partial_1 A(2 \text{ sech}^3 X_0 - \text{sech } X_0)$$
$$- i\omega_0 D_1 A \text{ sech } X_0 \tanh X_0$$
$$+ i\omega_0 VA(2 \text{ sech}^3 X_0 - \text{sech } X_0), \quad (20)$$

and

$$(\mathcal{L} - 6)\varphi_2^{(2)} = -3A^2 \text{ sech}^2 X_0 \tanh^3 X_0. \quad (21)$$

(The functions $\varphi_2^{(0)}$, $\varphi_2^{(1)}$, and $\varphi_2^{(2)}$ can also depend, parametrically, on $X_1, X_2, \dots$ and $T_1, T_2, \dots$.)

The homogeneous solutions of the first two of these equations are given by the eigenfunctions of the operator $\mathcal{L}$, Eqs. (11) and (12). According to the Fredholm alternative, the nonhomogeneous equations admit bounded solutions if and only if their right-hand sides are orthogonal to the corresponding homogeneous solutions. For this to be the case, we must set $D_1 V = 0$ and $D_1 A = 0$. The variation of parameters yields then

$$\varphi_2^{(0)} = 2|A|^2 \text{sech}^2 X_0 \tanh X_0 + \left(\frac{V^2}{2} - 3|A|^2\right) X_0 \text{ sech}^2 X_0, \quad (22)$$

and

$$\varphi_2^{(1)} = -(\partial_1 A + i\omega_0 VA) X_0 \text{ sech } X_0 \tanh X_0. \quad (23)$$

Although the function $\varphi_2^{(1)}$ decays to zero as $|X_0| \to \infty$, the product $\epsilon \varphi_2^{(1)}$ becomes greater than the first-order perturbation $y_1(X_0)$ for each fixed ϵ and sufficiently large $|X_0|$. Consequently, the term $\epsilon^2 \phi_2$ in expansion (4) becomes greater than the previous term, $\epsilon^1 \phi_1$, leading to nonuniformity of the expansion. In order to obtain a uniform expansion, we set this "quasisecular" term to zero,

$$\partial_1 A + i\omega_0 VA = 0, \quad (24)$$

whence

$$A = \tilde{A}(X_2, X_3, \dots; T_2, T_3, \dots)e^{-i\omega_0 VX_1}. \quad (25)$$

We also note the terms proportional to $X_0 \text{ sech}^2 X_0$ in Eq. (22). These terms do not grow bigger than the previous term, $\phi_0 = \tanh X_0$, yet they become larger than the difference $\phi_0 - 1$ as $X_0 \to \infty$ and $\phi_0 + 1$ as $X_0 \to -\infty$. If we attempted to construct the asymptotic expansion of the function $\phi - 1$ at the right infinity or the function $\phi + 1$ at the left infinity, the terms in question would cause nonuniformity of these expansions. Since the function $X_0 \text{ sech}^2 X_0$ is nothing but the derivative of $\tanh(kX_0)$ with respect to k, these terms represent the variation in the kink's width. Hence the potential nonuniformity of the expansion can be avoided simply by incorporating them in the variable width [see Eq. (48) below].

We now turn to the remaining nonhomogeneous equation [Eq. (21)]. The variation of parameters gives

$$\varphi_2^{(2)} = A^2 f_1(X_0), \quad (26)$$

with

$$f_1(X_0) = \frac{1}{8}\{6 \tanh X_0 \text{ sech}^2 X_0$$
$$+ (2 + ik_0 \tanh X_0 + \text{sech}^2 X_0)[J_2^*(X_0) - J_2^\infty]e^{ik_0 X_0}$$
$$+ (2 - ik_0 \tanh X_0 + \text{sech}^2 X_0)J_2(X_0)e^{-ik_0 X_0}\}. \quad (27)$$

Here the function $J_2(X_0)$ is defined by the integral

$$J_2(X_0) = \int_{-\infty}^{X_0} e^{ik_0 \xi} \text{sech}^2 \xi \, d\xi, \quad (28)$$

with $k_0 = \sqrt{8}$. The constant J_2^∞ is the asymptotic value of $J_2(X_0)$ as $X_0 \to \infty$,

$$J_2^\infty = \lim_{X_0 \to \infty} J_2(X_0). \quad (29)$$

The two constants of integration were chosen such that solution (26)–(27) describes right-moving radiation for positive X_0 and left-moving radiation for negative X_0. It is not difficult to show that f_1 is an odd function; we will use this fact in what follows.

