Decay of Loschmidt echo in a Bose-Einstein condensate at a dynamical phase transition

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We study the quantum Loschmidt echo (LE) in a Bose-Einstein condensate (BEC) in a double-well potential. The BEC may undergo a dynamical phase transition between two phases: a tunneling phase and a self-trapping phase. For sufficiently weak perturbation, the LE has Gaussian decay in both phases. While, for relatively strong perturbation, the LE has a Gaussian decay in the self-trapping phase and has a stretched exponential decay in the tunneling phase. This qualitative difference in the decaying law of the LE in the two phases provides a characterization of the dynamical phase transition of the BEC. The semiclassical theory is used to explain the numerically observed behaviors of the LE decay.

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I. INTRODUCTION

Since Bose-Einstein condensate (BEC) was realized experimentally [1], a new epoch for studying its dynamical properties has also been launched [2]. An interesting phenomenon of a BEC in a double-well potential [3] is a dynamical phase transition between a tunneling phase and a self-trapping when the strength of the interaction among the atoms is changed. This phenomenon was first predicted theoretically [4,5], then observed experimentally [6] and studied in detail analytically [7,8]. The dynamical phase transition can be characterized by the difference in the average numbers of the atoms in the two wells, as well as by entanglement entropy [8].

Recently, a fidelity approach to quantum phase transitions [9], i.e., fundamental changes in properties of the ground states of systems, has attracted much attention. It has been found that quantum phase transitions can be characterized by a quantity widely used in the field of quantum information, namely, the fidelity. Two types of fidelity have been studied: the overlap of ground states of neighboring systems [10,11] and the so-called quantum Loschmidt echo (LE) or Peres fidelity [12,13]. It has been found that both types of fidelity have some dramatic changes in the neighborhood of the quantum phase transitions.

It would be natural to study the possibility of using fidelity to characterize the dynamical phase transition of BEC mentioned above. In particular, one may consider the LE, which is a dynamical quantity, defined by the overlap of the time evolution of the same initial state under two slightly different Hamiltonians,

$$M(t) = |\langle\Psi_0|\exp(iHt/\hbar)\exp(-iH_0t/\hbar)|\Psi_0\rangle|^2, \qquad (1)$$

where H_0 and H are the unperturbed and perturbed Hamiltonians, respectively, $H=H_0+\epsilon V$, with ϵ a small quantity and V a generic perturbation. This quantity was first introduced by Peres more than twenty years ago in the study of the stability of quantum motion [14], and has been extensively studied in recent years [15–25]. Meanwhile, it is also of relevance in the study of decoherence [26–28] and the concurrence entanglement of a two-qubit system coupled to a spin chain [29].

The LE in quantum systems, whose classical counterparts have strong chaos with exponential instability, has been found to have the following main features in its decay, related to the perturbation strength: (i) in the perturbative regime in which the typical transition matrix element is smaller than the mean level spacing, the LE has a Gaussian decay. (ii) Above the perturbative regime, the LE has an exponential decay with a rate proportional to ϵ^2, usually called the Fermi-golden-rule (FGR) decay of LE. (iii) Above the FGR regime is the Lyapunov regime, in which the LE $M(t)$ usually has an approximate exponential decay with a perturbation-independent rate. The LE decay in regular systems with quasiperiodic motion in the classical limit has also attracted much attention. For a single initial Gaussian wave packet, the main feature is a Gaussian decay for times not too long [16,24,25].

Indeed, as shown in Ref. [30], the LE can be employed to characterize the dynamical phase transition of a BEC system in a double-well potential. Numerically the LE was found to have qualitatively different decaying behaviors in the two phases: a Gaussian decay in the self-trapping phase and a stretched exponential decay in the tunneling phase. This is in contrast to what has been found in quantum phase transitions, where the LE decay on both sides of a quantum phase transition obeys the same decaying law.

In this paper, we perform a more thorough investigation in the decaying behavior of the LE at the dynamical phase transition of the BEC system. We do this mainly for three reasons: (i) Ref. [30] gives only numerical evidence for the above mentioned difference in the decaying behavior of the LE in the two phases of the BEC, while the underlying mechanism of such a difference is still unclear. (ii) After the publication of Ref. [30], we found that, when the perturbation strength is decreased to sufficiently small values, the LE has Gaussian decay in both phases of the BEC. It should be of interest to give an explanation to the qualitative change in the decaying behavior of the LE with perturbation strength. (iii) Stretched exponential decay of the LE is still an unex-

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plained phenomenon. It has also been observed numerically in two other models: namely, a quantum system whose classical counterpart lies in a region between chaotic sea and regular island [31], and a quantum triangle map whose classical counterpart has a linear instability but is ergodic and mixing [32]. The origin of this stretched exponential decay is still not clear, therefore, an explanation of this phenomenon is of interest by itself.

In order to explain the above mentioned three points, we are to make use of a semiclassical approach to the LE. We show that the previously predicted Gaussian decay of the LE in regular systems, which was given in Refs. [16,24,25], is valid for relatively weak perturbation. (The BEC model we study here has a regular classical counterpart.) When the perturbation is strong enough, the LE may have a non-Gaussian decay, as a result, a stretched exponential decay may appear. We then give further semiclassical analysis to the LE decay and show that the semiclassical theory can indeed explain the observed stretched exponential decay.

