Saffman-Taylor instability for generalized Newtonian fluids

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We study theoretically the linear Saffman-Taylor instability for non-Newtonian fluids in a Hele-Shaw cell. After introducing the notion of generalized Newtonian fluid we calculate the associated Darcy's law. We derive the relation governing the growth rate of normal modes for a large class of non-Newtonian flows. For shear-thinning fluids at high shear rate our theory provides Darcy's laws free of the nonphysical divergences appearing in the classical approaches. We characterize fluids which develop instabilities faster than Newtonian fluids under the same hydrodynamical conditions. Another primary result that this paper provides is that for some shear-thickening fluids, all normal modes are stable.

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I. INTRODUCTION

The study of the formation and evolution of dynamical structures is one of the most exciting and active areas of nonlinear sciences. Pattern formation is one aspect of these dynamical structures and it is present in many areas of physics. Our attention here will be focused on hydrodynamics and more specifically in the evolution of interfaces. One of the best studied cases is that of patterns formed by the interface between two fluids of different viscosities. An interfacial instability arises when the more viscous fluid is displaced by the less viscous one in a very narrow gap between two parallel plates. This geometry is called a Hele-Shaw cell and the phenomenon is nowadays known as the Saffman-Taylor instability [1–4]. This instability has been extensively studied both theoretically and experimentally. Most of the studies refer to the case of Newtonian fluids and pattern formation for these flows are at present more quite well understood.

The Saffman-Taylor instability may also arise when one or both fluids are complex such as liquid crystals [5,6], polymer solutions and melts [7], clays [8], and foams. The pattern dynamics of these flows is at present poorly understood. Numerous experiments have been performed [9–13], as this problem is of considerable technological and economical importance (oil recovery, injection molding, and device display design). Analytical and/or numerical works have successfully related rheological properties of fluids (such as shear thinning or shear thickening) to pattern formation [14–17]. However a general theoretical framework is still lacking.

The aim of the present paper is to derive a more accurate theory than the previous ones [14,16,17]. This paper is organized as follows. In Sec. II we recall the notion of generalized Newtonian fluid (GNF), whereupon our formalism relies. Next in Sec. III A we introduce a generalized Darcy law and in Sec. III B we perform the linear stability analysis. The analysis leads to a very general relation wave-number/growth ratio. This allows us to predict how (linear) stability or instability is affected by the rheological properties for a large class of GNF. In Secs. IV A–IV C we investigate the particular cases of GNF.

II. PRELIMINARY DISCUSSION

A GNF is a fluid whose constitutive equation can be written as [18,19]

$$\sigma = 2\eta(S)D \quad \text{with} \quad S = 2\,\mathrm{tr}(D^2), \tag{1}$$

where σ is the shear stress, $D=(\nabla v+\nabla v^T)/2$ is the rate of strain tensor, and η is the viscosity. Equation (1) gives the relation between the strain and the stress of the fluid. This is in any case an approximation whose validity depends on the flow.

In general when a viscoelastic fluid characterized by a relaxation time λ is placed in a Hele-Shaw cell whose plate separation is b, it is displaced at a rate corresponding to a very small Weissenberg number W_e ($W_e=U\lambda/b$ with U as the lateral characteristic velocity). In this case the elastic nature of the fluid will not show up, and it will behave as a purely shear-thinning or shear-thickening fluid, just as it were a GNF. This is true provided the flow is not so slow as to inhibit the non-Newtonian aspects.

In [17], Fast *et al.* theoretically studied the Saffman-Taylor instability for a standard viscoelastic fluid model (the Johnson-Segalman-Oldroyed model which is a generalization of the Oldroyd-B model). The authors have shown that in a thin gap cell, there is a distinguished limit (small Weissenberg number) where shear-thinning or shear-thickening as well as normal stress differences are apparent, but elastic response is negligible. So under certain assumptions, a flow of a complex viscoelastic fluid in a Hele-Shaw cell turns out to be adequately described by the flow of a GNF, entailing a simpler description. It was found in [17] that in the weakly non-Newtonian limit shear thinning decreases the wave number of maximum growth, increases the maximum growth rate, and tightens the band of unstable modes. For shear-

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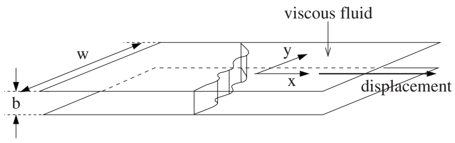

FIG. 1. Sketch of the Hele-Shaw cell. Definition of the axis for the stability analysis.

thickening fluids, the results are reversed; the growth rate for the wave number of maximum growth is decreased and the wave numbers of maximum and critical growth are increased. But in the general case of a non-Newtonian fluid (i.e., not necessarily *weakly* non-Newtonian), no explicit general expression for the growth rate was found.

III. DISPERSION RELATION FOR THE INSTABILITY INVOLVING ANY GENERALIZED NEWTONIAN FLUID

A. Generalized Darcy law

We consider a single fluid flowing in a rectangular Hele-Shaw cell formed by two parallel plates separated by a narrow gap of thickness b (Fig. 1). The relation between mean velocity and pressure is given by Darcy's law. In the Newtonian case it reads

$$\langle \mathbf{v}(x,y,z)\rangle = b^2 \nabla P(x,y)/(12\eta), \tag{2}$$

where η is the viscosity, $P(x,y)$ is the pressure, and $\langle \mathbf{v}\rangle$ is the velocity averaged over the gap b. Note that neither $\nabla P(x,y)/(12\eta)$ nor $\langle \mathbf{v}\rangle$ have to be constant with respect to x or y. In order to study the Saffman-Taylor instability for GNF, it is necessary to generalize the Darcy law for these fluids.

