

Conservative model for synchronization problems in complex networksC. E. La Rocca,¹ L. A. Braunstein,^{1,2} and P. A. Macri¹¹*Instituto de Investigaciones Físicas de Mar del Plata (IFIMAR)-Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata-CONICET, Funes 3350, (7600) Mar del Plata, Argentina*²*Center for Polymer Studies, Boston University, Boston, Massachusetts 02215, USA*

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In this paper we study the scaling behavior of the interface fluctuations (roughness) for a discrete model with conservative noise on complex networks. Conservative noise is a noise which has no external flux of deposition on the surface and the whole process is due to the diffusion. It was found that in Euclidean lattices the roughness of the steady state W_s does not depend on the system size. Here, we find that for scale-free networks of N nodes, characterized by a degree distribution $P(k) \sim k^{-\lambda}$, W_s is independent of N for any λ . This behavior is very different than the one found by Pastore y Piontti *et al.* [Phys. Rev. E **76**, 046117 (2007)] for a discrete model with nonconservative noise, which implies an external flux, where $W_s \sim \ln N$ for $\lambda < 3$, and was explained by nonlinear terms in the analytical evolution equation for the interface [La Rocca *et al.*, Phys. Rev. E **77**, 046120 (2008)]. In this work we show that in these processes with conservative noise the nonlinear terms are not relevant to describe the scaling behavior of W_s .

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I. INTRODUCTION

It is known that many physical and dynamical processes employ complex networks as the underlying substrate. For this reason many studies on complex networks are focused not only in their topology but also in the dynamic processes that run over them. Some examples of these kinds of dynamical processes on complex networks are cascading failures [1], traffic flow [2,3], epidemic spreadings [4], and synchronization [5,6]. In particular, synchronization problems are very important in the dynamics and fluctuations in task completion landscapes in causally constrained queuing networks [7], in supply-chain networks based on electronic transactions [8], brain networks [9], and networks of coupled populations in the synchronization of epidemic outbreaks [10]. For example, in the problem of the load balance on parallel processors the load is distributed between the processors. If the system is not synchronized, few processors have low load and they will have to wait for the most loaded processors to finish the task. The nodes (processors) of the system have to synchronize with their neighbors to ensure causality on the dynamics. The computational time will be given by the most loaded processors, thus synchronizing the system is equivalent to reduce or optimize the computational time. Synchronization problems deal with the optimization of the fluctuations of some scalar field h (load in processing) in the system that will be optimally synchronized minimizing those fluctuations. To analyze synchronization problems is customary to study the height fluctuations of a nonequilibrium surface growth. If the scalar field on the nodes represents the interface height at each node, its fluctuations are characterized by the average roughness $W(t)$ of the interface at time t , given by $W \equiv W(t) = \{1/N \sum_{i=1}^N (h_i - \langle h \rangle)^2\}^{1/2}$, where $h_i \equiv h_i(t)$ is the height of node i at time t , $\langle h \rangle$ is the mean value on the network, N is the system size, and $\{\cdot\}$ denotes an average over configurations. Pastore y Piontti *et al.* [11] studied this mapping in scale-free (SF) networks [12] of broadness λ and size N using a surface relaxation growth

model [13] with nonconservative noise and found that for $\lambda < 3$ the saturation roughness W_s scales as $W_s \sim \ln N$. Later, the evolution equation for the interface in this model was derived analytically [14] for any complex networks. The derived evolution equation has nonlinear terms as a consequence of the heterogeneity of the network that together with the nonconservative noise are necessary to explain the $W_s \sim \ln N$ behavior for $\lambda < 3$. However, there exist many physical processes where the noise is conservative and cannot be modeled as a flux deposition on a surface. In models without external flux, where particles are moved by diffusion, the total volume of the system remains unchanged. Examples of this kind of process are thermal fluctuations, diffusion by an external agent such as an electric field, and load balance of parallel processors where the total load in the system is constant over a certain time interval. For the last example, the only flux is due to diffusion of the load from a processor to another. Though not extensively, conservative noise has already been studied in Euclidean lattices [15,16] and it was found that W_s does not depend on the system size L . The evolution equation of this process in Euclidean lattices can be well represented by an Edwards-Wilkinson process [17] with conservative noise. In this paper we study this model in SF networks by simulations of the discrete model (Sec. II) and derive analytically its evolution equation for any complex networks (Sec. III). Those networks represent better the heterogeneity in the contacts in real systems, such as the internet, the world wide web, the networks of routers, etc. We applied the mean-field (MF) approximation to the evolution equation and show that the scaling behavior of W_s with N (Sec. IV) is only due to finite-size effects. To our knowledge this class of model was never studied before in complex networks.