This paper is organized as follows: in Sec. II, the double-well potential BEC model is introduced and the dynamical phase transition is briefly discussed. Numerical results for the decaying behavior of the LE in the neighborhood of the dynamical phase transition are presented and discussed in Sec. III. In Sec. IV, we explain the numerical results by making use of the semiclassical theory. Conclusions are given in Sec. V.

II. BEC IN A DOUBLE-WELL POTENTIAL

A. Model

We consider a N-atom BEC in a double-well potential, with the wells indicated by A and B, respectively. The Hamiltonian in the second quantization form is written as [33,34],

$$H = \frac{\gamma}{2}(a^\dagger a - b^\dagger b) + \frac{c}{2N}(a^\dagger a^\dagger aa + b^\dagger b^\dagger bb) - \frac{v}{2}(a^\dagger b + b^\dagger a). \tag{2}$$

Here γ is the level separation of the two wells, c is the interaction constant determined by the scattering of the atoms, v is the coupling strength between the two wells, and $a^\dagger$ and $b^\dagger$ (a,b) are boson creation (annihilation) operators for the two wells, respectively.

In this paper, we consider the case of a symmetric double-well potential with $\gamma=0$ and $c>0$ corresponding to a repulsive interaction between atoms. In this case, the controlling parameters of the dynamics are c, v, and N [35]. Here we fix the particle number $N=1000$. In addition, a scaling can make the Hamiltonian depend on only one of the two parameters v and c, hence, we set unit the coupling strength v and consider the variation of c. Experimentally, the atom-atom interaction may be adjusted via Feshbach resonances [36].

Time evolution of the state of the BEC is given by Schrödinger equation

$$i\frac{d|\varphi(t)\rangle}{dt} = H|\varphi(t)\rangle, \tag{3}$$

where the Planck constant is set unit, $\hbar=1$. We employ the Fock states $|n\rangle$ as the basis states, where n is the number of the atoms in the well A, with $(N-n)$ atoms in the well B. In this basis, nonzero matrix elements of the Hamiltonian are

$$H_{n,n} = \frac{c}{2N}[n(n-1) + (N-n)(N-n-1)], \tag{4}$$

$$H_{n,n-1} = H_{n-1,n} = -\frac{v}{2}[n(N-n+1)]^{\frac{1}{2}}, \tag{5}$$

and the state of the BEC is written as

$$|\varphi(t)\rangle = \sum_{n=0}^{N} a_n(t)|n\rangle. \tag{6}$$

In the study of LE decay, it is convenient to adopt the direct numerical diagonalization method as used in Ref. [39]. Meanwhile, it is worth mentioning that in some limiting cases this BEC model can be (approximately) solved by making use of the algebraic Bethe ansatz [37,38].

When the particle number is sufficiently large, the system can be well described by a mean-field approximation and has a classical counterpart with the following Hamiltonian, expressed in terms of a pair of canonical variables (θ,s) [4]:

$$H_{cl} = -\frac{1}{2}cs^2 + \sqrt{1-s^2}\cos\theta. \tag{7}$$

The two variables s and θ have the following physical meaning: writing the probability amplitudes of the atoms in the two wells as $a=|a|\exp(i\theta_a)$ and $b=|b|\exp(i\theta_b)$, respectively, $s=|b|^2-|a|^2$ is the population difference between the two wells and $\theta=\theta_b-\theta_a$ is the phase difference. The equations of motion of the canonical variables θ and s are

$$\dot{s} = \sqrt{1-s^2}\sin\theta, \quad \dot{\theta} = -cs - \frac{s}{\sqrt{1-s^2}}\cos\theta. \tag{8}$$

The Hamiltonian in Eq. (2) can also be written in terms of the generators L_x, L_y, and L_z of the group SU(2) [40],

$$L_x = (a^\dagger b + b^\dagger a)/2, \quad L_y = (a^\dagger b - b^\dagger a)/2i,$$

$$L_z = (a^\dagger a - b^\dagger b)/2. \tag{9}$$

These generators obey the same commutation rules as for angular momentum. The Fock states $|n\rangle$ are also eigenstates of L_z, $L_z|n\rangle=(n-\frac{N}{2})|n\rangle$, since $L_z=a^\dagger a-N/2$ and the total number of the atoms is conserved.

An important class of states in a system with the dynamical group SU(2) is coherent state [41]. A SU(2) coherent state $|\alpha\rangle$ centered at a point in the sphere with polar angle ϕ and azimuthal angle θ is given by

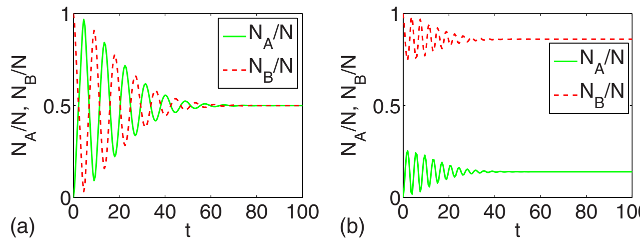

FIG. 1. (Color online) An illustration of the two phases of the BEC. N_A and N_B are the average numbers of the atoms in the wells A and B, respectively. The initial state is $|0\rangle$, for which $c_{\rm cr}=2$. Panel (a): $c=1.8$ in the tunneling phase. Panel (b): $c=2.3$ in the self-trapping phase. In this and the following figures, $N=1000$.

$$|\alpha\rangle \equiv e^{\alpha^* L_+ - \alpha L_-}|0\rangle, \quad \text{with} \quad \alpha = \frac{\pi-\phi}{2}e^{-i\theta}. \tag{10}$$

The Fock state $|n=0\rangle$ is a coherent state with $\alpha=0$. To see the relation of other Fock states to coherent states, one can expand $|\alpha\rangle$ in the Fock states,

$$|\alpha\rangle = \sum_{n=0}^{N} \frac{z^{*n}}{(1+zz^*)^{N/2}}\left[\frac{N!}{n!\,(N-n)!}\right]^{1/2}|n\rangle, \tag{11}$$

where $z=-e^{-i\theta}\cot(\phi/2)$. Considering the limit $|z|\to\infty$, it is easy to check that $|N\rangle$ is also a coherent state. For α sufficiently far from the two poles, the expansion of $|\alpha\rangle$ in $|n\rangle$ has approximately a Gaussian form.