From the linearized and steady Navier-Stokes equations we have

$$\frac{\partial P}{\partial x_i} = \frac{\partial}{\partial x^k}\left[\eta\left(\frac{\partial v_k}{\partial x^i} + \frac{\partial v_i}{\partial x^k}\right)\right], \tag{3}$$

where η is the generalized viscosity defined by the constitutive equation Eq. (1) and $(x^1,x^2,x^3)=(x,y,z)$. Owing to the smallness of the plate separation b one can use the approximations $\frac{\partial}{\partial x},\frac{\partial}{\partial y} \ll \frac{\partial}{\partial z}$ to obtain

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial z}\left[\eta\frac{\partial v_x}{\partial z}\right], \tag{4}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial z}\left[\eta\frac{\partial v_y}{\partial z}\right]. \tag{5}$$

Integrating with respect to z and taking the integration constants equal to zero (by considerations of symmetry), we find

$$\eta\left(\frac{\partial v_x}{\partial z},\frac{\partial v_y}{\partial z}\right)\frac{\partial v_x}{\partial z} = \frac{\partial P}{\partial x}z, \tag{6}$$

$$\eta\left(\frac{\partial v_x}{\partial z},\frac{\partial v_y}{\partial z}\right)\frac{\partial v_y}{\partial z} = \frac{\partial P}{\partial y}z. \tag{7}$$

It then appears that v_x and v_y are the solutions of two coupled differential equations where the independent variable is z. In these equations, $\partial P/\partial x$ and $\partial P/\partial y$ do not depend on z. They are just parameters and one can draw the conclusion that v_x and v_y depend only on z, $\partial P/\partial x$, and $\partial P/\partial y$. Upon averaging over z implies that the average velocity $\langle \mathbf{v}\rangle$ is a function of $\partial P/\partial x$ and $\partial P/\partial y$. Finally, we remark that the relation between $\langle \mathbf{v}\rangle$ and $\nabla \mathbf{P}$ has to be isotropic. Therefore, our ansatz is that a generalized Darcy 's law can then be written for a GNF as

$$\langle \mathbf{v}\rangle = -\mathcal{V}(|\nabla P|)\frac{\nabla P}{|\nabla P|}. \tag{8}$$

The function $\mathcal{V}$ depends on the behavior of the GNF. In the particular case of a Newtonian fluid we have

$$\mathcal{V}(|\nabla P|) = |\nabla P| b^2/(12\eta). \tag{9}$$

The generalized Darcy's law [Eq. (8)] is similar to the generalized Darcy's law of [17,20,21]. Although the physical bases are the same, we have not, in the present paper, tried to insert explicitly the shear-rate dependence of the viscosity in the generalized Darcy's law. It will be shown, in the following, that function $\mathcal{V}(|\nabla P|)$ (which can be computed using the expression of the shear-rate-dependent viscosity, see Sec. IV) contains all the information needed to predict analytically the stability of an interface in a Hele-Shaw cell without restriction, contrary to the previous derivations.

We can see in Eq. (8) that $\nabla\cdot\mathbf{v}=0$ does not imply $\Delta P=0$. In general for a GNF we have

$$\Delta P \neq 0. \tag{10}$$

The lacking of P to satisfy a Laplace equation is the central problem in the linear analysis of the Saffman-Taylor instability [22,23].

B. Linear stability analysis

Let us consider the linear stability of Eq. (8). The initial unperturbed state is a two-fluid flow system GNF(1) and GNF(2). GNF(2) is driven out by GNF(1) via a superimposed constant pressure gradient along the axis Ox. Unperturbed pressures and mean velocities are noted by $P_{o,j}$ and $v_{o,j}$, and are given by

$$\frac{\partial P_{o,j}}{\partial x} = -G_j, \tag{11}$$

$$\langle v_{o,j}\rangle = \mathcal{V}(G_j) \quad \text{with } j=1,2, \tag{12}$$

where G_1 and G_2 are positive constants. Deviations from this almost flat initial interface are obtained by adding small perturbations,

$$P_j^*(x,y,t), \quad j=1,2 \tag{13}$$

to the initial pressure gradient. In turn, this causes perturbations to the initial velocities which are denoted by

$$v_j^*(x,y,t), \quad j=1,2. \tag{14}$$

So the perturbed pressures and mean perturbed velocities are given by

$$P_j(x,y,t) = P_{o,j} + P_j^*(x,y,t), \tag{15}$$

$$\langle v_j \rangle = \langle v_{o,j} \rangle + \langle v_j^*(x,y,t) \rangle. \tag{16}$$

The fluid is incompressible, that is to say

$$\langle v_1 \rangle = \langle v_2 \rangle. \tag{17}$$

Let the perturbations to the pressure be

$$P_j^*(x,y,t) = \epsilon A_j(x,t)\cos(ky) \quad \text{with} \quad \epsilon \ll 1. \tag{18}$$