II. SIMULATIONS OF THE DISCRETE MODEL

In this model, at each time step a node i is chosen with probability $1/N$. If we denote by v_i the nearest-neighbor

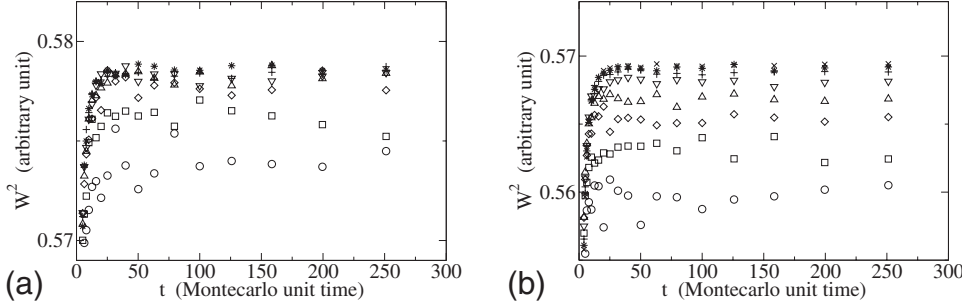


FIG. 1. W^2 as a function of t for the discrete model for (a) $\lambda = 3.5$ for $N=64$ (\circ), 128 (\square), 256 (\diamond), 512 (\triangle), 1024 (∇), 2048 ($+$), 3072 (\star), and 4096 (\times) and (b) $\lambda=2.5$ for $N=64$ (\circ), 128 (\square), 256 (\diamond), 512 (\triangle), 768 (∇), 1024 ($+$), and 1280 (\star). Each curve was obtained with 10.000 realizations.

nodes of i , then (1) if $h_i < h_j$, $\forall j \in v_i \Rightarrow$ the scalar fields remains unchanged, else (2) if $h_j < h_n$, $\forall n \neq j \in v_i \Rightarrow h_j = h_j + 1$ and $h_i = h_i - 1$. In that way the total height of the interface is conserved and we have that the average height is constant. We measure the roughness W for SF networks, characterized by a power law tail in the degree distribution $P(k) \sim k^{-\lambda}$, where $k_{\max} \geq k \geq k_{\min}$ is the degree of a node, k_{\max} is the maximum degree, k_{\min} is the minimum degree, and λ measures the broadness of the distribution [12]. To build the SF network we use the Molloy-Reed (MR) [18] algorithm or configurational model.

In Fig. 1 we plot W^2 as a function of the time t for different system sizes and in Fig. 2 the steady state W_s^2 as a function of N for (a) $\lambda=3.5$ and (b) $\lambda=2.5$. We can see that W_s^2 increases with N but, as we will show later, this dependence in the system size is only due to finite-size effects introduced by the correlated nature (dissortative) of the MR algorithm [19]. For all the results we use $k_{\min}=2$ in order to ensure that the network is fully connected [20] and assume that the initial configuration of $\{h_i\}$ is randomly distributed in the interval $[-0.5, 0.5]$. Then, we have that $\langle h \rangle = 0$.

III. DERIVATION OF THE STOCHASTIC CONTINUUM EQUATION

Next we derive the analytical evolution equation for the scalar field h_i for every node i in the conservative model in random graphs. The procedure chosen here is based on a coarse-grained (CG) version of the discrete Langevin equations obtained from a Kramers-Moyal expansion of the master equation [21–23]. The discrete Langevin equation for the evolution of the height in any growth model is given by [22,23]

$$\frac{\partial h_i}{\partial t} = \frac{1}{\tau} K_i^1 + \eta_i, \quad (1)$$

where K_i^1 takes into account the deterministic growth rules that produce the evolution of the scalar field h_i on node i ,

