Emergence of social cooperation in threshold public goods games with collective risk

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In real situations, people are often faced with the option of voluntary contribution to achieve a collective goal, for example, building a dam or a fence, in order to avoid an unfavorable loss. Those who do not donate, however, can free ride on others' sacrifices. As a result, cooperation is difficult to maintain, leading to an enduring collective-risk social dilemma. To address this issue, here we propose a simple yet effective theoretical model of threshold public goods game with collective risk and focus on the effect of risk on the emergence of social cooperation. To do this, we consider the population dynamics represented by replicator equation for two simplifying scenarios, respectively: one with fair sharers, who contribute the minimum average amount versus defectors and the other with altruists contributing more than average versus defectors. For both cases, we find that the dilemma is relieved in high-risk situations where cooperation is likely to persist and dominate defection in the population. Large initial endowment to individuals also encourages the risk-averse action, which means that, as compared to poor players (with small initial endowment), wealthy individuals (with large initial endowment) are more likely to cooperate in order to protect their private accounts. In addition, we show that small donation amount and small threshold (collective target) can encourage and sustain cooperation. Furthermore, for other parameters fixed, the impacts of group size act differently on the two scenarios because of distinct mechanisms: in the former case where the cost of cooperation depends on the group size, large size of group readily results in defection, while easily maintains cooperation in the latter case where the cost of cooperation is fixed irrespective of the group size. Our theoretical results of the replicator dynamics are in excellent agreement with the individual based simulation results.

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I. INTRODUCTION

A cooperator always benefits others with a cost to itself. Hence, everyone is faced with the temptation to defect. However, cooperation is ubiquitous in human and animal societies [1,2]. The puzzle why cooperative behavior can emerge in the real world attracts much attention recently [3–7]. The classical metaphors for investigating this social problem are the prisoner's dilemma game (PDG) [8,9] and the public goods game (PGG) [10]. The PDG gives emphasis to the cooperative behavior through pairwise interactions; whereas PGG, which captures the group interactions, focuses on the origin of cooperation in the conflict between individual interest and collective interest. In PGG, each cooperator contributes to the public pool with a cost to itself while each defector contributes nothing. The accumulated contribution is multiplied by an enhancement factor, and the total amount is distributed equally among all the individuals. With in a certain group, the payoff for a cooperator is always lower than that for a defector, it is better off defecting than cooperating, which leads to a social dilemma [4]. To solve such a dilemma, a number of mechanisms to enhance cooperation have been proposed, including repeated interactions, direct reciprocity, punishment [11–13], spatially structured populations [14–16], and voluntary participation in social interactions [17–19]. It is worthy of noting that this line of research has received increasing interests from the physics community (see, for example, a recent review [20]).

However, there are some social dilemmas which have the features of the classical PGG but cannot be characterized exactly by PGG, such as constructing a dam for the flood prevention and building a fence to avoid the aggression. In these mentioned social phenomena, each individual or community can also choose to contribute (cooperate) or not (defect). In order to meet a final collective target (which is also referred to as threshold), large scale social cooperation is required. In other words, the provision of public goods is completed if the total contribution meets or exceeds the threshold; otherwise, all individuals suffer with nothing irrespective of whether they contributed or not. For these social dilemmas, the effective framework commonly used is the threshold public goods game (TPGG) [21,22]. The TPGG has been intensively studied both theoretically [23,24] and experimentally [25,26]. Plenty of experimental results reveal that social cooperation in TPGG can be promoted in some situations, including sequential contribution mechanism [27], continuous contribution mechanism [28], and high step return mechanism [29,30].

Here, we aim to study the collective risk instead of deterministic loss if the collective goal is missed. Collective risk might be encountered when the provision of public goods fails [31]. In this case, the private goods are at stake with a certain probability. The mechanism of collective risk is substantially different from the above referred ones as it is based on the incentive to avoid a loss but not to obtain a gain. Motivated by this mechanism that has been studied experi-

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mentally in [31], we propose a model of TPGG incorporating the effect of collective risk to theoretically study the evolution of social cooperation.

