

Traffic-flow cellular automaton: Order parameter and its conjugated fieldA. M. C. Souza¹ and L. C. Q. Vilar²¹*Departamento Física, Universidade Federal de Sergipe, Sao Cristovao 49100-000, SE, Brazil*²*Instituto de Física, Universidade do Estado do Rio de Janeiro, Rua Sao Francisco Xavier 524, Rio de Janeiro 20550-013, RJ, Brazil*

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We use a cellular automaton traffic model in order to study a nonequilibrium phase transition. We define an order parameter and show that its conjugated field is a parameter of randomness of the model. We analyze the symmetries of the free (unbroken) and of the jammed (broken) phases. Our results are consistent with a second-order phase transition at $p=0$. Nontrivial critical exponents have also been obtained.

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I. INTRODUCTION

Nonequilibrium phase transitions are present in a high number of complex dynamical systems, as, for example, reaction-diffusion systems [1,2], sedimentation [3], mass aggregation [4–6], among others. The steady states of these systems may undergo nonequilibrium phase transitions [7] determined by underlying microscopic dynamical processes. The study of these transitions allows us to open a perspective on the characterization of such phenomena and how they may become distinct from those obtained by thermodynamical phase transitions.

A simple and useful approach for the study of nonequilibrium steady states and their transition mechanisms is the cellular automaton (CA) theory. The use of CA to simulate complex systems has grown in the last years [8–11]. Among the causes of its growth, we can highlight the adequacy of the simulations (discrete space time), simplicity, flexibility, and superiority to other methods for problems with complex geometry. The CA brought the possibility to improve our understanding of the phase-transitions dynamical processes: the attractors, critical order parameters, phase diagrams, exponents, and universality classes.

In the present paper, we study the CA introduced by Nagel and Schreckenberg [12] as a simulator of highway traffic flow [12–19]. Some years ago, we used this model to study real traffic conditions [14]. This approach allowed us to explain an old question on the density-flow relation obtained from measurements on roads under the stop-and-go conditions. Now, our aim is to explore theoretically this CA as a system in a nonequilibrium state under a possible phase transition, defining then its order parameter, conjugated field, and critical exponents.

The question of the transition of the Nagel-Schreckenberg CA from free flow to jammed traffic has been investigated by several authors [14–17]. However, there is still a great controversy whether this transition can be described as a critical phenomenon (see [18] and references therein).

We will improve the study of the second-order phase transition exhibited by the Nagel-Schreckenberg CA observed earlier [14,17]. We start by reinforcing the already known fact that one cannot obtain the transition to nonvanishing p . This has been considered a problem in describing the criticality of this phenomenon, and this fact is the root of the mentioned controversy [15,16]. However, up to now, the lit-

erature still has not identified p as the conjugated field of the order parameter, what we intend to show quite clearly. This will allow us to define an associated susceptibility. Then, we will be able to proceed with the analysis of the critical exponents. They will meet a standard scaling law, a nontrivial result in view of the nonequilibrium nature of this problem.

This paper is organized as follows. In Sec. II we introduce the CA approach. The criticality of the CA is presented in Sec. III. Finally, in Sec. V we conclude.

II. MODEL

The CA is constituted of an array of L sites, where each site S_i can either be empty or occupied by one vehicle with velocity $v=0, 1, \dots, v_{\max}$. The position and the velocity v_j of the j th vehicle are updated simultaneously (parallel update) according to the following:

- (1) $v_j=v_j+1$ if the distance of the j th car to the next car is greater than v_j+1 and taking $v_j=v_{\max}$ as a limit.
- (2) $v_j=D-1$ if the distance to the next car is $D\leq v_j$.
- (3) $v_j=v_j-1$ with a random probability p if $v_j\geq 1$.
- (4) The j th car is advanced v_j sites.

We consider a line with a periodic boundary condition (closed circuit) and with a random initial distribution of vehicle positions and null initial velocities. The rules ensure that the total number (N) of cars is conserved under the dynamics.