$\tau = N\delta t$ is the mean time of attempts to change the scalar fields of the interface, and η_i is a noise with zero mean and covariance given by [22,23]

$$\{\eta_i(t) \eta_j(t')\} = \frac{1}{\tau} K_{ij}^2 \delta(t - t'). \quad (2)$$

More explicitly, K_i^1 and K_{ij}^2 are the two first moments of the transition rate and they are given by

$$K_i^1 = \sum_{j=1}^N A_{ij} [P_{ij} - P_{ji}], \quad (3)$$

$$K_{ij}^2 = \frac{1}{\tau} K_i^1 \delta_{ij} - \frac{1}{\tau} \sum_{n=1}^N A_{in} (P_{in} + P_{ni}) (\delta_{nj} - \delta_{ij}), \quad (4)$$

where $\{A_{ij}\}$ is the adjacency matrix ($A_{ij}=1$ if i and j are connected and zero otherwise) and P_{ij} is the rule that represents the growth contribution to node i by relaxation from its neighbor j . In our model the network is undirected, then $A_{ij}=A_{ji}$. As the rules for this model are very complex if we allow degenerate scalar fields, we simplify the problem taking random initial conditions (see discrete rules in Sec. II). Thus,

$$P_{ij} = \Theta(h_j - h_i) \prod_{n \in v_j} [1 - \Theta(h_i - h_n)],$$

where Θ is the Heaviside function given by $\Theta(x)=1$ if $x \geq 0$ and zero otherwise, with $x=h_i - h_s \equiv \Delta h$. Without loss of generality, we take $\tau=1$.

In the CG version $\Delta h \rightarrow 0$; thus after expanding an analytical representation of $\Theta(x)$ in Taylor series around $x=0$ to first order in x , we obtain

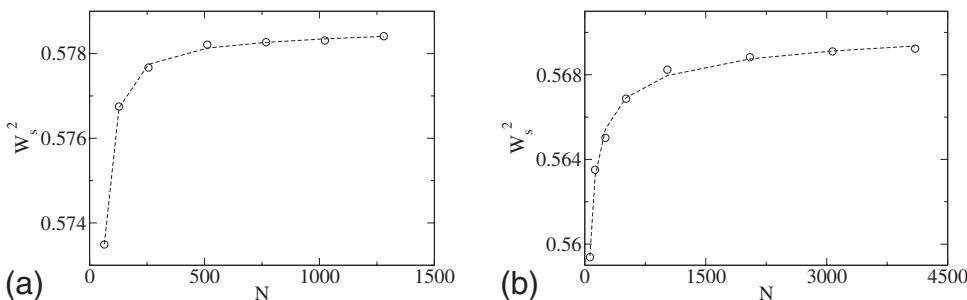


FIG. 2. W_s^2 as a function of N for (a) $\lambda=3.5$ and (b) $\lambda=2.5$ in symbols for the same system sizes of Fig. 1. The dashed lines represent the fitting with Eq. (12), obtained in the MF approximation by considering the finite-size effects introduced by the MR construction.

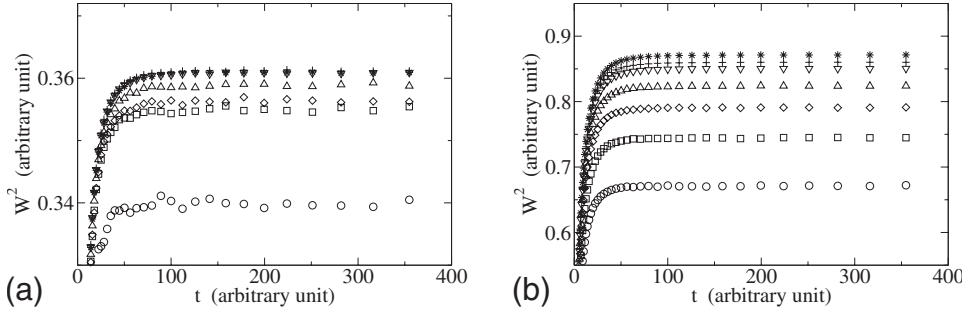


FIG. 3. W^2 as a function of t from the integration of the evolution equation: (a) $\lambda=3.5$ for $N=64$ (\circ), 128 (\square), 256 (\diamond), 512 (\triangle), 1024 (∇), 1536 ($+$), and 2048 (\star) and (b) $\lambda=2.5$ for $N=64$ (\circ), 128 (\square), 256 (\diamond), 512 (\triangle), 768 (∇), 1024 ($+$), and 1280 (\star). For all the integrations we use $U=0.5$ and typically 1.000 realizations of networks.

$$\begin{aligned}
 K_i^1 &= c_0 \sum_{j=1}^N A_{ij} [\Omega(k_j) - \Omega(k_i)] + c_1 \sum_{j=1}^N A_{ij} [\Omega(k_j) + \Omega(k_i)] \\
 &\times (h_j - h_i) + \frac{c_1 c_0}{(1 - c_0)} \sum_{j=1}^N A_{ij} \Omega(k_j) \left[\sum_{n=1, n \neq i}^N A_{jn} (h_n - h_i) \right] \\
 &+ O[(\Delta h)^2], \quad (5)
 \end{aligned}$$

where c_0 and c_1 are the first two coefficients of the expansion of $\Theta(x)$ and $\Omega(k_i) = (1 - c_0)^{k_i - 1}$ is the weight on the link ij introduced by the dynamic process.