V. RADIATION IN THE FAR FIELD

Function (26) is bounded but does not decay to zero as $|X_0| \to \infty$. This fact presents a problem, both for the consistency of our method and for the physical interpretation of the resulting solution. Mathematically, the term $\epsilon^2 \phi_2$ turns out to be greater than the previous term in expansion (4) for sufficiently large $|X_0|$. As we have mentioned in connection with the term $\varphi_2^{(1)}$, this leads to nonuniformity of the expansion. Physically, the problem is that any variation in the amplitude of the wobbling mode A on the time scale T_2 will result in a simultaneous change in the amplitude of the radiation tail for all values of X_0 from the origin to the plus and minus infinities. This is obviously in contradiction with the finiteness of the velocity of signal propagation in a relativistic theory [which is bounded by 1 in the dimensionless units of Eq. (1)].

The problem stems from the fact that Eq. (16) and, therefore, Eq. (21), were obtained under the assumption that, in expansion (4), the second term is smaller than the first one, the third one is smaller than the second, and so on—more precisely, that $\epsilon \phi_1 / \phi_0 \to 0$, $\epsilon^2 \phi_2 / (\epsilon \phi_1) \to 0$, and so on, as $\epsilon \to 0$. This assumption turns out to be only valid on the short scale, and therefore, Eq. (21) is only meant to hold for distances $X_0 = O(1)$ but not $X_0 = O(\epsilon^{-1})$ or longer. The interval of X_0 where $\epsilon \phi_{n+1} / \phi_n \to 0$ as $\epsilon \to 0$ will be referred to as the "inner" region in what follows. Equations (16) and (21) are, therefore, valid in the inner region.

To obtain a uniform expansion on the whole axis, we also consider two "outer" regions—one with $X_0 > 0$ and the other one with $X_0 < 0$. We define the outer regions by the requirement that $|X_0|$ be greater than $\frac{1}{2}\ln \epsilon^{-1}$. Note that the outer

regions overlap with the inner region. For example, the values $X_0 = \pm \frac{2}{3} \ln \epsilon^{-1}$ are obviously in the outer regions; on the other hand, we have $\epsilon \phi_1 / \phi_0 \to 0$, $\epsilon^2 \phi_2 / (\epsilon \phi_1) \to 0$, etc. for these X_0 and so they belong to the inner region as well.

In the right outer region, we expand ϕ in the power series

$$\phi = 1 + \epsilon^2 \phi_2 + \epsilon^4 \phi_4 + \cdots, \tag{30a}$$

and in the left outer region, we let

$$\phi = -1 + \epsilon^2 \phi_2 + \epsilon^4 \phi_4 + \cdots. \tag{30b}$$

Substituting these, together with expansion (6), in Eq. (3), the order ϵ^2 gives

$$\frac{1}{2} D_0^2 \phi_2 + \mathcal{L} \phi_2 = 0,$$

where $\mathcal{L} = -\frac{1}{2} \partial_0^2 + 2$ is the far-field asymptotic form of operator (8). The solutions of this equation in the right and left outer regions are, respectively,

$$\phi_2 = \mathcal{J} B_+ \, e^{i(\omega_+ T_0 - k_+ X_0)} + \text{c.c.}, \tag{31a}$$

and

$$\phi_2 = -\mathcal{J} B_- \, e^{i(\omega_- T_0 - k_- X_0)} + \text{c.c.}, \tag{31b}$$

where $\omega_\pm^2 = k_\pm^2 + 4$, and the amplitudes $B_\pm$ are functions of the "slow" variables: $B_\pm = B_\pm(X_1, \ldots; T_1, \ldots)$. The normalization constant $\mathcal{J}$ will be chosen at a later stage, and the negative sign in front of B_- is also introduced for later convenience.