The investigation of the traffic flow CA is based on the analysis of the fundamental traffic parameters: density, defined as

$$\rho_i = \frac{1}{T} \sum_{t=t_0+1}^{t_0+T} n_i(t), \quad (1)$$

where $n_i(t)$ is zero if S_i is empty and one if it is occupied at time t , and flux, defined as

$$q_i = \frac{1}{T} \sum_{t=t_0+1}^{t_0+T} m_i(t), \quad (2)$$

where $m_i(t)$ is one if at time $t-1$ there was a car behind or at S_i and at t it is found after S_i (i.e., a car is detected passing by S_i) and zero otherwise. The average is taken over a time period T after a relaxation time t_0 . As the system is homogeneous due to the considered boundary conditions, if we take

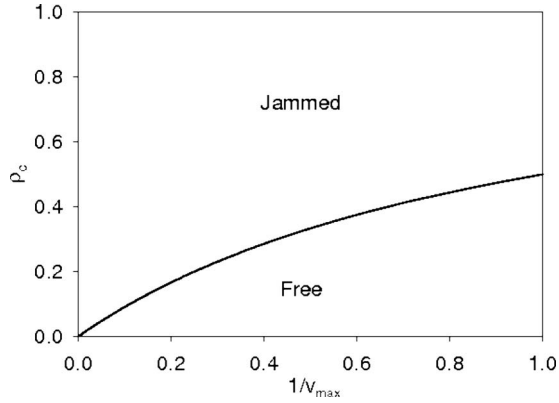


FIG. 1. Steady-state phase diagram of critical density vs $1/v_{\max}$ showing the free and jammed phases.

T sufficiently long, no parameter will be position dependent such that the density and the flux can be labeled, respectively, according to $\rho_i = \rho$ and $q_i = q$. In this stationary state, the results of this CA can be compared directly with measurements of real traffic [12]. At low density, this model exhibits a laminar flow (free) phase. At high density, one expects a start-stop waves (jammed) phase. In this sense, with increasing car density, the notion of a phase transition emerges [14].

III. CRITICALITY OF THE TRAFFIC CA

A. Order parameter and its conjugated field

To look for a possible transition between a free and a jammed phase, we define an order parameter

$$M = 1 - \frac{q}{v_{\max}\rho}. \quad (3)$$

This choice of parameter is made on the consideration that the free regime happens when all cars have a maximal velocity. If we take the analytical solution for the particular case of $p=0$ [14], we find

$$M = \begin{cases} 0 & \text{if } \rho \leq \rho_c \\ \frac{1}{v_{\max}} \frac{\rho - \rho_c}{\rho \rho_c} & \text{otherwise,} \end{cases} \quad (4)$$

where $\rho_c = 1/(1+v_{\max})$. There are two distinct regions. The first, where the parameter is zero, is associated with a free regime. After the transition point, the order parameter is not zero anymore, representing the region where the cars tend to get grouped in long clusters, associated with a jammed phase. Figure 1 presents the phase diagram. We observe that the bigger the maximal velocity gets, the smaller the region of the free phase becomes and, consequently, the region of the bottled phase enlarges.

The general analytical solution of this CA for an arbitrary p is still missing up to now. Only in the limit $v_{\max}=1$, this model is exactly solved [13]. The analytical expression of M for the particular case of $v_{\max}=1$ is given by