Consider an infinite well-mixed population. From time to time, an interacting group of N agents is chosen at random among the whole population. Each of these N agents is provided with a fixed endowment W. Within such a group, there is a target to be accomplished, for example, constructing a dam. Specifically, building the dam requires a final contribution T collected by donating. For simplicity, we restrict participants to binary contributions (a fixed donation amount or nothing). Namely, each player can choose to donate either a fixed amount H (cooperate) or nothing at all (defect). Herein the donation amount H is also referred to as the cost of cooperation although the actual cost (final payout) depends on whether the collective target can be achieved. If the final target is reached, the dam can be constructed and the remaining private goods of each individual is prevented from losing. In this case, cooperators can keep whatever is left in their private account, $W-H$, and defectors own the whole endowment W. If the target is not completed, the dam cannot be built and the risk happens with probability p $(0 \leq p \leq 1)$. Once the danger occurs, all participants, including cooperators and defectors, lose their whole private goods. Whereas the danger does not happen, cooperators can hold what they had not invested in their private account, $W-H$, and defectors can keep the whole endowment W. Hence, cooperation is always worse off than defection in any given group. Because of this apparent disadvantage of cooperation, under what conditions social cooperation can emerge? To answer this question, we analyze the evolutionary population dynamics using the replicator equations [32–35] for two different scenarios. One scenario focuses on defectors versus fair sharers, who donate the fair share ($H=T/N$, which depends on the group size), and the other consists of defectors and altruists, who donate a fixed amount more than fair share ($H>T/N$). We shall investigate the impacts of risk (parameter p) and other factors (e.g., the initial amount of endowment) on the evolution of cooperation. We find that high risk as well as large initial endowment can significantly promote social cooperation. In addition, small cost of cooperation (donation amount) and small collective target also maintain cooperative behavior. Most interestingly, we observe different effects of group size on the two scenarios because of distinct mechanisms: a large group size tends to inhibit the emergence of cooperation in the case of fair sharers and defectors (where the collective target is only fulfilled when all individuals donate; thus the cost of cooperation is dependent of the group size) but enhances the sustainment of cooperators in the other situation of defectors versus altruists, where the cost of cooperation is fixed and irrespective of the group size. Furthermore, we confirm the validity of our theoretical results by individual based simulations.

The paper is organized as follows. For the case of fair sharers and defectors, the TPGG model with collective risk is proposed and discussed in Sec. II. In Sec. III, the corresponding model is introduced and investigated in detail for the case of altruists and defectors. Numerical simulations are presented in Sec. IV. Finally, conclusions are drawn in Sec. V

II. DYNAMICS OF POPULATIONS OF FAIR SHARERS AND DEFECTORS

First, we consider a simple situation that donators contribute minimum average amount, i.e., the amount $H=T/N$. In this case, the target T can be reached in the limit of all donating. Note that the initial endowment W needs to be larger than T/N (otherwise, public goods can never be provided). In a group consisting of n cooperators and $N-n$ defectors, the remainder in the private account for a cooperator and a defector is, respectively, given by

$$P_C(n) = \begin{cases} W - \dfrac{T}{N}, & n = N \\ (1-p)\left(W - \dfrac{T}{N}\right), & 0 < n < N \end{cases} \tag{1}$$

and

$$P_D(n) = (1-p)W, 0 \leq n < N. \tag{2}$$

It is easy to see that if the danger happens with certainty ($p=1$), individuals are better off donating than defecting. If the danger never happens ($p=0$), defection is the best choice. Nevertheless, if the danger happens with a probability $0<p<1$, some individuals want to save their interests by cooperating, whereas others are willing to gamble for the danger. If the donation amount for a cooperator exceeds the expected loss for a defector, i.e., $T/N > pW$ $[p<T/(NW)]$, defection dominates cooperation. The strategy profile of all defection is the unique Nash equilibrium. If $T/N<pW$ $[p>T/(NW)]$, everyone is better off if the public goods is provided than not. The "all fair sharers" set of strategies is a Nash equilibrium as no one can increase its remainder by changing its own strategy while the others stay the same. However, it is an "unstable" equilibrium because once one player changes its strategy, the remaining players may increase their expected remainders by altering their strategies. These changes lead to another "stable" Nash equilibrium, the profile of "all defectors."