Equations (31) should be matched to the solution in the inner region [Eq. (18)] with coefficients as in Eqs. (22), (23), and (26). To this end, we take the values $X_0 = \pm \frac{2}{3} \ln \epsilon^{-1}$ (which, as we remember, belong to the overlap regions). For these X_0, we have $|X_1| = O(\epsilon \ln \epsilon^{-1})$, $|X_2| = O(\epsilon^2 \ln \epsilon^{-1})$,..., and so $X_1 \to 0$, $X_2 \to 0$,..., as $\epsilon \to 0$. Solutions (31) become, in this limit:

$$\phi_2 = \pm \mathcal{J} B_\pm(0, 0, \ldots; T_1, T_2, \ldots) e^{i(\omega_\pm T_0 - k_\pm X_0)} + \text{c.c.}$$

On the other hand, letting $|X_0| = \frac{2}{3} \ln \epsilon^{-1}$ and sending $\epsilon \to 0$ in Eqs. (22), (23), and (26), we get

$$\phi_2 = \pm (2 - ik_0) J_2^\infty A^2(0, 0, \ldots; T_2, T_3, \ldots) \, e^{i(2\omega_0 T_0 \mp k_0 X_0)} + \text{c.c.},$$

where the top and bottom signs pertain to the positive and negative X_0, respectively. Choosing $\mathcal{J} = (2 - ik_0) J_2^\infty$ and equating the above two expressions, we obtain $\omega_\pm = 2\omega_0$, $k_\pm = \pm k_0$, and

$$B_\pm(0, 0, \ldots; T_1, T_2, \ldots) = A^2(0, 0, \ldots; T_2, T_3, \ldots). \tag{32}$$

Equation (32) can be regarded as the boundary conditions for the amplitude fields B_+ and B_-. Equations governing the evolution of these functions of slow variables can be derived at higher orders of the (outer) perturbation expansion. Namely, the solvability condition at the order ϵ^3 yields

$$(\partial_0 \partial_1 - D_0 D_1 + V \partial_0 D_0) \phi_2 = 0. \tag{33}$$

Substituting from Eq. (31), this becomes

$$D_1 B_\pm + \frac{k_\pm}{2\omega_0} \partial_1 B_\pm + i k_\pm V B_\pm = 0, \tag{34}$$

whence

$$B_\pm = e^{-2i\omega_0 V X_1} \mathcal{B}_\pm(X_1, X_2, \ldots; T_1, T_2, \ldots), \tag{35}$$

where $\mathcal{B}_\pm$ satisfy a pair of linear transport equations

$$D_1 \mathcal{B}_+ + c_0 \partial_1 \mathcal{B}_+ = 0, \quad X_1 > 0, \tag{36a}$$

$$D_1 \mathcal{B}_- - c_0 \partial_1 \mathcal{B}_- = 0, \quad X_1 < 0, \tag{36b}$$

with $c_0 = k_0 / (2\omega_0)$. Note that c_0 is nothing but the group velocity of the radiation waves with the wavenumber k_0: $c_0 = (d\omega/dk)|_{k=k_0}$, where $\omega = \sqrt{4 + k^2}$.