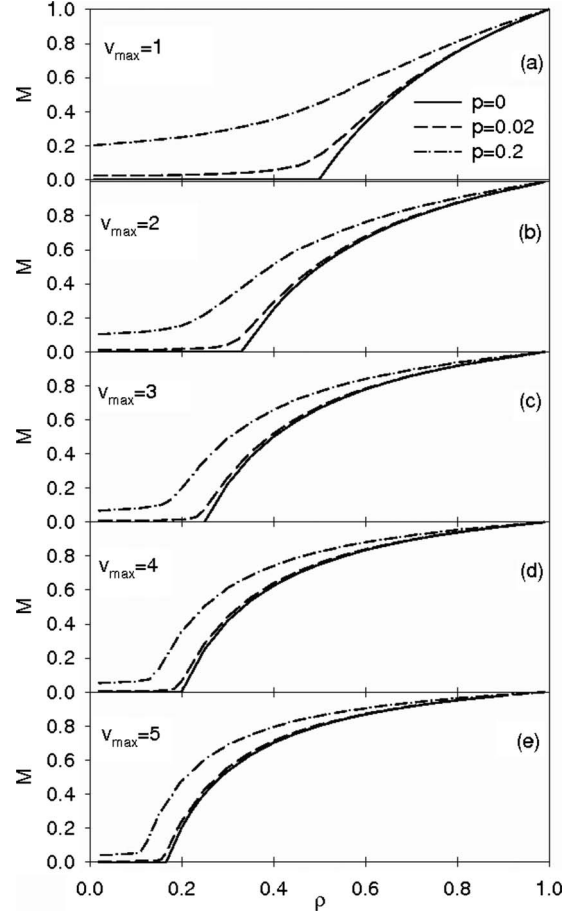


FIG. 2. Order parameter M as a function of the density for different p and (a) $v_{\max}=1$, (b) $v_{\max}=2$, (c) $v_{\max}=3$, (d) $v_{\max}=4$, and (e) $v_{\max}=5$. We perform 100 experiments with $L=10^4$, each running over a time period $T=10^5$ after a relaxation time $t_0=10L$.

$$M = \frac{2\rho - 1 + \sqrt{1 - 4\rho(1-\rho)(1-p)}}{2\rho}. \quad (5)$$

The cases with $v_{\max} > 1$ can be carried out through numerical simulations using the Monte Carlo method. We will present results obtained through simulations in which we have performed about 100 and 50 experiments, respectively, each running over a time period $T=10^5$ and $T=10^6$ with $L=10^4$ and $L=10^5$, after a relaxation time $t_0=10L$. The order parameter is shown in Fig. 2, where it is plotted as a function of the density ρ for $v_{\max}=1, 2, 3, 4$, and 5, for some typical values of p and $L=10^4$. In particular, for $v_{\max}=1$, we observe that the analytical result of Eq. (5) is reproduced numerically. For $p=0$, we have $M=0$ if $\rho \leq \rho_c$ and $M \neq 0$ if $\rho > \rho_c$. For $p \neq 0$, the M parameter is not zero anymore. In all cases, p destroys the continuous phase transition observed at $p=0$. This clear relationship among p , M , and the phase characterization suggests that the probability p is the *conjugated parameter* of the order parameter of this CA.

Furthermore, to gain one more quantitative picture of this phase transition we can define the associated susceptibility

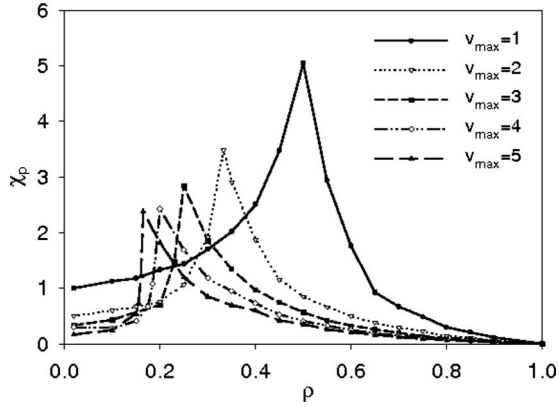


FIG. 3. Associated susceptibility versus density for $p=0.01$ and $v_{\max}=1, 2, 3, 4,$ and 5 .

$$\chi_p = \left. \frac{\partial M}{\partial p} \right|_{p=0}. \quad (6)$$

This quantity is relevant in order to evaluate a continuous transition, considering that it diverges where the phase transition occurs.

From Eq. (5), for $v_{\max}=1$, we find

$$\chi_p = \frac{(1-\rho)\rho_c}{|\rho-\rho_c|}, \quad (7)$$

confirming that when $\rho \rightarrow \rho_c = 1/2$ we have $\chi_p \rightarrow \infty$.

Figure 3 shows the numerical result for the associated susceptibility as a function of the density considering the cases $v_{\max}=1, 2, 3, 4,$ and 5 for $p=0.01$ and $L=10^4$. We can see that it has a maximum for $\rho=\rho_c$. The peak's values increase with decreasing p . Figure 4 shows the associated susceptibility versus the density considering the case $v_{\max}=4$, $L=10^4$ for $p=0.01$ and $p=0.1$. We see that the peak of χ_p , when $\rho \rightarrow \rho_c$, tends to diverge for $p \rightarrow 0$.