The above discussion is only applied to one-shot game. Using replicator dynamics, we study the evolutionary behavior of repeated game. Denote the fraction of cooperators by x and that of defectors by y. We have $x+y=1$. The time evolution of this system is governed by the following differential equations

$$\begin{aligned} \dot{x} &= x(f_C - \bar{f}) \\ \dot{y} &= y(f_D - \bar{f}), \end{aligned} \tag{3}$$

where f_C is the expected remainder for a cooperator in a group of N players and f_D is that for a defector, $\bar{f}=xf_C+yf_D$ is the average remainder in the population.

In the well-mixed population, a group of N agents is chosen randomly, resulting in a random population composition. For a given fair sharer, the probability to find him in an N persons group consisting of j other fair sharers and $N-1-j$ defectors is

$$\binom{N-1}{j}x^j y^{N-1-j}.$$

In this case, the expected remainder in the private account for a fair sharer is

$$f_C = \sum_{j=0}^{N-1}\binom{N-1}{j}x^j y^{N-1-j}P_C(j+1). \tag{4}$$

Similarly, the expected remainder for a defector is

$$f_D = \sum_{j=0}^{N-1}\binom{N-1}{j}x^j y^{N-1-j}P_D(j). \tag{5}$$

Substituting equation $x+y=1$ and using

$$\sum_{j=0}^{N-1}\binom{N-1}{j}x^j(1-x)^{N-1-j} = 1,$$

Equations (4) and (5) can be transformed to

$$\begin{aligned} f_C &= (1-p)\left(W-\frac{T}{N}\right) + x^{N-1}p\left(W-\frac{T}{N}\right) \\ &= W-\frac{T}{N} - p\left(W-\frac{T}{N}\right) + x^{N-1}p\left(W-\frac{T}{N}\right) \end{aligned} \tag{6}$$

and

$$f_D = (1-p)W. \tag{7}$$

Denote the expected payout for a cooperator and a defector by $g_C=\frac{T}{N}+p(W-\frac{T}{N})-x^{N-1}p(W-\frac{T}{N})$ and $g_D=pW$, respectively. We can find that the payout for a cooperator is composed of the cost of cooperation T/N and the expected loss arising from the risk, whereas the payout for a defector is only due to the expected loss.

Further substituting $x+y=1$ into the first equation of Eq. (3), the dynamics of $x(t)$ is given by

$$\begin{aligned} \dot{x} &= x(1-x)(f_C-f_D) \\ &= x(1-x)(g_D-g_C) \\ &= x(1-x)\left[x^{N-1}p\left(W-\frac{T}{N}\right)-\frac{T}{N}(1-p)\right]. \end{aligned} \tag{8}$$

We focus on the steady state to which the population evolves. Let $\dot{x}=0$, we get all fixed points of the system. According to the value of the risk rate p, we distinguish three situations as follows:

(1) If $p=0$, Eq. (8) reduces to

$$\dot{x} = -\frac{T}{N}x(1-x).$$

This system has only two fixed points, $x=0$ and $x=1$. At $x=0$, the Jacobian is $J(x=0)=-T/N<0$, leading to a stable equilibrium. At $x=1$, the Jacobian is $J(x=1)=T/N>0$, leading to an unstable equilibrium. In this case, the population system converges to the state of all defectors. This result can also be derived from the comparison between the payout (or remainder) for a cooperator and a defector. The relationship $g_D-g_C=-T/N<0$ indicates that defection is always better off than cooperation. Thus, all individuals choose to defect. No one wants to contribute its savings to a dam for the flood prevention in a desert.

(2) If $p=1$, Eq. (8) becomes

$$\dot{x} = \left(W-\frac{T}{N}\right)x^N(1-x).$$

There are also only two fixed points $x=0$ and $x=1$. It is worth noting that the payout for a cooperator is less than that for a defector as $g_D-g_C=(W-\frac{T}{N})x^{N-1}$ is always positive in the whole interval (0,1), resulting in x as an increasing function of t irrespective of the initial state. Departure of the trajectory from the point $x=0$ shows that $x=0$ is an unstable equilibrium. At $x=1$, the Jacobian is $J(x=1)=-(W-\frac{T}{N})<0$. Thus, the fixed point $x=1$ is stable, leading to extinction of defectors. Accordingly, the population ends up with the steady state of all cooperators. In fact, if the danger happens with certainty, the best response for each individual is to donate unconditionally.