The pressure gradients are at order ϵ,

$$\frac{\nabla \mathbf{P_j}}{|\nabla P_j|} = \begin{pmatrix} -1 \\ -\epsilon \dfrac{kA_j \sin(ky)}{G_i} \end{pmatrix}. \tag{19}$$

We then obtain the mean velocities by application of the generalized Darcy's law,

$$\langle v_{xj}^* \rangle = -\epsilon \frac{\partial A_j}{\partial x} \cos(ky) \mathcal{V}_j'(G_j),$$

$$\langle v_{yj}^* \rangle = \epsilon \frac{kA_j \sin(ky)}{G_i} \mathcal{V}_j(G_j). \tag{20}$$

The incompressibility of the fluids $(\nabla \cdot v^* = 0)$ yields

$$-\frac{\partial^2 A_j}{\partial x^2} \mathcal{V}_j'(G_j) + A_j \frac{k^2}{G_j} \mathcal{V}_j(G_j) = 0, \tag{21}$$

and then

$$\alpha_j^2 \frac{\partial^2 A_j}{\partial x^2} - k^2 A_j = 0, \quad \text{where} \quad \alpha_j^2 = \frac{G_j}{\mathcal{V}_j} \frac{d\mathcal{V}_j}{dG_j}. \tag{22}$$

C. The different cases according to the sign of α

The solutions of Eq. (22) depend strongly of the sign of α_j^2. Let us consider first of all the case $\alpha_j^2 > 0$. Since the perturbation must not diverge for $x \to \pm\infty$, one obtains for the perturbation of the pressure

$$P_1^*(x,y;t) = \epsilon \widetilde{B}_1(t) e^{k/\alpha_1 (x-\zeta_0)} \cos(ky),$$

$$P_2^*(x,y;t) = \epsilon \widetilde{A}_2(t) e^{-k/\alpha_2 (x-\zeta_0)} \cos(ky), \tag{23}$$

where ζ_0 is a constant. We now choose for ζ_0 the basic position of the interface, $\zeta_0 = \langle v_{01} \rangle t = \langle v_{02} \rangle t$. The position of the disturbed interface is $x = \zeta_0 + \zeta^*(y;t)$. The continuity of the normal components of the velocity at the interface can be approximated in the limit of small deformations by the continuity of the velocities along Ox,

$$\frac{\partial(\zeta_0 + \zeta^*)}{\partial t} = \langle v_{x1} \rangle (\zeta_0 + \zeta^*) = \langle v_{x2} \rangle (\zeta_0 + \zeta^*), \tag{24}$$

keeping only the first-order terms

$$\frac{\partial \zeta^*}{\partial t} = \epsilon \overbrace{\left(\widetilde{A}_2 \frac{k}{\alpha_2} \mathcal{V}_2'(G_2) \right)}^{\dot{X}(t)} \cos(ky). \tag{25}$$

One can thus infer that $\zeta^*(y;t)$ can be written as

$$\zeta^*(y;t) = \epsilon X(t) \cos(ky), \tag{26}$$

with

$$\dot{X}(t) = -\widetilde{B}_1 \frac{k}{\alpha_1} \mathcal{V}_1'(G_1) = \widetilde{A}_2 \frac{k}{\alpha_2} \mathcal{V}_2'(G_2). \tag{27}$$

Using the definition of α_i in Eq. (27) one obtains

$$\dot{X}(t) = -k\widetilde{B}_1 \sqrt{\frac{\mathcal{V}_1}{G_1} \frac{d\mathcal{V}_1}{dG_1}} = k\widetilde{A}_2 \sqrt{\frac{\mathcal{V}_2}{G_2} \frac{d\mathcal{V}_2}{dG_2}}. \tag{28}$$

The formulation of the problem is completed by the interfacial effects due to capillarity. They create a pressure jump $\delta P = P_2 - P_1$ given by Laplace's law across the two sides of a curved interface,

$$\delta P = \gamma(\kappa_1 + \kappa_2), \tag{29}$$

where γ is the surface tension and κ_1 and κ_2 are the two main curvatures of the interface. We have

$$\delta P = [(G_1 - G_2)X(t) + \widetilde{A}_2(t) - \widetilde{B}_1(t)]\epsilon \cos(ky). \tag{30}$$

The curvature of the interface perpendicular to the plane of the cell κ_2 is of the order of $\kappa_2 \simeq 2/b$ and constant along the interface, so that its dynamical effect can be neglected. Only the curvature in the plane of the cells is taken into account. We can write

$$\delta P = -\gamma k^2 \epsilon X(t) \cos(ky), \tag{31}$$

and then

$$\widetilde{A}_2(t) - \widetilde{B}_1(t) = [(G_2 - G_1) - \gamma k^2] X(t). \tag{32}$$

Now, from Eqs. (28) and (32), the equation for $X(t)$ is

$$\frac{dX}{dt} = \frac{(G_2 - G_1 - \gamma k^2)k}{\sqrt{\frac{G_2}{\mathcal{V}_2}\frac{dG_2}{d\mathcal{V}_2}} + \sqrt{\frac{G_1}{\mathcal{V}_1}\frac{dG_1}{d\mathcal{V}_1}}} X(t). \tag{33}$$

Therefore the solution $X(t)$ is

$$X = X_0 \exp[(i\Omega + M)t], \tag{34}$$

from Eq. (33) we obtain

$$\Omega = 0,$$

$$M = \frac{(G_2 - G_1 - \gamma k^2)k}{\sqrt{\frac{G_2}{\mathcal{V}_2}\frac{dG_2}{d\mathcal{V}_2}} + \sqrt{\frac{G_1}{\mathcal{V}_1}\frac{dG_1}{d\mathcal{V}_1}}}. \tag{35}$$

This relation is one of the main result of this paper. It gives the amplification rate as a function of the wave vector k for any GNF for which $\mathcal{V}$ can be computed.