We also analyzed the time needed to get to the steady state called equilibrium time t_{eq} . For this study, we have taken, for $p=0$, the averages over the positions S_i , assuming $\zeta(t) = \frac{1}{L} \sum_{i=1}^L m_i(t)$ and $\mu(t) = 1 - \frac{\zeta(t)L}{Nv_{\max}}$. We consider t_{eq} as the

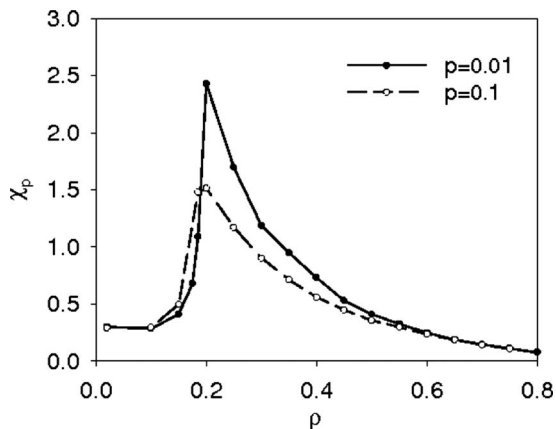


FIG. 4. Associated susceptibility versus density at $v_{\max}=4$ for $p=0.01$ and 0.1 .

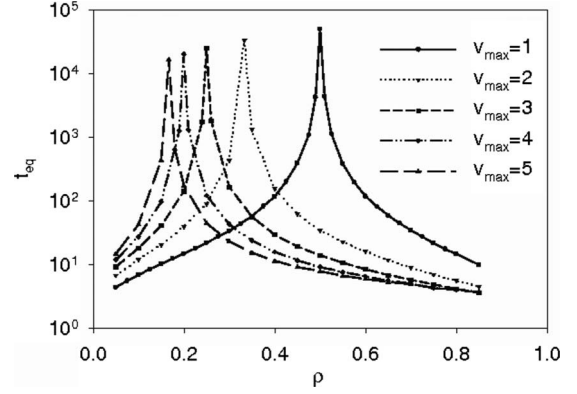


FIG. 5. Equilibrium time as a function of ρ for $L=10^5$ and different v_{\max} .

time t obtained when $|\mu(t)-M| < 0.000001$, where M is given by Eq. (4). Figure 5 presents t_{eq} as a function of ρ for typical v_{\max} values with $L=10^5$. We observe that close to the transition, the relaxation to the steady state becomes quite slow. For $\rho=\rho_c$, we obtain $t_{eq}=L/(1+v_{\max})=\rho_c L$, such that increasing the system size L delays the steady state. In the thermodynamic equilibrium limit, when $L \rightarrow \infty$, the equilibrium time t_{eq} will diverge at $\rho=\rho_c$.

All these results are consistent with a second-order phase transition and they are analogous to an equilibrium phase transition.

Probability has also been found to be a conjugated field of the spreading of damage in the Ising model [20] and in the Domany-Kinzel CA [21]. In Sec. IV, we will give a possible explanation for this fact based on the Nagel-Schreckenberg CA.

B. Critical exponents

We take the thermodynamical phase transition as a guide and we assume power laws for M and χ near the nonequilibrium transition. In the limit of zero “field” ($p=0$), we define

$$M \sim (\rho - \rho_c)^\beta, \quad \chi \sim (\rho - \rho_c)^{-\gamma}. \quad (8)$$

At the transition ($\rho=\rho_c$) as the field p goes to zero, we write

$$M \sim p^{1/\delta}. \quad (9)$$

The β , γ , and δ are called critical exponents. From Eq. (4), it follows that $\beta=1$ independent of v_{\max} . In particular, for $v_{\max}=1$ we can obtain exact expressions for δ and γ . In this situation, from Eqs. (5) and (7), we obtain that $\gamma=1$ and $\delta=2$. We also found exact values for δ and γ in the limit $v_{\max} \rightarrow \infty$, by noticing that, in this case, $\rho_c \rightarrow 0$. The exact relation between M and p , in the low-density limit, can be obtained for any v_{\max} and p [14], without further difficulties as $M=p/v_{\max}$. From Fig. 3 we can observe, in agreement with this analytical result, that $\chi_p(\rho=0)=1/v_{\max}$. We have $\gamma=0$ and $\delta=1$.