(3) If $0<p<1$, we also obtain the two boundary fixed points $x=0$ and $x=1$. At $x=0$, the Jacobian

$$J(x=0) = -\frac{T}{N}(1-p) < 0$$

shows that $x=0$ is a stable equilibrium. At $x=1$, the Jacobian is

$$J(x=1) = \frac{T}{N} - pW,$$

which is below zero in the case $p>T/(NW)$ and exceeds zero in the case $p<T/(NW)$. Thus, $x=1$ is a stable equilibrium if $p>T/(NW)$, and an unstable equilibrium, otherwise. Therefore, if the cost of cooperation T/N is lower than the expected loss of defection pW, cooperation can prevail and take over the population, depending on the initial abundance of cooperators. However, whenever $pW<T/N$, the expected loss of defection is less than the cost of cooperation. Thus cooperation is not a social optimum in this case. Individuals better do nothing rather than contribute. Even so, we still call these zero contributors as defectors in order to simplify the terminology.

To gain a complete picture of the dynamics, let us further analyze the existence and stability of the interior equilibrium. For $p<T/(NW)$, there is no interior equilibrium and for $p>T/(NW)$, there is a unique one in the interval (0,1). In order to prove this, we set $F(x)=f_C-f_D$. It is obvious that $F(0)=-\frac{T}{N}(1-p)<0$, $F(1)=p(W-\frac{T}{N})-\frac{T}{N}(1-p)$, and $F'(x)=(N-1)x^{N-2}p(W-T/N)>0$. For $p<T/(NW)$, $F(1)<0$ holds, and thus there is no interior equilibrium (see Fig. 1); for $p>T/(NW)$, we have $F(1)>0$, $F(0)<0$, and $F'(x)>0$, and hence there exists a unique root of $F(x)=0$ as $x^*=\sqrt[N-1]{\frac{T}{N}(1-p)/[p(W-\frac{T}{N})]}$ in (0,1). At the interior equilibrium x^*, the Jacobian is

$$J(x=x^*) = \frac{T}{N}(1-x^*)(1-p)(N-1) > 0,$$

resulting in an unstable equilibrium.

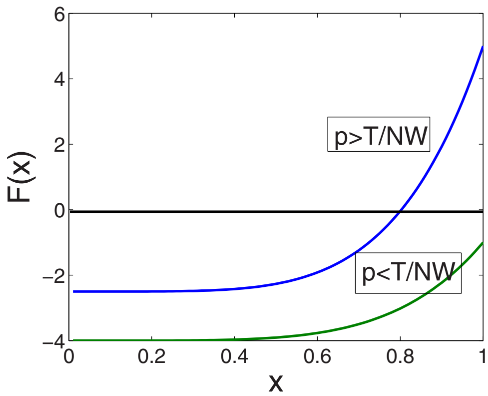

FIG. 1. (Color online) Difference between the expected remainder for a cooperator and a defector as a function of the fraction of cooperators.

To sum up, the collective risk brings in rich dynamics (see Table I). With increasing the risk rate, evolutionary dynamics transits from the dominance of defectors $[0 \leq p < T/(NW)]$ to the bistability between defectors and fair sharers $[T/(NW) < p < 1]$ and further to the dominance of fair sharers $(p=1)$ (Fig. 2).