Notice that the network is undirected and the noise is conservative, thus the average noise correlation [see Eq. (2)] is $\langle \eta_i(t) \eta_j(t') \rangle = 0$, where $\langle \rangle$ represents average over all the nodes of the network. Notice that in Eq. (5) the nonlinear terms are disregarded. As we will show below, for this conservative noise model these terms are not necessary to explain the scaling behavior of W_s with N .

We numerically integrate our evolution equation [Eq. (1)] in SF networks using the Euler method with a representation of the Heaviside function given by $\Theta(x) = (1 + \tanh[U(x + z)])/2$, where U is the width and $z = 1/2$ [23]. With this representation, $c_0 = (1 + \tanh[U/2])/2$ and $c_1 = (1 - \tanh^2[U/2])U/2$. We assume that the initial configuration of $\{h_i\}$ is randomly distributed in the interval $[-0.5, 0.5]$ and for the conservative noise we used the algorithm described in [24]: at each time step, for every node in the network and for any of its nearest neighbor we add a random number in the interval $[-0.5, 0.5]$ and remove this amount to one of the nearest-neighbor nodes.

In Fig. 3 we plot W^2 as a function of t from the integration of Eq. (1) for (a) $\lambda=3.5$ and (b) $\lambda=2.5$ and different values of N with $k_{\min}=2$. For the time step integration we chose $\Delta t \ll 1/k_{\max}$ according to Ref. [25]. In Fig. 4 we plot the steady state W_s^2 as a function of N for (a) $\lambda=3.5$ and (b) $\lambda=2.5$. We can see that W_s^2 depends weakly on N , but as

shown below this size dependence is due to finite-size effects introduced by the MR construction. Next we derive the mean-field approximation in order to explain the nature of the corrections to the scaling.

IV. MEAN-FIELD APPROXIMATION FOR THE EVOLUTION EQUATION

We apply a MF approximation to the linear terms of Eq. (5). In this approximation we consider $1 \ll k_{\min} \ll k_{\max}$. Taking $C_{ij} = A_{ij} \Omega(k_j)$ and $T_{ijn} = A_{ij} A_{jn} \Omega(k_j)$, then

$$\begin{aligned}
 K_i^1 &= c_0 [C_i - k_i \Omega(k_i)] + c_1 C_i \left[\sum_{j=1}^N \frac{C_{ij} h_j}{C_i} - h_i \right] \\
 &+ c_1 \Omega(k_i) k_i \left[\sum_{j=1}^N \frac{A_{ij} h_j}{k_i} - h_i \right] \\
 &+ \frac{c_1 c_0}{(1 - c_0)} T_i \left[\sum_{j=1}^N \sum_{n=1, n \neq i}^N \frac{T_{ijn} h_n}{T_i} - h_i \right], \quad (6)
 \end{aligned}$$

where

$$\begin{aligned}
 C_i &= \sum_{j=1}^N C_{ij}, \\
 T_i &= \sum_{j=1}^N \sum_{n=1, n \neq i}^N T_{ijn}. \quad (7)
 \end{aligned}$$

Disregarding the fluctuations, we take $\sum_{j=1}^N A_{ij} h_j / k_i \approx \langle h \rangle$, $\sum_{j=1}^N C_{ij} h_j / C_i \approx \langle h \rangle$, and $\sum_{j=1}^N \sum_{n=1, n \neq i}^N T_{ijn} h_n / T_i \approx \langle h \rangle$. From Eq. (7), we can approximate C_i by $C_i(k_i) \approx k_i \int_{k_{\min}}^{k_{\max}} P(k|k_i) \Omega(k) dk$ [6], where $P(k|k_i)$ is the conditional probability that a node with degree k_i is connected to another with degree k . For uncorrelated networks $P(k|k_i) = kP(k)/\langle k \rangle$ [12], then $C_i(k_i)$