Solution of Eq. (36) with boundary condition (32) is a textbook exercise. Assume that the functions $\mathcal{B}_\pm$ satisfy the initial conditions $\mathcal{B}_+(X_1, 0) = \mathcal{B}^{(0)}(X_1)$ (for $X_1 > 0$) and $\mathcal{B}_-(X_1, 0) = \mathcal{B}^{(0)}(X_1)$ (for $X_1 < 0$), with some function $\mathcal{B}^{(0)}(X_1)$ defined on the whole axis $-\infty < X_1 < \infty$, with $\mathcal{B}^{(0)}(X_1) \to 0$ as $|X_1| \to \infty$. (We have suppressed the dependence on the variables $X_2, X_3, \ldots; T_2, T_3, \ldots$ for notational convenience.) In the region $X_1 > c_0 T_1$, the solution to Eq. (36a) with the above initial condition is given by $\mathcal{B}_+(X_1, T_1) = \mathcal{B}^{(0)}(X_1 - c_0 T_1)$. This solution represents an envelope of a group of second-harmonic radiation waves, moving to the right with the velocity c_0. Importantly, the amplitude $\mathcal{B}_+$ in this region is not related to the wobbling amplitude A and so no information from the core of the kink can reach this region. In the region $0 < X_1 < c_0 T_1$, the solution to Eq. (36a) is determined by the boundary condition instead: $\mathcal{B}_+(X_1, T_1) = A^2(0; 0)$. This result implies that the moving envelope has the form of a propagating front, leaving $\mathcal{B}_+$ flat and stationary in its wake. In a similar way, on the negative semiaxis we have a front moving with the velocity $-c_0$ and leaving $\mathcal{B}_-(X_1, T_1)$ equal to the constant $A^2(0; 0)$ in its wake.

The above analysis has two shortcomings. One drawback is that we have restricted ourselves to groups of radiation waves with the characteristic length and time scale of order ϵ^{-1}. A natural question, therefore, is whether variations with larger space and time scales (e.g., variations on X_2 and T_2 scales) could not propagate faster than c_0. Another latent defect is that the solutions for $\mathcal{B}_\pm(X_1, T_1)$ that we have constructed will generally be discontinuous along the lines $X_1 = \pm c_0 T_1$. To address both of these issues, we proceed to the order ϵ^4 of the outer expansion where the solvability condition for the second harmonic gives

$$i(2\omega_0 D_2 + k_0 \partial_2) B_\pm + \frac{1}{2}(D_1^2 - \partial_1^2) B_\pm$$

$$+ iV(k_0 D_1 - 2\omega_0 \partial_1) B_\pm - \frac{1}{2} V^2 k_0^2 B_\pm = 0.$$

Eliminating $D_1 B_\pm$ using Eq. (34), this becomes

$$iD_2B_\pm \pm ic_0\partial_2 B_\pm - iV\partial_1 B_\pm - \frac{\omega_{kk}}{2}\partial_1^2 B_\pm = 0, \qquad (37)$$

where $\omega_{kk} \equiv (d^2\omega/dk^2)|_{k_0} = (4\omega_0^2 - k_0^2)/(8\omega_0^3)$ is the dispersion of the group velocity of the radiation waves. Combining Eq. (37) with Eq. (34), we obtain a pair of equations in the original space and time variables:

$$i\partial_t B_\pm \pm ic_0\,\partial_x B_\pm \mp vk_0\,B_\pm - \frac{\omega_{kk}}{2}\partial_x^2 B_\pm = 0. \qquad (38)$$

The pair of linear Schrödinger equations [Eq. (38)] govern the evolution of the radiation amplitudes over times and distances as large as ϵ^{-2}; if we want to have a description on even a larger scale, we simply need to include equations from higher orders of the outer expansion. Solutions of Eqs. (38) with the boundary conditions $B_\pm = A^2$ at $x=vt$ and $B_\pm = 0$ at $x = \pm\infty$ have the form of slowly dispersing fronts propagating at the velocities $\pm c_0$ and interpolating, *continuously*, between A^2 and 0. As in our previous description exploiting the transport equations [Eqs. (36)] and valid on a shorter space-time scale, perturbations of A^2 cannot travel faster than c_0, the group velocity of radiation.

Thus, by introducing the long-range variables $B_\pm$, "untied" from the short-range amplitude A, we have restored the finiteness of the velocity of the radiation wave propagation. By introducing the outer expansions, we have also prevented the breakdown of the asymptotic expansion at large distances.