The critical exponents for $v_{\max} > 1$ were obtained through simulations using $L=10^5$. The results for M as a function of

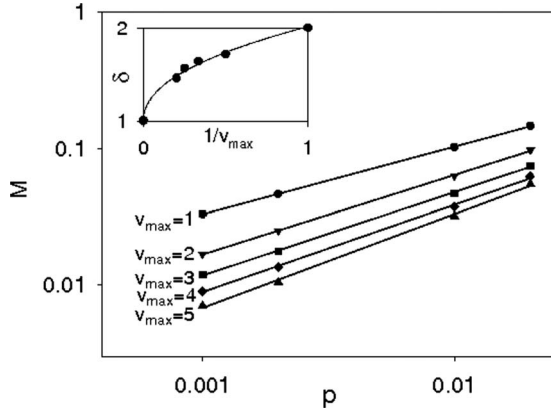


FIG. 6. Order parameter as a function of p for different v_{\max} and $\rho = \rho_c$. Lines are guides for the eyes. In the inset: exponent δ versus $1/v_{\max}$.

p at $\rho = \rho_c$ for different v_{\max} are displayed in Fig. 6 in logarithmic scale in order to find δ . In the inset of Fig. 6, the exponent δ is plotted vs $1/v_{\max}$, where we infer that it decreases smoothly with v_{\max} . We have used the function $\delta = a + b(1/v_{\max})^c$ to fit this relation. Then, we found approximately that $\delta - 1 = (1/v_{\max})^{0.46}$. We have also obtained the numerical estimation for the exponent γ . We summarize our results concerning the critical exponents in Table I. The final nontrivial observation is the agreement of the three exponents with Widom scaling law $\gamma = \beta(\delta - 1)$. Indeed, we expect this scaling over the complete range of the CA parameters.

IV. SYMMETRY-BREAKING PROCESS

Along this work, we have been describing the properties of a phase transition on the Nagel-Schreckenberg CA. Obviously, the existence of a parameter, such as M , that is null in a given range of the variables of the theory is a necessary—but not sufficient—condition for assuring a phase transition. It is the whole complexity of divergent quantities, as the susceptibility and the equilibrium time, together with a series of critical exponents, culminating with a Widom scaling law, all shown to be present in this CA that makes it rather plausible that this CA effectively suffers a phase transition in the vicinity of the critical density.

Anyway, one point is still missing. The notion of a phase transition is always associated to the noninvariance of the stable (vacuum) configuration in the ordered phase of the

physical system under transition in relation to certain symmetries explicitly present in the disordered phase, what is called a symmetry-breaking process. Without such description, the picture of the Nagel-Schreckenberg CA phase transition cannot be claimed to be complete. We will try now to convince the reader that this blank can be filled.

As we do not have a Hamiltonian representation for this system, we can only have an idea of which symmetries are being broken by carefully analyzing the steady-state configurations of the CA for $p = 0$, prior and after the phase transition. In what we have called the free phase, the distinguishing property of its steady states is that all cars move with the same maximum velocity v_{\max} allowed by the CA. In the jammed phase, there is at least one car moving with a velocity lower than v_{\max} . So, the symmetry that is being broken is certainly related to this loss of homogeneity among the car velocities.

In order to get a more profound understanding of this symmetry breaking, we need to properly characterize the steady-state configurations of the broken jammed phase. We will derive rules that are sufficient to determine a steady state in this phase. Then we begin by claiming that such states are fully described by an array with each velocity entering in the respective position occupied by the car at a given time step of the CA. As an example, in the case of N cars, we represent a steady state by $(v_1, v_2, v_3, \dots, v_N)$, where by v_1 we mean the velocity of an arbitrary car chosen as the first one and, subsequently, writing the velocities of the cars to its right at this step. If two representations describe the same steady state in different time steps, they are said to be equivalent. Obviously, by the periodic boundary conditions of the CA, any two arrays that can be obtained by cyclic permutations will be equivalent. The first law to construct a steady-state array of the broken phase comes from the defining property that if $\rho \geq \rho_c$, the number of vacant sites in front of a car is its velocity in each given step of the CA. As the total number of sites L equals the number of vacant sites plus the number of occupied sites, it follows immediately the first law (1) $L = \text{number of vacant sites} + \text{the number of occupied sites} = \sum_{i=1}^N v_i + N$, where, again, v_i means the velocity of the i th car, and N is the total number of cars. The second law comes from a specific information of the Nagel-Schreckenberg CA that a car can only accelerate by one unit at each time step, following the first rule of the CA (although it can break as necessary in order not to hit the car ahead, following the second rule). Then we establish that for a state in a steady flow, we have (2) $v_{i+1} \leq v_i + 1$. It is not difficult to understand the necessity of this law. As the velocity v_i of the i th car is the number of vacant sites ahead in a given step, in the next step the car ahead will leave v_{i+1} vacant sites behind, implying that v_i will have to assume the value v_{i+1} . Then, as this i th car can only accelerate by one unit in one time step, it follows the second law.