It should be noted that within the limit of Newtonian fluids $\mathcal{V}_j(G) \to \frac{Gb^2}{12\eta_{0j}}$ (η_{0j} is the viscosity of fluid j), we find the well-known wave vector dependence of the amplification rate for Newtonian fluids,

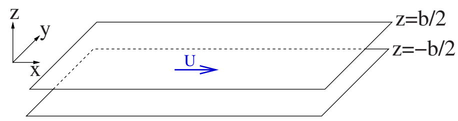

FIG. 2. (Color online) Steady unidirectional Poiseuille flow. The channel has a width in the z direction of b. Velocity is parallel to the x direction and depends only on z.

$$M = \frac{1}{\eta_{02}+\eta_{01}}\frac{kb^2}{12}(G_2 - G_1 - \gamma k^2). \qquad (36)$$

Suppose now that $\alpha_j^2 < 0$ for $j=1,2$. We have shown (see the Appendix for details) that in this case the perturbation to the surface $\zeta^*(y;t)$ is stable, i.e., the amplification rate is zero. It oscillates in time with frequency [see Eq. (A11) in the Appendix],

$$\Omega = \frac{(G_1 - G_2 + \gamma k^2)k}{\sqrt{\left|\frac{G_2}{\mathcal{V}_2}\right|\left|\frac{dG_2}{d\mathcal{V}_2}\right|} - \sqrt{\left|\frac{G_1}{\mathcal{V}_1}\right|\left|\frac{dG_1}{d\mathcal{V}_1}\right|}}. \qquad (37)$$

The mixed cases were also studied (see the Appendix for details). We were able to calculate the amplification rate and/or the oscillatory frequency explicitly. For example, in the limit case of air being pushed by a highly shear-thickening fluid [$\alpha_1^2 \sim 0$ (air), $\alpha_2^2 < 0$ (shear-thickening fluid)], we have found that the amplification rate goes to zero and $\zeta^*(y;t)$ oscillates with the limit frequency [see Eq. (A20) in the Appendix],

$$\Omega = \frac{(G_2 - G_1 - \gamma k^2)k}{\sqrt{\left|\frac{G_2}{\mathcal{V}_2}\right|\left|\frac{dG_2}{d\mathcal{V}_2}\right|}}. \qquad (38)$$

IV. APPLICATION FOR PARTICULAR FLUIDS

In order to investigate the Saffman-Taylor instability for particular GNF, we calculate the function $\mathcal{V}(\nabla P)$ in the case of a purely unidirectional Poiseuille flow in a Hele-Shaw cell (Fig. 2). Since it has been demonstrated that $\mathcal{V}(\nabla P)$ depends only on ∇P, the expression of $\mathcal{V}$ will be valid even if the flow is not unidirectional. For a homogeneous pressure gradient P_x along the x axis, the x component of the velocity can be calculated for a GNF following Kondic *et al.* [20,21] (no slip condition at the top and bottom edges of the channel),

$$v_x(z) = \frac{1}{P_x}\int_{\sigma=b/2P_x}^{\sigma=zP_x} \dot{\gamma}\, d\sigma. \qquad (39)$$

In this equation $\dot{\gamma}$ is the shear rate defined as

$$\dot{\gamma} = \frac{\partial v_x}{\partial z}, \qquad (40)$$

and σ is the shear stress. The relation between σ and $\dot{\gamma}$ fully characterizes the GNF [since the function η in Eq. (1) is equal to $\sigma/\dot{\gamma}$]. It is given by a flow curve obtained by purely steady rheological experiments. Let us now use Eq. (39) to compute $\mathcal{V}$.

A. General expression for shear-thinning fluids

We suppose that the relation between σ and $\dot{\gamma}$ can be written as [18]

$$\dot{\gamma} = \frac{\sigma}{\eta_0}\left[1 + \sum_{n=1}^{\infty}\left(\frac{\sigma}{\sigma_n}\right)^{2n}\right] = \frac{\sigma}{\eta_0}\sum_{n=0}^{\infty}\left(\frac{\sigma}{\sigma_n}\right)^{2n}. \qquad (41)$$