B. Two phases

The BEC system may have two phases: a tunneling phase and a self-trapping phase. For relatively weak nonlinear term, i.e., small value of c, if all the atoms are initially in one of the two wells, they can travel to the other well by tunneling. On the other hand, for sufficiently large c, the tunneling effect can be suppressed, resulting in a self-trapping phase. Since the two phases are not required to be stationary, transition between them is called a dynamical phase transition.

Separation of the two phases is sharp in the large N limit. For any initial coherent state, which can be represented approximately by its center (θ_i, s_i) in the classical phase space, the critical value $c_{\rm cr}$ separating the two phases can be conveniently evaluated by the classical dynamics, giving [4,8]

$$c_{\rm cr} = \frac{2}{s_i^2}(1+\sqrt{1-s_i^2}\cos\theta_i). \tag{12}$$

The system has a tunneling phase for $c<c_{\rm cr}$ and a self-trapping phase for $c\geq c_{\rm cr}$. For example, for the initial state $|0\rangle$ with all the atoms in the well B, $s_i=1$, as a result, $c_{\rm cr}=2$.

To illustrate the difference in the behavior of the system in the two phases, one can consider the time evolution of the numbers of the atoms in the two wells, denoted by N_A and N_B, respectively,

$$N_A = \langle\varphi(t)|a^\dagger a|\varphi(t)\rangle, \quad N_B = \langle\varphi(t)|b^\dagger b|\varphi(t)\rangle. \tag{13}$$

Some examples are shown in Fig. 1 for the initial state $|0\rangle$.

III. DECAY OF LOSCHMIDT ECHO-NUMERICAL RESULTS

A. General discussions

The classical counterpart of the BEC system discussed above has a time-independent Hamiltonian in Eq. (7), with one degree of freedom, hence, it is integrable. The semiclassical theory predicts a Gaussian decay for the LE in regular systems, when the initial states are Gaussian wave packets or coherent states with approximate Gaussian shape [16,24,25]. Therefore, it is natural to expect a Gaussian decay of the LE in both phases of the BEC model with the Hamiltonian in Eq. (2).

However, numerical results given in Ref. [30] show that the LE of the initial state $|0\rangle$ has qualitatively different decaying behaviors on the two sides of the dynamical phase transition for certain intermediate perturbation strength: it has a Gaussian decay in the self-trapping phase and has a stretched exponential decay in the tunneling phase. Reference [30] is a letter presenting some numerical results only, without any analysis in the mechanism of this difference in the decay law of the LE, while such an analysis is obviously important for understanding the LE decay at the dynamical phase transition. Moreover, the origin of a stretched exponential decay of LE is still not clear.

For the above reasons, in what follows, we perform a more complete investigation in the decaying behavior of the LE in the BEC system. We first study the case of very weak perturbation and show that the LE indeed has the analytically expected Gaussian decay on both sides of the dynamical phase transition. Then, making use of the semiclassical theory, we analyze the LE's decaying behaviors under intermediate perturbation strength. We also study the decaying behavior of the LE for initial coherent states other than $|0\rangle$.

Below, we use $H(c)$ and $H(c')$ to denote the unperturbed and perturbed Hamiltonians, respectively, with $c'=c+\delta c$, hence, $\epsilon=\delta c$. Writing the time evolution of the states in the Fock basis,

$$|\varphi(c,t)\rangle = \exp[-iH(c)t]|\varphi_0\rangle = \sum_{n=0}^{N} a_n(t)|n\rangle$$

$$|\varphi(c',t)\rangle = \exp[-iH(c')t]|\varphi_0\rangle = \sum_{n=0}^{N} a'_n(t)|n\rangle, \tag{14}$$

the LE is

$$M(t) = |\langle\varphi(c',t)|\varphi(c,t)\rangle|^2 = \left|\sum_{n=0}^{N} a'^*_n(t)a_n(t)\right|^2. \tag{15}$$

B. LE decay for initial state $|0\rangle$

We have performed extensive numerical experiments in the LE decay at weak perturbation. For sufficiently small δc, specifically, $\delta c \leq 10^{-5}$, we indeed found the Gaussian decay of LE, $\ln M(t) \propto -t^2$, in both the tunneling phase and the self-trapping phase (see Fig. 2 for an example in the tunneling phase). Hence, when the perturbation is sufficiently

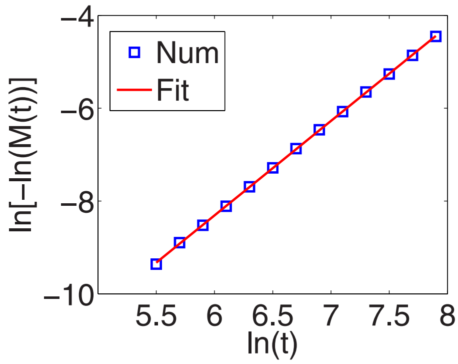

FIG. 2. (Color online) Decay of the LE $M(t)$ at very weak perturbation for the initial state $|0\rangle$, $c=1.8$ (tunneling phase) and $\delta c=10^{-5}$. The straight line is the best fitting to the squares obtained numerically. The slope of the straight fitting line is 2, showing that the LE has a Gaussian decay, $\ln M(t) \propto -t^2$. Similar Gaussian decay has also been found in the self-trapping phase.

weak, the LE obeys the same decaying law on both sides of the dynamical phase transition, as expected from previous semiclassical analysis.