Now, with laws 1 and 2, together with the cyclic property determining an equivalent class, we are able to show that in the thermodynamic limit $L \rightarrow \infty$, steady states of the jammed phase are degenerate in the sense that given a density $1 > \rho > \rho_c$, we can always find at least two nonequivalent steady states. Let us choose as an example for this reasoning a CA with $L = 17$, $v_{\max} = 3$, and $N = 5$. Following the first law, the

TABLE I. Critical exponents.

v_{\max}	β	δ	γ
1	1	2	1
2	1	1.73	0.73
3	1	1.61	0.60
4	1	1.54	0.54
5	1	1.48	0.47
∞	1	1	0

sum of the velocities of the cars is $L-N=12$. Now, with the second law it is easy to find three nonequivalent steady states labeled by (1,2,3,3,3), (2,2,2,3,3), and (2,2,3,2,3). Other possibilities are easily seen as cyclic permutations of these elements and are then understood as belonging to one of the equivalent classes identified by them. It is now comprehensible that there will always be degenerate steady states in the thermodynamic limit for densities finitely above the critical density. Once different velocities are allowed for the different cars, nonequivalent classes will appear, and this is the symmetry-breaking process that we were looking for. Resuming this idea, the Nagel-Schreckenberg CA changes from a nondegenerate steady-state phase called the free phase, where all cars move with the same v_{\max} velocity, to a jammed phase which has degenerate steady-state configurations once the critical density is crossed.

From this point of view, if in the broken jammed phase the steady-state configurations still have a remaining invariance under a group of cyclic permutations of the car's velocities, the unbroken free phase steady-state configurations are in fact invariant under a larger group of arbitrary permutations of car's velocities. It is the choice of a specific steady-state (vacuum) configuration among all possible ones in the jammed phase which makes the symmetry breaking.

Naturally, as expected from a classical system, this choice is predictable from the initial conditions of the car's velocities and positions. Even this point can be understood more profoundly from a closer analogy with the usual example of a magnetic phase transition, also enlightening the role played by the Nagel-Schreckenberg CA conjugated parameter.

We remember that the p parameter, which makes any car suddenly break with probability p , is a factor of randomness in the velocities of the cars. Its presence obviously destroys the cyclic symmetries both at the jammed and at the free phases. This is why we have mathematically seen it as a conjugated parameter in the previous section. So if we di-

minish p until we make it vanish in a jammed phase, the system will choose a steady-state configuration in a quite unpredictable way, and it will undergo a symmetry-breaking phase transition at this moment.

V. CONCLUSIONS

We show in this work that the random probability p introduced in one of the rules of update of the Nagel and Schreckenberg CA can be considered as the field conjugated to the order parameter that characterizes the free and jammed phases of the model. This means that for this CA the probability plays the role analogous to that of an external field in ferromagnetism, which can destroy the ferromagnet phase transition.

We should also call the reader's attention to the nontriviality of the critical exponents listed in Table I. They are intrinsically different from those derived from mean-field theory, as they do not belong to the same universality class. A better understanding of this point would allow an insight on the intrinsic nature of the system, as its possible symmetries, degrees of freedom, or even the nature of the interaction itself.

Finally, in the last section, through the observation of the structure of the steady-states configurations for $p=0$, we were able to identify the symmetries of the free (unbroken) and of the jammed (broken) phases. Then, the degeneracy of the configurations in the jammed phase, together with the understanding of the mechanism triggered by p , completed the phase-transition picture.

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