First let σ be small; $\sigma \ll \sigma_n$ for all n. Equation (41) becomes $\dot{\gamma}=\sigma/\eta_0$. Since the viscosity if defined by the ratio between the stress σ and the shear rate $\dot{\gamma}$ [18], it appears that η_0 is the zero-shear viscosity of the fluid. For a higher value of σ, the ratio $\sigma/\dot{\gamma}$ decreases, so the fluid is shear thinning. Using Eq. (39) we obtain

$$v(z) = \sum_{n=0}^{\infty}\frac{P_x^{2n+1}}{(2n+2)\eta_0\sigma_n^{2n}}\left(z^{2n+2} - \frac{b^{2n+2}}{2^{2n+2}}\right), \qquad (42)$$

the mean velocity is

$$\langle v\rangle = \sum_{n=0}^{\infty}\frac{1}{2n+3}\frac{P_x^{2n+1}}{\eta_0\sigma_n^{2n}}\left(\frac{b}{2}\right)^{2n+2}, \qquad (43)$$

and then

$$\mathcal{V}(\nabla \mathbf{P}) = -\sum_{n=0}^{\infty}\frac{1}{2n+3}\frac{|\nabla P|^{2n+1}}{\eta_0\sigma_n^{2n}}\left(\frac{b}{2}\right)^{2n+2}. \qquad (44)$$

B. Power law at high shear rate

In the case where

$$\dot{\gamma} = \frac{\sigma}{\eta_0}\left[1 + \left(\frac{\sigma}{\sigma_n}\right)^{2n}\right], \qquad (45)$$

with η_0 being the zero-shear viscosity (viscosity for small shear rates) we find [from Eq. (44)]

$$\langle v\rangle = -\frac{P_x b^2}{12\eta_0}\left[1 + \frac{3}{3+2n}\left(\frac{P_x b}{2\sigma_n}\right)^{2n}\right]. \qquad (46)$$

Two particular cases may be investigated.

(1) Newtonian fluid $\sigma_n=\infty$, and Eq. (46) is the well-known Darcy law for Newtonian fluids.

(2) The Ostwald–de Waele power-law fluids whose viscosity can be written as

$$\eta(\dot{\gamma}) = k_1\dot{\gamma}^{m-1}, \qquad (47)$$

and thus, for the constitutive equation,

$$\sigma = k_1\dot{\gamma}^m \text{ or } \dot{\gamma} = \left(\frac{\sigma}{k_1}\right)^{1/m}. \qquad (48)$$

Linear analysis of the Saffman-Taylor instability has been studied for such GNF [14] because of the large amount of fluids that are well modeled by this constitutive equation at large shear rate [24].

A fundamental problem with this model for GNF is that it exhibits a nonphysical divergence at low shear rate ($\eta\to\infty$). Since low shear rates are necessarily present in the unperturbed Hele-Shaw velocity profile (zero-shear rate at $z=0$), a

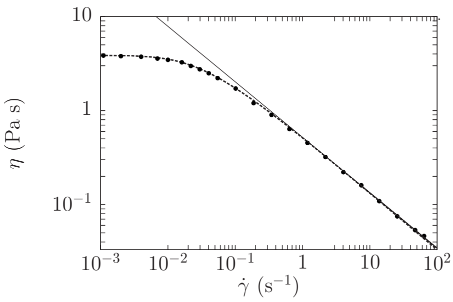

FIG. 3. Viscosity as a function of shear rate of a Xanthan solution (measured with a Contraves low shear Couette rheometer). The experimental viscosity (circles) is approximately constant (zero-shear viscosity) for $\gamma < \gamma_1$, whereas its decrease is well described using a power law for $\gamma > \gamma_2$ (solid line). Viscosity calculated from Eq. (45) fit well the data in the full range of γ (dashed line).

theory where no nonphysical divergence appears is needed.

The comparison with the expression of Eq. (45) shows that asymptotic behaviors for high shear rates (or equivalently high stresses) are the same with

$$m = \frac{1}{2n+1} \quad \text{and} \quad k_1 = (\eta_0 \sigma_n^{2n})^{1/(2n+1)}. \tag{49}$$

So, our theory can address the problem of the Ostwald–de Waele fluids without any unphysical divergence. Figure 3 clearly shows that Eq. (45) fits well real fluids such as polymer solutions when Eq. (48) failed.

Darcy's laws computed from Eq. (45) [see Eq. (46)] or from Eq. (48) are given in Fig. 4. We now consider the experimental situation where a gas forces a GNF described by Eq. (45) into motion. What happens to the amplification rate due to the non-Newtonian properties of this kind of GNF? Does some wavelength become unstable compared with the case of a Newtonian fluid? In which way? In order to answer these important questions, let us calculate the amplification rate. From Eqs. (35) and (46),

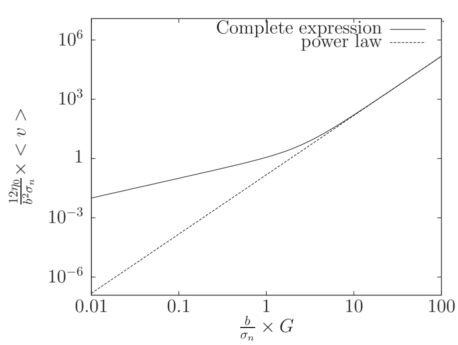

FIG. 4. Darcy's laws for to different models of GNF: for an Ostwald–de Waele power-law fluid (with $m=0.3$) and for a fluid whose constitutive equation is given by Eq. (45). The parameters are chosen so that the fluids have the same high shear-rate behavior. The nonphysical divergence of the viscosity for the Ostwald–de Waele power-law fluid leads to a deviation in the Darcy's law (dashed line). The right Darcy's law (solid line) needs to be used in order to study Saffman-Taylor instability.