Further, let us study the dependence of the resulting dynamics on the model parameters. If the expected loss of defection is larger than the cost of cooperation, i.e., $p > T/NW$, the steady states $x=0$ and $x=1$ are both stable. Which state that the system evolves to eventually depends on which attraction basin the initial state is located in. The interior equilibrium x^* separates the attraction basins of $x=0$ and $x=1$. If the initial state meets $x_0 < x^*$, the expected remainder for a cooperator in the sampled groups is less than that for a defector, that is, the payout for a cooperator surpasses that for a defector. Hence, the system ends up with all individuals donating nothing. Otherwise $(x_0 > x^*)$, the payout for a cooperator is below that for a defector (the remainder for a cooperator exceeds that for a defector). Thus, each individual chooses to donate the fair share. As a result, decreasing the equilibrium x^* broadens the attraction basin of $x=1$ and makes it easier to reach full cooperation, promoting the emergence of cooperation. Note that the interior equilibrium x^* is a decreasing function of the risk rate p. With an increasing rate p, the attraction basin of the state $x=0$ is reduced, and therefore a lower initial abundance of cooperators is needed to maintain cooperation. In this situation $(x_0 > x^*)$, both remainders for a cooperator and a defector decrease when the risk rate p increases. However, the payout for a defector may exceed that for a cooperator as the increasing rate of the former with respect to the risk rate p is higher than the latter. Accordingly, the remainder for a cooperator is more likely to be larger than that for a defector in a more risky environment, inducing the prevalence of cooperation. Overall, cooperation may be favored under high risk situations.

TABLE I. Stability of equilibria in systems of fair sharers and defectors.

			$0<p<1$	
	$p=0$	$p=1$	$p<\frac{T}{NW}$	$p>\frac{T}{NW}$
$x=0$	Stable	Unstable	Stable	Stable
x^*				Unstable
$x=1$	Unstable	Stable	Unstable	Stable

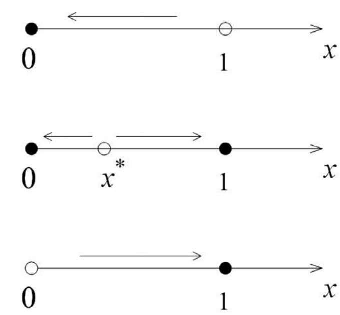

FIG. 2. Evolutionary dynamics of the population consisting of fair sharers and defectors. Filled and open circles represent the stable and unstable fixed points, respectively. Arrows indicate the evolutionary direction. (a) $0 \leq p < T/(NW)$; (b) $T/(NW) < p < 1$; and (c) $p=1$.

In addition, the attraction basin of the state $x=0$ shrinks with the increase in the endowment W and the decrease in the final target T. Hence, it is easier for the population to reach all cooperators with larger endowment and smaller target sum. Besides, noticeably, the interior equilibrium x^* approaches to $x=1$ when the group size N increases to infinite. Namely, for sufficiently large group size, the attraction basin of the state $x=1$ is vanishing. Therefore, large group size has a negative impact on the emergence of cooperation in the case of fair sharers and defectors.

This result is consistent with previous studies of PGG involving voluntary participation in Ref. [19] where the authors stated that cooperation is favored for small effective interaction group size. Since with decreasing the effective interaction group size S, condition $r > S$ (r is the enhancement factor) is easily satisfied, and hence the net payoff for a cooperator readily tends to exceed that for a defector. In our model, although decreasing the group size raises the cost of cooperation, it becomes more likely to pick up N cooperators, which form an interaction group and can complete the collective target. Consequently, cooperation can gain a foothold in the population for a small group size compared with a large group size, as the expected remainder for a cooperator is lightly larger than that for a defector.

III. DYNAMICS OF POPULATIONS OF ALTRUISTS AND DEFECTORS

Let us consider the situation of altruists versus defectors. Altruist is a type of player who contributes more than the fair share $(H > T/N)$. In order to analyze the emergence of social cooperation in this case, we have to distinguish two scenarios below.

A. Case 1: T can be divisible by H

In this case, an altruist in an N-player group has reminder in its private account

$$P_A(n) = \begin{cases} W - H, & m \le n \le N \\ (1-p)(W-H), & 0 < n < m, \end{cases} \quad (9)$$

where n is the number of altruists in the group and $m = T/H < N$. The remainder for a defector in the same group is given by

$$P_D(n) = \begin{cases} W, & m \le n < N \\ W(1-p), & 0 \le n < m. \end{cases} \quad (10)$$