With increasing perturbation strength, the phenomenon reported in Ref. [30] appears. Namely, for the initial state $|0\rangle$, the LE still has a Gaussian decay in the self-trapping phase, but has a stretched exponential decay in the tunneling phase. Both decay can be fitted by the following function:

$$M_f(t) = exp(-kt^{\beta}), \tag{16}$$

with $\beta=2$ corresponding to a Gaussian decay and $\beta<2$ to a stretched exponential decay. Some examples are presented in Fig. 3, where $\beta=2$ for the fitting line in the self-trapping case of $c=2.3$ and $\beta=1.85$ for the tunneling case of $c=1.8$.

In order to have a whole picture for the decaying behavior of the LE in the neighborhood of the dynamical phase transition, in Fig. 4 we plot the variation of β with c for the initial state $|0\rangle$. The value of β is smaller than 2 for $c<2$, indicating a stretched exponential decay of the LE in the tunneling phase. With c approaching 2, the stretched exponential decay becomes closer and closer to the Gaussian decay. For c approximately between 2 and 2.3, there is an oscillation of the value of β around $\beta=2$. Finally, for $c>2.3$, β is quite close to 2, showing a good Gaussian decay of the LE. This figure shows clearly that the value of β can characterize the dynamical phase transition of the BEC.

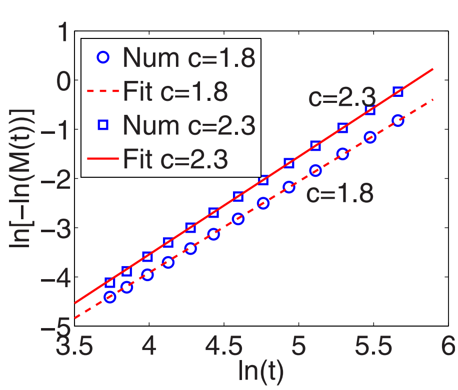

FIG. 3. (Color online) The LE decay for the initial state $|0\rangle$ with $\delta c=5\times10^{-4}$, $c=1.8$ (tunneling phase) and $c=2.3$ (self-trapping phase). The squares and circles are numerically computed LE and the solid and dotted straight lines are fitting curves of the form in Eq. (16): $\beta=1.85$ for $c=1.8$ and $\beta=2.0$ for $c=2.3$.

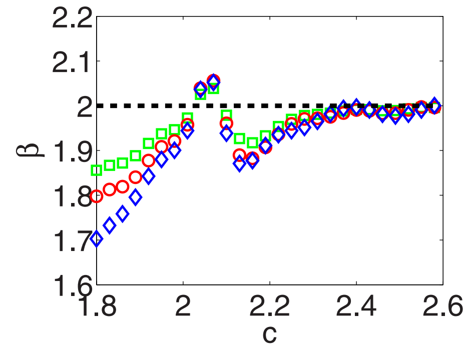

FIG. 4. (Color online) Variation in the fitting parameter β in Eq. (16) with c for the initial state $|0\rangle$. The straight dashed line is shown for guiding eyes. Empty squares, circles, and diamonds correspond to the perturbation strength $\delta c=5\times10^{-4}$, 8×10^{-4}, and 10^{-3}, respectively.

We have also performed extensive numerically experiments for other values of δc. The LE has been found to have similar decaying behaviors for δc in the region $[5\times10^{-4},2\times10^{-3}]$. In particular, the same type of oscillation of β for c between 2 and 2.3 has been found for different perturbation strengths δc (see Fig. 4). For $\delta c>2\times10^{-3}$, the LE has been found to have quite large fluctuations, which make it difficult to study the decaying law of the LE.

One may guess that the oscillation of β for c between 2 and 2.3 might be related to some finite-size effect. In fact, this oscillation lies in the transition region between the two phases. For a finite N, the transition is not sharp and has a finite region. Due to competition of the two phases, behaviors of the system in the finite transition region are more complex than those in the region of either of the two pure phases, hence, explanation of the former is more difficult. If in the large N limit the phase transition becomes sharp, with the transition region shrinking to zero, then, it seems reasonable to expect that the oscillation might disappear. However, numerically it is difficult to confirm this point, since it might need a quite large N. In fact, in our numerical simulations, no obvious signature of this mechanism has been observed for N from 500 to 1500. Therefore, the real mechanism of the oscillation of β in the transition region is still an open problem.

C. LE decay for initial coherent states

Dynamical phase transition may happen not only for the initial state $|0\rangle$, but for all initial coherent states with high population of the particles in one of the two wells. The behavior of the LE of initial coherent states other than $|0\rangle$ is not discussed in Ref. [30]. Therefore, we study this more general situation here.