$$M = k(G_2 - \gamma k^2)\sqrt{\frac{\mathcal{V}_2}{G_2}\frac{d\mathcal{V}_2}{dG_2}}, \tag{50}$$

$$M = \frac{b^2}{12\eta_0}k(G_2 - \gamma k^2) \times \sqrt{\left(\frac{3(2n+1)}{2n+3}\left(\frac{G_2 b}{2\sigma_n}\right)^{2n} + 1\right)} \times \sqrt{\left(\frac{3}{2n+3}\left(\frac{G_2 b}{2\sigma_n}\right)^{2n} + 1\right)}, \tag{51}$$

$$= k(G_2 - \gamma k^2)\frac{\mathcal{V}}{G_2}\sqrt{\frac{2n+3+6n\left(\frac{G_2 b}{2\sigma_n}\right)^{2n}}{2n+3+3\left(\frac{G_2 b}{2\sigma_n}\right)^{2n}}}, \tag{52}$$

Equations (51) and (52) have to be compared to the case of a purely Newtonian fluid (of viscosity η_0) for which

$$M = \frac{\mathcal{V}}{G_2}k(G_2 - \gamma k), \tag{53}$$

$$= \frac{b^2}{12\eta_0}k(G_2 - \gamma k). \tag{54}$$

The comparison between Eqs. (51) and (54) show that the growth rate is larger in the case of a GNF described by Eq. (45) than the growth rate for a Newtonian fluid, with the two fluids having the same zero-shear viscosity. The range of pressure gradient for which the instability occurs is exactly the same in both cases. This result has been predicted by earlier theories [17]. Its physical explanation is simple, remarking that for a Newtonian fluid the growth rate increases for decreasing viscosities and since the viscosity decreases from the zero-shear viscosity as the shear increases for the GNF we consider, it is straightforward to draw the conclusion that the growth rate is larger for the GNF.

Let us now have a look at a more interesting situation. We want to compare the growth rate of the GNF and the Newtonian fluid, with the average velocities being equal. The viscosity of the Newtonian fluid is chosen so that the pressure gradients are equal. So, in these two situations, pressure gradients and averaged velocities are the same. But the comparison between Eqs. (52) and (53) shows that the two growth rates are not equal even if stable and unstable wave vectors are the same. To be more precise, the growth rate is larger in the case of the GNF if $n > 1/2$, and it is smaller in the opposite case. To conclude, only GNF with high n exponent ($n > 1/2$) tend to develop the instability faster than a Newtonian fluid in the same hydrodynamical condition (given by G and $\langle v \rangle$).

C. Highly shear-thickening fluids

The relation between $\dot{\gamma}$ and σ is nonmonotonic for some very shear-thickening fluids [25,26] (see Fig. 5). We will show that in this case $\mathcal{V}$ may be a nonmonotonic function of

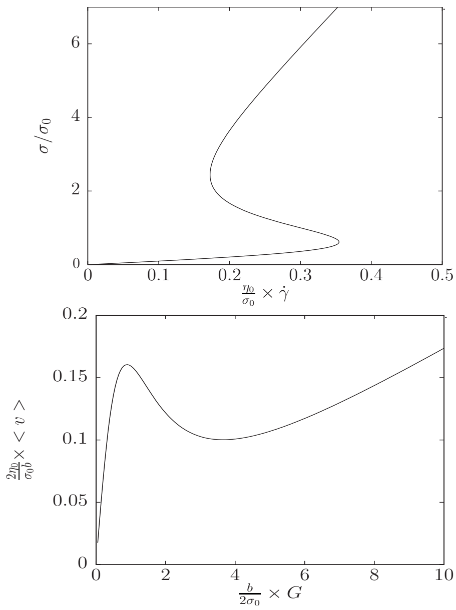

FIG. 5. Top: stress σ as a function of the shear rate $\dot{\gamma}$ for a highly shear-thickening fluid, calculated from Eq. (55) with $\eta_1 = 20\eta_0$. Bottom: generalized Darcy's law for this fluid (average velocity as a function of G). Note the range of G for which $\langle v \rangle$ decreases as G increases.

G with a range of pressure gradient where $d\mathcal{V}/dG < 0$. In the case, the grow rate for the Saffman-Taylor instability is purely imaginary and then the interface is stable.

Let us consider a shear-thickening fluid for which (see Fig. 5)

$$\dot{\gamma}(\sigma) = \frac{1}{\eta_0}\frac{\sigma}{\left[\left(\frac{\sigma}{\sigma_0}\right)^2+1\right]^2} + \frac{\sigma}{\eta_1}. \tag{55}$$

This fluid is so shear thickening that the flow curve is nonmonotonic. With $G=-P_x$ and $X=Gb/(2\sigma_0)$ the generalized Darcy Law we find [using Eq. (39)] is

$$\langle v \rangle = \frac{\sigma_0 b}{2}\left\{\frac{1}{2\eta_0}\frac{1}{X}\left(\frac{\tan^{-1}X}{X} - \frac{1}{X^2+1}\right) + \frac{X}{3\eta_1}\right\}. \tag{56}$$

It appears that $d\mathcal{V}/dG$ is negative for a certain range of pressure gradient. We consider an experimental situation in which a gas is pushing this shear-thickening GNF imposing a pressure gradient G_2 corresponding to negative values of $d\mathcal{V}/dG$. We find in Eq. (38) a purely imaginary amplification rate for any wave vector k. The important result is that the amplification rate is never positive; the interface is always stable.