Similarly, we get the expected remainder in the personal account for an altruist and a defector, respectively, as follows:

$$f_A = W - H - p(W-H)\sum_{j=0}^{m-2} \binom{N-1}{j} x^j (1-x)^{N-1-j},$$

$$f_D = W - pW \sum_{j=0}^{m-1} \binom{N-1}{j} x^j (1-x)^{N-1-j}. \quad (11)$$

The payout for an altruist and a defector is, respectively, given by

$$g_A = H + p(W-H)\sum_{j=0}^{m-2} \binom{N-1}{j} x^j (1-x)^{N-1-j},$$

$$g_D = pW \sum_{j=0}^{m-1} \binom{N-1}{j} x^j (1-x)^{N-1-j}.$$

The corresponding replicator equation is

$$\dot{x} = x(1-x)(f_A - f_D) = x(1-x)(g_D - g_A).$$

Let $F(x) = f_A - f_D (= g_D - g_A)$. We get

$$F(x) = -H + pW\binom{N-1}{m-1} x^{m-1}(1-x)^{N-m} + Hp\sum_{j=0}^{m-2} \binom{N-1}{j} x^j (1-x)^{N-1-j}. \quad (12)$$

The boundary fixed point $x=1$ is an unstable equilibrium as the Jacobian is $J(x=1) = H > 0$, while the other boundary fixed point $x=0$ is a stable equilibrium as a result of the Jacobian $J(x=0) = -H + Hp < 0$, with $p < 1$. Let us consider the interior fixed points which are roots of the function $F(x)$. We only concentrate on the real roots of the function $F(x)$ in the interval $[0,1]$ and the situation of no multiple root. Labeling these real roots according to an increasing rank order, we claim that all fixed points appear alternately between stable and unstable as the first derivative of $x(1-x)F(x)$ at these fixed points alternately changes signs between negative and positive (the proof can be found in the Appendix). The stability of the interior fixed points is summarized in Table II.

In the case of $p=1$, the replicator equation is simplified as

TABLE II. Stability of all fixed points in systems of altruists and defectors (case IIIA). k is the number of interior fixed points. It is a positive even number or zero in the case of $p<1$ and $p=1$ $[F(x) < 0]$, while it is a positive odd number in the situation of $p=1$ $[F(x) > 0]$. $0 < x_1 < x_2 < \cdots < x_k < 1$. The stability of the interior fixed points listed in the table is obtained under the assumption of no multiple root. If there exists a multiple root, it may be stable, unstable, and saddle node. The inequalities $F(x) > 0$ and $F(x) < 0$ need to be satisfied in the right neighborhood of $x=0$.

		$p=1$	
	$p<1$	$F(x)<0$	$F(x)>0$
$x=0$	Stable	Stable	Unstable
$x=x_1$	Unstable	Unstable	Stable
$x=x_2$	Stable	Stable	Unstable
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x=x_k$	Stable	Stable	Stable
$x=1$	Unstable	Unstable	Unstable

$$\begin{aligned} \dot{x} &= x(1-x)\left[-H + W\binom{N-1}{m-1} x^{m-1}(1-x)^{N-m} + H\sum_{j=0}^{m-2} \binom{N-1}{j} x^j (1-x)^{N-1-j} \right] \\ &= x(1-x)\left[W\binom{N-1}{m-1} x^{m-1}(1-x)^{N-m} - H\sum_{j=m-1}^{N-1} \binom{N-1}{j} x^j (1-x)^{N-1-j} \right]. \end{aligned} \quad (13)$$

Note that if $F(x) < 0$ holds in the right neighborhood of $x=0$, i.e., $\dot{x} = x(1-x)F(x) < 0$, then a sufficiently small fraction of cooperators will decrease to zero, resulting in a stable equilibrium $x=0$. Whereas if $F(x) > 0$ is always satisfied in the right neighborhood of $x=0$, the departure of the small fraction of cooperators from the state $x=0$ shows that the fixed point $x=0$ is no longer stable, implying that cooperation cannot disappear. Thus, elements which can lead to $F(x) > 0$ in the right neighborhood of $x=0$ also promote the emergence of cooperation. By observing Eq. (13), we find that cooperation can be sustained by increasing the initial endowment W and the group size N, or decreasing the target sum T and the cost of cooperation H for a fixed number of donators needed to accomplish the final goal. In fact, if the danger happens consequentially, for larger initial endowment W and smaller cost of cooperation H, more individuals consider it worthy of saving their private goods by donating, thus promoting the emergence of cooperation. The decrease in the target sum T gives rise to more free riders and fewer donators that are needed to collect the goal. In this situation, the smaller number of donators is more readily satisfied to fulfill the goal, and therefore more persistent cooperation is found. Regarding the group size, let us explain why its increase enhances the maintenance of cooperation. The prob-