VI. DECAY LAW FOR THE WOBBLING AMPLITUDE

Returning to the original, "inner," expansion (4) and collecting terms of order ϵ^3 gives the equation

$$\frac{1}{2}D_0^2\phi_3 + \mathcal{L}\phi_3 = F_3, \qquad (39a)$$

where

$$F_3 = (\partial_0\partial_1 - D_0D_1)\phi_2 + (\partial_0\partial_2 - D_0D_2)\phi_1$$

$$+ \frac{1}{2}(\partial_1^2 - D_1^2)\phi_1 - \phi_1^3 - 6\phi_0\phi_1\phi_2 + VD_0\partial_0\phi_2$$

$$+ VD_0\partial_1\phi_1 + VD_1\partial_0\phi_1 + \frac{1}{2}D_2V\partial_0\phi_0 - \frac{1}{2}V^2\partial_0^2\phi_1. \qquad (39b)$$

Having evaluated F_3 using the known functions ϕ_0, ϕ_1, and ϕ_2, we decompose the solution ϕ_3 into simple harmonics as we did at $O(\epsilon^2)$. The solvability condition for the zeroth harmonic in Eq. (39b) gives $D_2V=0$, which means that V remains constant up to times $t \sim \epsilon^{-3}$. The solvability condition for the first harmonic produces

$$i\frac{2\omega_0}{3}D_2A + \zeta\,|A|^2A - V^2A = 0, \qquad (40)$$

where

$$\zeta = 6\int_{-\infty}^{\infty} \mathrm{sech}^2\,X_0\,\tanh^3 X_0\left[\frac{5}{2}\mathrm{sech}^2\,X_0\,\tanh X_0\right.$$

$$\left. - 3X_0\,\mathrm{sech}^2\,X_0 + f_1(X_0)\right]dX_0. \qquad (41)$$

Out of the real and imaginary parts of ζ, the imaginary part is more important; it can be easily evaluated analytically,

$$\zeta_I = \frac{3\pi^2 k_0}{\sinh^2(\pi k_0/2)} = 0.046\,36. \qquad (42)$$

The real part was computed numerically,

$$\zeta_R = -0.8509. \qquad (43)$$

Denoting $\epsilon\tilde{A} \equiv a$ the "natural" (unscaled) amplitude of the wobbling mode, and recalling that $v = \epsilon V$ and $A_t = \epsilon^2 D_2 A + O(\epsilon^3)$, we express amplitude equation (40) in terms of the original variables,

$$ia_t = -\frac{\omega_0\zeta}{2}|a|^2a + \frac{\omega_0}{2}v^2a + O(|a|^5). \qquad (44)$$

Equation (44) contains solvability conditions at all orders covered so far—they arise simply by expanding the derivative d/dt as in Eq. (6). Unlike the amplitude equation $D_1A = 0$, which only governs the evolution for times $t \sim \epsilon^{-1}$, and unlike Eq. (40), which only holds on the time scale $t \sim \epsilon^{-2}$, the "master equation" (44) is applicable for all times, from $t=0$ to $t \sim \epsilon^{-2}$.

Master equation (44) is the final result of the asymptotic analysis. All the conclusions about the behavior of the wobbler's amplitude shall be made on the basis of this equation. We could extend the range of applicability of the master equation beyond times of order ϵ^{-2} by continuing our perturbation analysis to higher orders of ϵ. However, corrections to Eq. (44) obtained in this way would be smaller than the terms that are already in the right-hand side of Eq. (44) and would not affect our conclusions based on Eq. (44) in its present form.

The absolute value of a is governed by the equation

$$\frac{d}{dt}|a|^2 = -\omega_0\zeta_I|a|^4 + O(|a|^6). \qquad (45)$$

Previously this equation was obtained using heuristic considerations [21,23,24]. Since $\zeta_I > 0$, the amplitude of the wobbling is monotonically decreasing with time: a constant emission of radiation damps the wobbler. Dropping the $O(|a|^6)$ correction term from Eq. (45), the decay law is straightforward:

$$|a(t)|^2 = \frac{|a(0)|^2}{1 + \omega_0\,\zeta_I|a(0)|^2t} = \frac{|a(0)|^2}{1 + 0.08030\,|a(0)|^2t}. \qquad (46)$$