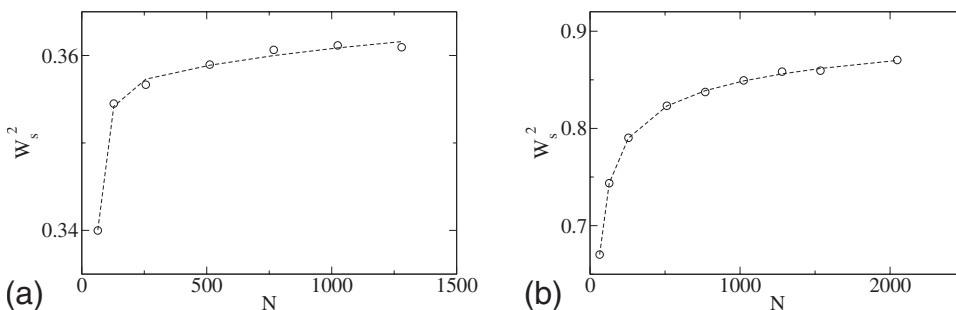


FIG. 4. W_s^2 as a function of N for (a) $\lambda=3.5$ and (b) $\lambda=2.5$ in symbols for the same system sizes of Fig. 3. The dashed lines represent the fitting with Eq. (12), obtained by considering the finite-size effects introduced by the MR construction.

$\approx I_1 k_i / \langle k \rangle$ with $I_1 = \int_{k_{\min}}^{k_{\max}} P(k) k \Omega(k) dk$. Making the same assumption for T_i , we obtain $T_i(k_i) \approx I_2 k_i / \langle k \rangle$ with $I_2 = \int_{k_{\min}}^{k_{\max}} P(k) k(k-1) \Omega(k) dk$. Then, the linearized evolution equation for the heights in the MF approximation can be written as

$$\frac{\partial h_i}{\partial t} = F_i(k_i) + \nu_i(k_i)(\langle h \rangle - h_i) + \eta_i, \quad (8)$$

where $F_i(k_i) = c_0 k_i [I_1 / \langle k \rangle - \Omega(k_i)]$ represents a local driving force and $\nu_i(k_i) = c_1 k_i [b + \Omega(k_i)]$ is a local superficial tension-like coefficient with $b = [I_1 + I_2 c_0 / (1 - c_0)] / \langle k \rangle$. This mean-field approximation reveals the network topology dependence through $P(k)$.

Taking the average over the network in Eq. (8), $\partial \langle h \rangle / \partial t = 1/N \sum_{i=1}^N F_i = 0$, then $\langle h \rangle$ is constant in time. The solution of Eq. (8) [21] is given by

$$\begin{aligned} h_i(t) &= \int_0^t e^{-\nu_i(t-s)} [F_i + \nu_i \langle h \rangle + \eta_i(s)] ds \\ &= \frac{F_i + \nu_i \langle h \rangle}{\nu_i} (1 - e^{-\nu_i t}) + \int_0^t e^{-\nu_i(t-s)} \eta_i(s) ds. \end{aligned} \quad (9)$$

Using Eq. (9) and the fact that in our model with the initial conditions we use $\langle h \rangle = 0$, we find the two-point correlation function,

$$\begin{aligned} \{h_i(t_1) h_j(t_2)\} &= \left(\frac{F_i}{\nu_i} \right) \left(\frac{F_j}{\nu_j} \right) (1 - e^{-\nu_i t_1}) (1 - e^{-\nu_j t_2}) \\ &\quad + \int_0^{t_2} \int_0^{t_1} e^{-\nu_i(t_1-s_1)} e^{-\nu_j(t_2-s_2)} \\ &\quad \times \{ \eta_i(s_1) \eta_j(s_2) \} ds_1 ds_2. \end{aligned}$$

For $t > \max\{1/\nu_i\}$, we can write W_s as

$$W_s^2 = \{ \langle h_i^2 \rangle \} = \frac{1}{N} \sum_{i=1}^N \left(\frac{F_i}{\nu_i} \right)^2 + \frac{1}{N} \sum_{i=1}^N \frac{K_{ii}^2}{\nu_i}, \quad (10)$$

where K_{ii}^2 [see Eq. (4)] is given by

$$K_{ii}^2 = \sum_{j=1}^N A_{ij} [P_{ij} + P_{ji}] \approx c_0 k_i \left[\frac{I_1}{\langle k \rangle} + \Omega(k_i) \right].$$

For SF networks it can be shown that $I_1, I_2 \sim \text{const} + k_{\max} \exp(-k_{\max} \text{const})$, where $k_{\max} \sim N^{1/(\lambda-1)}$ for MR networks; thus we can consider the quantities I_1 and I_2 as independent of N .