We have taken the example of a particular shear-thickening fluid described by Eq. (55). We have taken this example because Eq. (55) is probably one of the simplest ones for a shear-thickening fluid with a nonmonotonic flow curve for which it is easy to compute $\mathcal{V}$.

This result can be generalized for any fluid whose shear thickening is high enough. The important result is a gas can force a viscous fluid with a sable interface. The conditions are follows.

(1) The viscous fluid is highly shear thickening.

(2) The pressure gradient is in the range where $\mathcal{V}$ is negative.

This is an unexpected prediction that may have many implications for industrial processes. It now needs to be experimentally tested.

V. CONCLUDING REMARKS

We have developed a theory predicting the growth rate of normal modes for any GNF. This theory fills up a lack; the existing analytical theories only consider weakly nonlinear fluids. Our approach enables us to predict new effects, for instance, the stabilization of the air-liquid front in a Hele-Shaw cell in the case of some shear-thickening fluids. It should be interesting to test experimentally this prediction.

Another extension to this work should be to consider the case of viscoelastic fluids with long relaxation times. Two decades ago Wilson [14] proposed a theory that could be improved nowadays thanks to the developments of new analytical methods. This is precisely a subject of current interest, for instance, very recent experimental results show unexpected behaviors that are not yet elucidate [27].

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APPENDIX

In this appendix we explain in detail the cases for which both or only one α_j^2 is negative

1. Both $\alpha_j^2 < 0$

First at all let us suppose that we have

$$\frac{d\mathcal{V}_j}{dG_j} < 0 \quad \text{for} \quad j = 1,2, \tag{A1}$$

thus α_j^2 is negative in Eq. (22) and we can introduce β_j^2 as

$$\beta_j^2 = \left|\frac{G_j}{\mathcal{V}_j}\frac{d\mathcal{V}_j}{dG_j}\right|, \tag{A2}$$

such that $\alpha_j^2 = -\beta_j^2$ and $A_j(x,t)$ satisfies

$$\beta_j^2\frac{\partial^2 A_j(x,t)}{\partial x^2} + k^2 A_j(x,t) = 0. \tag{A3}$$

$A_j(x,t)$ can be obtained using the method of *normal modes* [28]. So, we look for an $A_j(x,t)$ in the form

$$A_j(x,t) = q_j\cos(k_j x + St) + r_j\sin(k_j x + St), \tag{A4}$$

with q_j and r_j as two arbitrary constants to be determined using initial condition and S as an arbitrary complex constant given the time evolution of the normal mode,

$$S = \Omega + iM. \tag{A5}$$

The equation for $A_j(x,t)$ determines k_j,

$$k_j = \frac{k}{\beta_j}, \tag{A6}$$

where k (real) is the wave number of the pressure perturbation in y. Through the change in variables from (q_j, r_j) to (a_j, b_j),

$$\frac{r_j}{q_j} = -\tan(b_j), \quad r_j^2 + q_j^2 = a_j^2, \tag{A7}$$

we can bring the solution $A_j(x,t)$ into the form

$$A_j(x,t) = a_j \cos(k_j x + St + b_j), \tag{A8}$$

with a_j, b_j as constants. The continuity of the normal components of the velocity at the interface determine $\zeta^*(y;t)$,

$$\zeta^*(y;t) = \epsilon X(t)\cos(ky), \tag{A9}$$

with $X(t)$ in this case is given by

$$X(t) = \frac{-a_1 k}{S}\cos(St + k_1\zeta_0 + b_1)\sqrt{\left|\frac{\mathcal{V}_1}{G_1}\right|\left|\frac{d\mathcal{V}_1}{dG_1}\right|} = \frac{-a_2 k}{S}\cos(St + k_2\zeta_0 + b_2)\sqrt{\left|\frac{\mathcal{V}_2}{G_2}\right|\left|\frac{d\mathcal{V}_2}{dG_2}\right|}. \tag{A10}$$

Finally the Laplace law allows us to determine S. It reads

$$S = \Omega + i0 = \Omega = \frac{(G_1 - G_2 + \gamma k^2)k}{\sqrt{\left|\frac{G_2}{\mathcal{V}_2}\right|\left|\frac{dG_2}{d\mathcal{V}_2}\right|} - \sqrt{\left|\frac{G_1}{\mathcal{V}_1}\right|\left|\frac{dG_1}{d\mathcal{V}_1}\right|}}. \tag{A11}$$

Since $M=0$ we have in this case a mode said to be neutrally stable which oscillates with frequency Ω. So, we have neither $A_j(x,t)$ nor the corresponding perturbed pressures $P_j^*(x,y,t)$ going to zero for $x \to \pm\infty$. They are oscillatory. The whole disturbed pressure is a Fourier superposition of oscillatory normal modes such as

$$P_j^*(x,y,t) = A_j(x,t) = a_j \cos(k_j x + St + b_j). \tag{A12}$$

Nevertheless if we want a disturbed pressure going to zero for $x \to \pm\infty$, it can be obtained by imposing to the Fourier kernel appropriated initial and/or boundary conditions.