Numerically, we found that the LE of initial coherent states other than $|0\rangle$ has behaviors qualitatively similar to

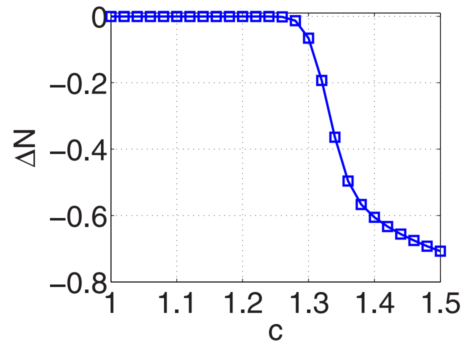

FIG. 5. (Color online) Variation in the averaged atom number difference, $\Delta N=\{N_A-N_B\}/N$, with the interaction strength c for the initial coherent state $|\alpha=1.3\rangle$. Here $\{.\}$ denotes average over time. $\Delta N\approx 0$ for $c\leq 1.28$ in the tunneling phase and $|\Delta N|\geq 0.5$ for $c\geq 1.36$ in the self-trapping phase. This is in agreement with the analytical prediction of Eq. (12) for the large N limit, which gives $c_{\rm cr}=1.32$ for this initial state. The finite region of transition is due to finite-size effect.

those of the initial state $|0\rangle$. As an example, let us consider the initial coherent state $|\alpha\rangle$ with $\alpha=1.3$. Equation (12) gives $c_{\rm cr}=1.32$ for $\alpha=1.3$, which is in agreement with a direct calculation of ΔN shown in Fig. 5. For sufficiently weak perturbation, the LE has been found to have Gaussian decay in both phases. While for relatively strong perturbation, as shown in Fig. 6, the LE has a stretched exponential decay in the tunneling phase ($c=1.2$) and has a Gaussian decay in the self-trapping phase ($c=5$).

In fact, we have another reason to study initial coherent states other than $|0\rangle$. In the classical limit, $|0\rangle$ corresponds to the south pole in the Bloch sphere. Direct application of the semiclassical theory to this state is not easy, due to the peculiarity of the pole. Now, the LE has qualitatively similar behaviors at dynamical phase transition for different initial coherent states, hence, we can study initial coherent states whose centers depart enough from the two poles. For example, in the derivation of the Gaussian decay of the LE of initial coherent states in regular systems, which is given in Ref. [16], the coherent states are assumed to have a Gaussian shape in the Fock basis. That derivation cannot be directly

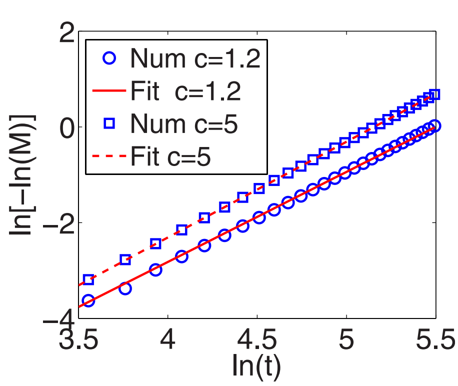

FIG. 6. (Color online) Variation in the LE for the initial coherent state $|\alpha\rangle$ with $\alpha=1.3$. The slopes of the fitting lines: $\beta=1.85$ for $c=1.2$ (tunneling phase) and $\beta=2$ for $c=5$ (self-trapping phase). The perturbation strength $\delta c=3\times 10^{-3}$.

applied to the initial state $|0\rangle$, which is a single Fock state, but can be applied to initial states like $|\alpha=1.3\rangle$.

IV. SEMICLASSICAL EXPLANATION

The most useful analytical tool in the study of the LE decay in regular systems seems the semiclassical approach. In this section, we use the semiclassical theory to explain the numerical results presented above, in particular, the following ones: (i) for initial coherent states with high population in one of the two wells, the LE has Gaussian decay in both phases of the BEC system when the perturbation is sufficiently weak, but obeys different decay laws in the two phases when the perturbation is relatively strong. (ii) The LE has a stretched exponential decay in the tunneling phase for relatively strong perturbation.

A. Semiclassical approach to LE: A preliminary explanation of the numerical results

Let us first briefly recall the main results of the approach. We present it in a general d-dimensional configuration space and consider the general expression of the LE given in Eq. (1). For an initial wave-function $\psi_0(\mathbf{r}_0)$, its time evolution propagated by the semiclassical Van Vleck-Gutzwiller propagator is written as

$$\psi_{\rm sc}(\mathbf{r};t)=\int d\mathbf{r}_0 K_{\rm sc}(\mathbf{r},\mathbf{r}_0;t)\psi_0(\mathbf{r}_0), \qquad (17)$$

where $K_{\rm sc}(\mathbf{r},\mathbf{r}_0;t)=\Sigma_c K_c(\mathbf{r},\mathbf{r}_0;t)$, with

$$K_c(\mathbf{r},\mathbf{r}_0;t)=\frac{C_c^{1/2}}{(2\pi i\hbar)^{d/2}}\exp\left[\frac{i}{\hbar}S_c(\mathbf{r},\mathbf{r}_0;t)-\frac{i\pi}{2}\mu_c\right]. \qquad (18)$$

Here, the label c [more exactly $c(\mathbf{r},\mathbf{r}_0;t)$] indicates classical trajectories starting from $\mathbf{r}_0$ and ending at $\mathbf{r}$ within the time t, $S_c(\mathbf{r},\mathbf{r}_0;t)$ is the action (the time integral of the Lagrangian $\mathcal{L}$) along the trajectory c, $S_c(\mathbf{r},\mathbf{r}_0;t)=\int_0^t dt'\mathcal{L}$, and $C_c=|\det(\partial^2 S_c/\partial r_{0i}\partial r_j)|$. μ_c is the Maslov index counting the conjugate points. (In this section, we write the Planck constant explicitly in the semiclassical expressions.)