From Eq. (10) and using the expressions for F_i , ν_i , and K_{ii}^2 , we have

$$W_s^2 = \left(\frac{c_0}{c_1} \right)^2 B^2 \frac{1}{N} \sum_{i=1}^N [f_-(k_i)]^2 + \frac{c_0}{c_1} B \frac{1}{N} \sum_{i=1}^N f_+(k_i), \quad (11)$$

where

$$f_{\pm}(k_i) = \frac{1 \pm \frac{\langle k \rangle}{I_1} \Omega(k_i)}{1 + \frac{\langle k \rangle}{I_2} \frac{c_0}{1 - c_0} \Omega(k_i)}$$

and $B = 1/[1 + c_0 I_2 / (1 - c_0) I_1]$. Taking the continuum limit we find another expression for Eq. (11) as

$$W_s^2 = \left(\frac{c_0}{c_1} \right)^2 B^2 \int_{k_{\min}}^{k_{\max}} p(k) [f_-(k)]^2 dk + \frac{c_0}{c_1} B \int_{k_{\min}}^{k_{\max}} p(k) f_+(k) dk.$$

The function $f_{\pm}(k)$ has a crossover at $k = k^*$, where k^* is the crossover degree between the two different behaviors, then

- (1) for $k < k^* \Rightarrow f_{\pm}(k) \approx \pm \langle k \rangle \Omega(k) / I_1 / 2$ and
- (2) for $k > k^* \Rightarrow f_{\pm}(k) \approx 1$.

As k^* is the crossover between two different behaviors of $f_{\pm}(k)$ and the numerator of the function diverges faster than the denominator, we have $1 \approx \langle k \rangle \Omega(k^*) / I_1$, thus $k^* \approx \ln(I_1 / \langle k \rangle) / \ln(1 - c_0)$. Then,

$$\begin{aligned} W_s^2 &= \left[\left(\frac{c_0}{c_1} \right)^2 B^2 + \frac{c_0}{c_1} B \right] \int_{k^*}^{k_{\max}} p(k) dk \\ &\quad + \left(\frac{c_0 B}{2c_1 I_1} \right)^2 \langle k \rangle^2 \int_{k_{\min}}^{k^*} p(k) [\Omega(k)]^2 dk + \left(\frac{c_0 B}{2c_1 I_1} \right) \\ &\quad \times \langle k \rangle \int_{k_{\min}}^{k^*} p(k) \Omega(k) dk. \end{aligned}$$

Even though k^* depends on k_{\max} , it can be demonstrated that the two last integrals depend weakly on N and can be considered as constant. Then, introducing the corrections due to finite-size effects through k_{\max} in $\langle k \rangle$, we obtain

$$W_s^2 \sim W_s^2(\infty) \left(1 + \frac{A_1}{N} + \frac{A_2}{N^{(\lambda-2)/(\lambda-1)}} + \frac{A_3}{N^{2[(\lambda-2)/(\lambda-1)]}} \right), \quad (12)$$

where A_1 , A_2 , and A_3 do not depend on k_{\max} .

In Figs. 2 and 4 the dashed lines represent the fitting of the curves with Eq. (12) considering finite-size effects introduced by the MR construction. We can see that this equation represents very well the finite-size effects of this model. This means that even though the networks is heterogeneous, the nonlinear terms are not necessary to explain the N independence of W_s when a conservative noise is used. Notice that even when our network is correlated in the degree, the expression for W_s^2 found describes very well the scaling behavior with N as shown in the insets of Figs. 2 and 4. This model suggests a useful load balance algorithm suitable for processor synchronization in parallel computation. Our results show that the algorithm could be useful when one want to increase the number of processors and its general behavior is well represented by a simple mean-field equation.

V. SUMMARY

In summary, in this paper we study a conservative model in SF networks and find that the roughness of the steady state

is a constant and its dependence on N for any λ it is only due to finite-size effects. We derive analytically the evolution equation for the model and retain only linear terms because they are enough to explain the scaling behavior of W_s . Finally, we apply the mean-field approximation to the equation and we calculate explicitly the corrections to scaling of W_s . This approximation describes very well the behavior of the

model and shows clearly that the corrections are due to finite-size effects.

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