2. Mixed cases

Now we consider the mixed cases $\alpha_1^2 > 0$, $\alpha_2^2 < 0$ and $\alpha_1^2 < 0$, $\alpha_2^2 > 0$.

a. $\alpha_1^2 > 0$, $\alpha_2^2 < 0$

These values of α_1^2 and α_2^2 bring to two different solutions A of Eq. (22): A_1 and A_2. A_1 satisfies

$$A_1(x,t) = C(t)\exp\left[\frac{k(x-\zeta_0)}{\alpha_1}\right] \quad \text{for} \quad (x-\zeta_0) \to -\infty, \tag{A13}$$

with $C(t)$ as a function of t to be determined and where we have chosen one of the integration constant equal to ζ_0. A_2 satisfies

$$A_2(x,t) = a_2 \cos(k_2 x + S_2 t + b_2) \quad \text{for} \quad x \to +\infty, \tag{A14}$$

with a_2 and b_2 as arbitrary constants,

$$k_2 = \frac{k}{\beta_2}, \quad \beta_2^2 = -\alpha_2^2 = \left|\frac{G_2}{\mathcal{V}_2}\right|\left|\frac{d\mathcal{V}_2}{dG_2}\right|, \tag{A15}$$

and

$$S_2 = \Omega_2 + iM_2. \tag{A16}$$

The continuity of the normal components of the velocity at the interface determine $\zeta^*(y;t)$,

$$\zeta^*(y;t) = \epsilon X(t)\cos(ky), \tag{A17}$$

with $X(t)$,

$$X(t) = -\frac{k}{\alpha_1}\left[\int^t C(t')dt'\right]\frac{d\mathcal{V}_1}{dG_1} = \frac{-a_2 k_2}{S_2}\cos(S_2 t)\left(\frac{d\mathcal{V}_2}{dG_2}\right). \tag{A18}$$

The Laplace equation gives [eliminating $X(t)$ and $C(t)$ using Eq. (A18)]

$$\cos(S_2 t)\left[1 + \frac{G_1 - G_2 + \gamma k^2}{S_2}\sqrt{\left|\frac{\mathcal{V}_2}{G_2}\right|\left|\frac{d\mathcal{V}_2}{dG_2}\right|}\right] = \left[\sqrt{\frac{G_1}{\mathcal{V}_1}\frac{dG_1}{d\mathcal{V}_1}}\sqrt{\left|\frac{\mathcal{V}_2}{G_2}\right|\left|\frac{d\mathcal{V}_2}{dG_2}\right|}\right]\sin(S_2 t). \tag{A19}$$

Finally separating the real and imaginary parts of Eq. (A19) we have

$$\Omega_2 = \frac{(G_2 - G_1 - \gamma k^2)k}{\sqrt{\left|\frac{G_2}{\mathcal{V}_2}\right|\left|\frac{dG_2}{d\mathcal{V}_2}\right|} + \frac{G_1}{\mathcal{V}_1}\frac{dG_1}{d\mathcal{V}_1}\sqrt{\left|\frac{\mathcal{V}_2}{G_2}\right|\left|\frac{d\mathcal{V}_2}{dG_2}\right|}}, \tag{A20}$$

$$M_2 = \frac{(G_2 - G_1 - \gamma k^2)k}{\left|\frac{G_2}{\mathcal{V}_2}\right|\left|\frac{dG_2}{d\mathcal{V}_2}\right|\sqrt{\frac{\mathcal{V}_1}{G_1}\frac{d\mathcal{V}_1}{dG_1}} + \sqrt{\frac{G_1}{\mathcal{V}_1}\frac{dG_1}{d\mathcal{V}_1}}}. \tag{A21}$$

b. $\alpha_1^2 < 0$, $\alpha_2^2 > 0$

This case is treated in a similar way and we obtain for

$$S_3 = \Omega_3 + iM_3, \tag{A22}$$

with

$$\Omega_3 = \frac{(G_1 - G_2 + \gamma k^2)\sqrt{\left|\frac{\mathcal{V}_1}{G_1}\right|\left|\frac{d\mathcal{V}_1}{dG_1}\right|}}{1 + \frac{G_2}{\mathcal{V}_2}\frac{dG_2}{d\mathcal{V}_2}\left|\frac{\mathcal{V}_1}{G_1}\right|\left|\frac{d\mathcal{V}_1}{dG_1}\right|}. \tag{A23}$$

$$M_3 = \frac{(G_2 - G_1 - \gamma k^2)\sqrt{\left|\frac{\mathcal{V}_1}{G_1}\right|\left|\frac{d\mathcal{V}_1}{dG_1}\right|}}{\sqrt{\left|\frac{G_1}{\mathcal{V}_1}\right|\left|\frac{dG_1}{d\mathcal{V}_1}\right|\frac{\mathcal{V}_2}{G_2}\frac{d\mathcal{V}_2}{dG_2}} + \sqrt{\frac{G_2}{\mathcal{V}_2}\frac{dG_2}{d\mathcal{V}_2}}\left|\frac{\mathcal{V}_1}{G_1}\right|\left|\frac{d\mathcal{V}_1}{dG_1}\right|}. \tag{A24}$$

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