Consider an initial Gaussian wave packet centered at $\tilde{\mathbf{r}}_0$, with dispersion ξ and mean momentum $\tilde{\mathbf{p}}_0$,

$$\psi_0(\mathbf{r}_0)=\left(\frac{1}{\pi\xi^2}\right)^{d/4}\exp\left[\frac{i}{\hbar}\tilde{\mathbf{p}}_0\cdot\mathbf{r}_0-\frac{(\mathbf{r}_0-\tilde{\mathbf{r}}_0)^2}{2\xi^2}\right]. \qquad (19)$$

When ξ is small enough, the amplitude $m(t)$ of the LE is written as [15,18,21]

$$m_{\rm sc}(t)\simeq(\pi w_p^2)^{-d/2}\int d\mathbf{p}_0\exp\left[\frac{i}{\hbar}\Delta S_{\mathbf{p}_0}-\frac{(\mathbf{p}_0-\tilde{\mathbf{p}}_0)^2}{w_p^2}\right], \qquad (20)$$

where $w_p=\hbar/\xi$ and $\Delta S_{\mathbf{p}_0}$ is the action difference between two nearby trajectories of the two systems H and H_0 starting at $(\mathbf{p}_0,\tilde{\mathbf{r}}_0)$. The semiclassical expression of the LE can then be calculated, $M_{\rm sc}(t)=|m_{\rm sc}(t)|^2$. We mention that, for not very narrow initial Gaussian packets, Eq. (20) may still hold with a redefinition of w_p [21].

The action difference can be calculated in the first-order classical perturbation theory

$$\Delta S_{\mathbf{p}_0} \simeq \epsilon \int_0^t dt' V[\mathbf{r}(t'), \mathbf{p}(t')], \tag{21}$$

with the perturbation V evaluated along one of the two trajectories. Equations (20) and (21) give quite accurate predictions even for relatively long times, much more accurate than what is usually expected for a first-order perturbation treatment [18,20]. The reason for the unexpected accuracy is explained in [19] by making use of a shadowing theorem.

For a regular system, Eq. (20) predicts an approximate Gaussian decay of the LE in the case of $\tau \gg T$. Here τ is the decay time of the LE, which can be defined by $M(\tau)=1/e$, and T is the period of the (approximately) periodic motion of the classical system. In fact, in this case, as shown in Ref. [24], due to the periodicity of the classical motion, ΔS can be divided into an average part and an oscillating part and, for times not too long, the LE has mainly a Gaussian decay determined by the average part of ΔS [42],

$$M_1(t) \simeq e^{-\Gamma t^2} \quad \text{with} \quad \Gamma = 2w_p^2 \epsilon^2 |\nabla_{\mathbf{p}_0} \langle \tilde{V} \rangle|^2/\hbar^2. \tag{22}$$

Here, $\langle V \rangle = \frac{1}{T}\int_0^T V(t)dt$ and tilde means evaluation at $\tilde{\mathbf{p}}_0$. This Gaussian decay was first derived in Ref. [16] for initial coherent states. Since τ increases with decreasing ϵ, $\tau \gg T$ is satisfied for sufficiently weak perturbation. This explains the phenomenon that the LE of the BEC system has Gaussian decay in both phases for sufficiently weak perturbation.

With increasing the perturbation strength ϵ, we enter into the region of perturbation strength in which $\tau \lesssim T$. The decay law of the LE in this case has not been discussed in the literature and the LE does not necessarily have a Gaussian decay. Numerically, we found that the LE of the BEC system in the tunneling phase begins to have a stretched exponential decay when $\tau \simeq 5T$, in agreement with the above analytical analysis. Therefore, the semiclassical theory predicts the possibility of non-Gaussian decay for relatively strong perturbation.

B. LE decay of the BEC system at relatively strong perturbation

Now, we use Eq. (20) to explain the following numerical observation in the system of a BEC in a double-well potential. That is, for relatively strong perturbation the LE obeys different decay laws in the two phases of the BEC: a stretched exponential decay in the tunneling phase and a Gaussian decay in the self-trapping phase.

For this purpose, we need to use a technique which is different from that used in Ref. [24]. In fact, for times $t < T$, the periodicity of the classical trajectories plays no role in determining the decay law of the LE. To calculate the LE in this case, one may write the LE given in Eq. (20) in the following form:

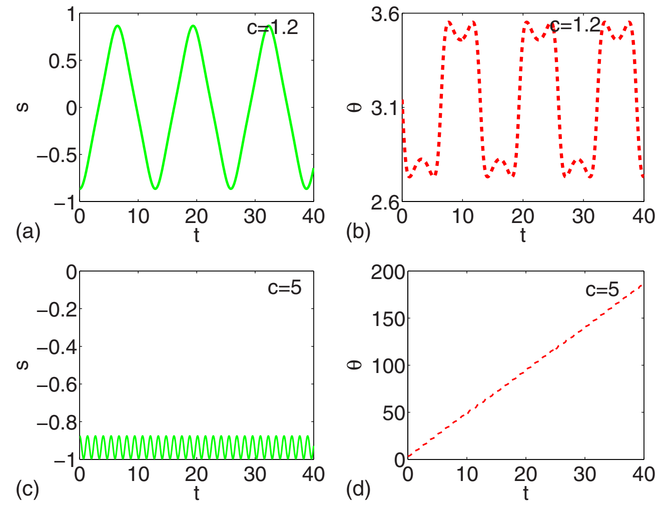

FIG. 7. (Color online) Panels (a) and (b): variation of s and θ with time, for the classical trajectory [see Eq. (8)] starting from the center of the coherent state $|\alpha=1.3\rangle$ with $c=1.2$ in the tunneling phase. Panels (c) and (d): the same as panels (a) and (b), for $c=5$ in the self-trapping phase.