When $a(0)$ is small, the decay becomes appreciable only after long times $t \sim |a(0)|^{-2}$. The decay is slow; for times $t \geq 12.5|a(0)|^{-2}$, Eq. (46) gives $|a| \sim t^{-1/2}$.
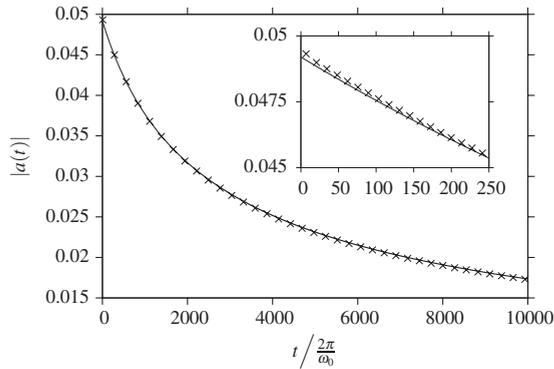
FIG. 1. The decay of the free wobbling kink. Crosses: $|a(t)|$ as obtained from the direct numerical simulations of Eq. (1) with the initial conditions in the form $\phi=\tanh x + 2a_0 \, \mathrm{sech}\, x \tanh x$, $\phi_t=0$, with $a_0=0.05$. Solid curve: Eq. (46) with $|a(0)|=0.0492$. The inset shows the first 250 periods of oscillation; during this time the amplitude drops by less than 10%.

We have verified the above decay law in direct numerical simulations of the full partial differential equation (1). (The details of our numerical algorithm have been relegated to the Appendix.) As the initial conditions, we took $\phi(x,0)=\tanh x + 2a_0 \, \mathrm{sech}\, x \tanh x$ with some real a_0 and $\phi_t(x,0)=0$. After a short initial transient, the solution was seen to settle to curve (46) with $|a(0)|$ close to a_0, see Fig. 1.

Equation (44) gives us the leading-order contributions to the frequency of the wobbling:

$$\omega = \omega_0 \left[1 - \frac{1}{2}v^2 + \frac{1}{2}\zeta_R |a|^2 + O(|a|^4) \right], \qquad (47)$$

with $\zeta_R < 0$ as in Eq. (43). (Note that ω is the frequency of oscillation of the "full" field ϕ, not just of the amplitude a.) The $|a|^2$ term here is a nonlinear frequency shift from the linear frequency $\omega_0 = \sqrt{3}$; as time advances, this term decays, slowly, to zero. The v^2 term comes from the transverse Doppler effect. We could have obtained this term simply by calculating the wobbling frequency in the rest frame and then multiplying the result by the relativistic time-dilation factor $\sqrt{1-v^2}$ (which becomes $1-\frac{1}{2}v^2$ for small v).

VII. CONCLUDING REMARKS

In this paper, we have formulated a singular perturbation expansion for the wobbling kink of the ϕ^4 model. Unlike the previously published singular perturbation theories based on the Krylov-Bogoliubov and Lindstedt methods, our approach exploits the existence of multiple space and time scales in the kink+radiation system. Some aspects of our scheme are standard to the method of multiple scales; some other ones (e.g., the appearance of the quasisecular terms) are less traditional. We particularly emphasize our treatment of the long-range radiation and the infinite propagation speed paradox. The final result of the asymptotic analysis is the amplitude equation for the wobbling mode [Eq. (44)]. Using this equation, we evaluate the nonlinear frequency shift and decay rate of the wobbler.

The coupling of a spatially localized temporally periodic excitation to radiation modes via a nonlinearity was discussed previously in several contexts. In particular, Ref. [30] described the decay of the internal mode of the nonlinear Schrödinger soliton, in the equation with a general nonlinearity. (For rigorous estimates, see, e.g., [31].) In Ref. [32], the dynamics of the soliton's internal mode was considered in the nonlinear Schrödinger equation with the parametric forcing and damping. Next, the authors of Ref. [33] studied the persistence of a localized linear impurity mode in the cubic Klein-Gordon equation. [We note that although our Eq. (1) can also be cast in the form of an equation with an impurity potential—by letting $\phi=\phi_0+\chi$—the resulting Klein-Gordon equation satisfied by χ does not fall into the class of systems covered by the analysis in that paper.] We also mention an earlier paper [34] where a similar problem was considered for the nonlinear wave equation.