$$M_{\rm sc}(t) \simeq \left| \int d\Delta S e^{i\Delta S/\hbar} P(\Delta S) \right|^2, \tag{23}$$

where $P(\Delta S)$ is the distribution of ΔS with the Gaussian weight taken into account,

$$P(\Delta S) = \int \frac{d\mathbf{p}_0}{(\pi w_p^2)^{-d/2}} e^{-(\mathbf{p}_0 - \tilde{\mathbf{p}}_0)^2/w_p^2} \delta(\Delta S - \Delta S_{\mathbf{p}_0}). \tag{24}$$

The decaying behavior of the LE is determined by the distribution of the action difference, $P(\Delta S)$. In the case of a classical system with strong chaos, $P(\Delta S)$ has a form close to a Gaussian distribution [17]. However, when the underlying classical motion is not strongly chaotic, $P(\Delta S)$ may deviate from the Gaussian form. For example, in the case of weak chaos in the sawtooth map [20,21] and in the triangle map [32], obvious deviation of $P(\Delta S)$ from the Gaussian form has been observed numerically, which leads to deviation of the decay law of the LE from that found in strongly chaotic systems.

Before calculating the distribution $P(\Delta S)$, it is useful to first have a look at features of the classical trajectories corresponding to the two phases. Some examples are given in Fig. 7, for classical trajectories starting from the center of coherent state $|\alpha=1.3\rangle$, computed by using Eq. (8). It is seen that the trajectory in the self-trapping phase is relatively simple, while that in the tunneling phase is more complex in θ, with more than one obvious frequencies.

In the sawtooth map, the central body of the distribution $P(\Delta S)$ was found to have roughly a Levy shape and this was used in explaining the LE decay there [20,21]. Hence, here we also study the relation between the distribution $P(\Delta S)$ and a Levy distribution. We note that, since a Levy distribution has an infinite variance, the distribution $P(\Delta S)$ cannot be of a Levy form in the long-tail region. This point is not serious, if we consider times not long such that the long tails of $P(\Delta S)$ do not give a significant contribution to the right-hand side of Eq. (23). In this case, we can focus on the central part of $P(\Delta S)$ and study whether it can be approximated by a Levy distribution.

We consider the following asymmetric form of the Levy distribution [43]:

$$L(x,\eta,\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(izx)\psi(z)dz. \quad (25)$$

Here the function $\psi(z)$ is

$$\psi(z) = \exp\{-igz - D_L|z|^{\eta}[1 + i\beta\,\mathrm{sgn}(z)\omega(z,\eta)]\}, \quad (26)$$

where

$$\omega(z,\eta) = \tan(\pi\eta/2) \quad \text{for} \quad \eta \neq 1, \quad (27)$$

$$\omega(z,\eta) = (2/\pi)\ln|z| \quad \text{for} \quad \eta = 1. \quad (28)$$

The parameter η, with $0<\eta<2$, determines the decay of long tails, i.e., $L(x) \sim |x|^{-(1+\eta)}$ for large $|x|$; the parameter β has the domain $[-1,1]$, with $\beta=0$ giving the symmetric distribution; the parameter g gives a shift along the x direction; and D_L is related to the width of the distribution.

If the Levy distribution can be used as an approximation to $P(\Delta S)$, with $x=\Delta S/\epsilon$, substituting Eq. (25) into Eq. (23), one obtains,

$$M_{sc}(t) \propto \exp\{-2(\epsilon/\hbar)^{\eta}D_L\}. \quad (29)$$

The dependence of M_{sc} on the time t is given by that of the quantity D_L. Therefore, if $D_L \propto t^{\beta}$, then, the semiclassical theory can explain the observed stretched exponential decay of the LE with the exponent β. Note that $M_{sc}(t)$ can also be written in the form of Eq. (29), when the distribution $P(\Delta S)$ has a Gaussian form. In this case, $\eta=2$ and D_L is proportional to the width of the Gaussian distribution.

Now we study whether the expression (29) can be used to explain the numerically found stretched exponential decay of the LE in the double-well BEC model. The distribution $P(\Delta S)$ can be calculated numerically by making use of Eq. (24) for initial coherent states with Gaussian shape. For this, we need to calculate classical trajectories starting from $(\tilde{\mathbf{r}}_0, \mathbf{p}_0)$ and use Eq. (21) to compute the action difference. As discussed in Sec. II A, the classical counterpart of the system has a pair of canonical variables (θ,s). For initial coherent states sufficiently far from the two poles, s and θ can be treated like the ordinary momentum and coordinate, respectively. Then, using Eq. (8) to calculate the classical trajectories and noticing that $V=-\frac{1}{2}s^2$ since the perturbation is taken for the parameter c [see Eq. (7)], we can compute the distribution $P(\Delta S)$ numerically.