We conclude our study by producing the perturbation expansion of the wobbling kink in terms of the original variables:

$$\phi(x,t) = \tanh\left(\frac{1-3|a|^2}{\sqrt{1-v^2}} \xi \right) + a \, \mathrm{sech}\, \xi \tanh \xi \, e^{i\omega_0(t-v\xi)} + \mathrm{c.c.}$$
$$+ 2|a|^2 \mathrm{sech}^2 \xi \tanh \xi + a^2 f_1(\xi) e^{2i\omega_0(t-v\xi)} + \mathrm{c.c.}$$
$$+ O(|a|^3). \qquad (48)$$

Here $\xi=x-vt$; the complex function $a(t)$ satisfies ordinary differential equation (44), and $f_1(\xi)$ is given by Eq. (27). Note that we have incorporated two $X_0 \, \mathrm{sech}^2 X_0$ terms of the sum (22) into the variable width of the kink. Expansions (48) is only valid at the length scale $|\xi|=O(1)$; for larger distances one has to use outer expansions (30) with coefficients determined in Sec. V.

The first term in Eq. (48) describes a moving nonoscillatory kink with the width decreasing (to the value of $\sqrt{1-v^2}$) on the time scale $t\sim|a|^{-2}$. The second term describes the wobbling mode; the third gives the quasistationary correction to the shape of the kink induced by the wobbling, and the last term accounts for the second-harmonic radiation from the wobbler.

The first term in Eq. (48) is manifestly Lorentz covariant. The other terms can also be cast in the relativistically covariant form if we replace ξ with $\xi/\sqrt{1-v^2}$ in $\mathrm{sech}\,\xi$ and $\tanh\xi$ (this is correct to the order of v^2), and write $ae^{i\omega_0(t-v\xi)}$ as

$$|a| \exp\left[i\omega_0 \left(1 + \frac{1}{2}\zeta_R |a|^2 \right) \frac{t-vx}{\sqrt{1-v^2}} \right].$$

Here we used Eq. (47) and neglected terms of order $|a|^4 x$ and $|a|^5 t$. (We remind the reader that v and $|a|$ are considered to be small quantities, of the same order of smallness.)

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APPENDIX A: NUMERICAL METHOD

In this Appendix we outline some relevant details of the numerical method that we used to verify predictions of our theoretical analysis.

Equation (1) was simulated using an explicit finite-difference scheme on a grid of step size $\Delta x = 0.1$ and $\Delta t = 0.05$. The simulations were performed on the interval $-L < x < L$, where L was chosen large enough to prevent the kink exiting the domain of integration. (Typical values of L were of the order of 1000.) We imposed the free-end boundary conditions.

In order to prevent the radiation reflecting back from the boundaries of the system, damping was introduced near the edges to absorb the radiation. That is, we added to the ϕ^4 equation an absorbing term $\tilde{\gamma}(x)\phi_t$, with

$$\tilde{\gamma}(x) = \begin{cases} \left[\dfrac{x-(L-100)}{100}\right]^4 & \text{for} \quad x \geq L-100, \\[4mm] \left[\dfrac{x+(L-100)}{100}\right]^4 & \text{for} \quad x \leq -L+100, \\[4mm] \quad\quad 0 & \text{otherwise}. \end{cases}$$

The position $x_0(t)$ of the wobbling kink was determined from the location of the zero crossing. The amplitude of the wobbling mode was measured by taking the profile $\phi(x,t)$, subtracting the reference kink $\tanh[x-x_0(t)]$, and assuming the odd component of what remains to be the first-harmonic wobbling mode, $a\,\text{sech}\,X_0 \tanh X_0 e^{i\omega_0\tau}+\text{c.c.}$. This technique, of course, furnishes only a first-order approximation to the amplitude because of the higher order terms in the perturbation expansion. Interpolation and smoothing were applied to counter the effects of the discreteness of the x values and the various oscillations occurring on the fast time scale.

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