As shown in Fig. 8, in the self-trapping phase with $c=5$, the numerically computed distribution $P(\Delta S)$ is quite close to a Gaussian distribution, in agreement with the numerical observation that the LE has a Gaussian decay in the self-trapping phase even for relatively strong perturbation. On the other hand, in the tunneling phase ($c=1.2$), the distribution $P(\Delta S)$ deviates notably from the Gaussian shape. Here, for a reason discussed previously, namely, $P(\Delta S)$ cannot have long tails of the Levy form, we consider a truncated Levy distribution, which is obtained from the Levy distribution (25) by taking a finite domain of the variable, $y \in [-3,3]$. The figure shows that the central part of $P(\Delta S)$ has a shape roughly close to a Levy distribution. We have further compared the Fourier transform of $P(\Delta S)$ and the function $\psi(z)$ in Eq. (26), which is the Fourier transform of the Levy distribution, and found that they are close.

We further study the dependence of D_L on the time t. For this, we first calculate numerically the Fourier transform of $P(\Delta S)$, then, use it as an approximation to $\psi(z)$ in Eq. (26) and calculate D_L. As shown in Fig. 9, $\ln D_L$ has a good linear dependence on $\ln t$ in both phases, implying that $D_L \propto t^{\gamma}$. In both phases, the numerically computed values of γ have been found quite close to the corresponding values of the exponent β, which were obtained from fitting of the LE decay by $M_f(t)$ in Eq. (16). That is, in both phases, $\gamma \simeq \beta$. Hence, $M_{sc}(t)$ in Eq. (29) decays in a way similar to the numerically computed LE, specifically, a stretched exponential decay in the tunneling phase and a Gaussian decay in the self-trapping phase in the case of relatively strong perturbation. Therefore, the semiclassical theory indeed can give explanations to the numerically observed decaying behaviors of the LE in the BEC system.

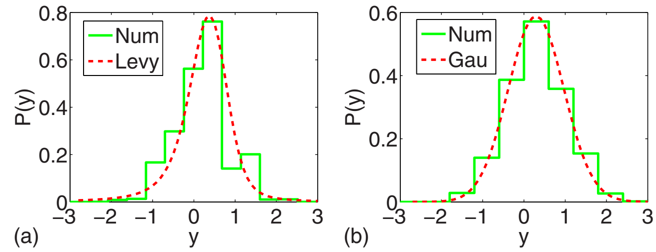

FIG. 8. (Color online) Histograms show the distribution $P(y)$ of the action difference ΔS for $t=80$, where $y=(\Delta S-\langle\Delta S\rangle)/\delta c$, with $\langle\Delta S\rangle$ the average value of ΔS. The distribution was calculated by taking 10^4 initial points in the phase space, corresponding to the initial coherent state $|\alpha=1.3\rangle$. (a): $c=1.2$ in the tunneling phase. The dotted curve is a fitting one calculated from a truncated ($y\in[-3,3]$) Levy distribution. (b): $c=5$ in the self-trapping phase. The dotted curve is a fitting Gaussian distribution.

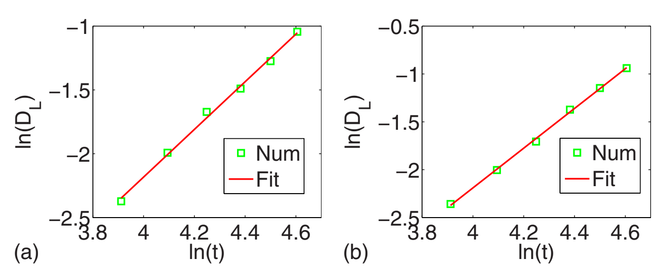

FIG. 9. (Color online) (a): Dependence of D_L on the time t for $c=1.2$ in the tunneling phase. Here D_L is the parameter in Eq. (29), calculated from fitting of $\psi(z)$ in Eq. (26) to the Fourier transform of the distribution $P(\Delta S)$. The squares represent numerically computed results and the solid straight line gives the linear fitting, $\ln D_L \propto \gamma \ln t$ with $\gamma=1.86$. (b): Similar to (a), except that $c=5$, D_L is the width of the fitting Gaussian distribution, and $\gamma=2.0$ for the fitting line. Note that in both cases $\gamma \simeq \beta$, where β is exponent of the stretched exponential decay of the LE shown in Fig. 6.

V. CONCLUSIONS

We have studied decaying behaviors of the LE in a BEC system in a double-well potential. This system has a classical counterpart with one pair of canonical variables, hence, is a regular system. The BEC system may undergo a dynamical phase transition between two phases: a tunneling phase and a self-trapping phase. Our numerical results show that the LE has Gaussian decay in both phases under sufficiently weak perturbation, which is in agreement with a previous semiclassical prediction. However, for relatively strong perturbation, the LE is found to obey different decaying laws on the two sides of the dynamical phase transition: Gaussian decay in the self-trapping phase and stretched exponential decay in the tunneling phase. This feature of the LE decay provides a proper characterization of the dynamical phase transition of the BEC system.

In order to understand the above mentioned numerical observations, we have performed a semiclassical analysis. The semiclassical theory shows that the LE indeed may have different types of decaying behaviors under weak and strong perturbation. Furthermore, the difference in the decaying law of the LE in the two phases at relatively strong perturbation, has its origin in the difference between the classical trajectories in the two cases. We show that certain properties of classical trajectories, specifically, properties of the distribution of certain action difference, are indeed responsible to the difference in the decaying law of the LE in the two phases. The semiclassical approach can quantitatively predict the exponents of the decay of the LE in the two phases, in particular, that of the stretched exponential decay in the tunneling phase.

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