TABLE III. Stability of all fixed points in systems of altruists and free riders (case IIIB). k is the number of interior fixed points. It is a positive even number or zero when one of $x=0$ and $x=1$ is stable, whereas the other one is unstable, while it is a positive odd number when both $x=0$ and $x=1$ are stable or unstable. $0<x_1<x_2<\cdots<x_k<1$. The stability of the interior fixed points listed in the table is obtained under the assumption of no multiple root. If there exists a multiple root, it may be stable, unstable, and saddle node. The inequalities $F(x)>0$ and $F(x)<0$ need to be satisfied in the right neighborhood of $x=0$.

	$p=1$				$p<1$		
	$\frac{T}{N-1}\geq H>\frac{T}{N}$		$H>\frac{T}{N-1}$		$\frac{T}{N-1}\geq H>\frac{T}{N}$		
	$F(x)<0$	$F(x)>0$	$F(x)<0$	$F(x)>0$	$p>\frac{H}{W}$	$p<\frac{H}{W}$	$H>\frac{T}{N-1}$
$x=0$	Stable	Unstable	Stable	Unstable	Stable	Stable	Stable
$x=x_1$	Unstable	Stable	Unstable	Stable	Unstable	Unstable	Unstable
$x=x_2$	Stable	Unstable	Stable	Unstable	Stable	Stable	Stable
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$x=x_k$	Unstable	Unstable	Stable	Stable	Unstable	Stable	Stable
$x=1$	Stable	Stable	Unstable	Unstable	Stable	Unstable	Unstable

ability to find an N_1 persons group consisting of m altruists and N_1-m defectors is

$$\binom{N_1}{m}x^m y^{N_1-m}.$$

Similarly, the probability to find an N_2 group ($N_1<N_2$) composed of m altruists and N_2-m defectors is

$$\binom{N_2}{m}x^m y^{N_2-m}.$$

Notice that when $x\ll 1$, the first probability is smaller than the second one. Namely, for sufficiently small x, large group size raises the probability to complete the final target, thus encouraging the cooperative behavior even when rare cooperators are present in the population. Hence, increasing the size of group can promote the emergence of cooperation in our model.

B. Case 2: *T* cannot be divisible by *H*

Similarly, in this case, an altruist in an N-player group has the remainder

$$P_A(n)=\begin{cases}W-H, & [m]+1\leq n\leq N\\ (1-p)(W-H), & 0<n\leq[m],\end{cases} \quad (14)$$

where n is the number of altruists in the group and "[]" represents the integralized Gauss mark. The remainder for a defector in the same group is given by

$$P_D(n)=\begin{cases}W, & [m]+1\leq n<N\\ W(1-p), & 0\leq n\leq[m].\end{cases} \quad (15)$$

Then, we obtain the expected remainder for an altruist and a defector, respectively, as follows:

$$f_A=W-H-p(W-H)\sum_{j=0}^{[m]-1}\binom{N-1}{j}x^j(1-x)^{N-1-j}=W-g_A$$

$$f_D=W-pW\sum_{j=0}^{[m]}\binom{N-1}{j}x^j(1-x)^{N-1-j}=W-g_D. \quad (16)$$

In this situation, the replicator equation is given by

$$\dot{x}=x(1-x)\left[-H+pW\binom{N-1}{[m]}x^{[m]}(1-x)^{N-[m]-1}+Hp\sum_{j=0}^{[m]-1}\binom{N-1}{j}x^j(1-x)^{N-1-j}\right]. \quad (17)$$

The stability of each fixed point is listed in Table III.

In what follows, we focus on investigating what ingredients can enhance the emergence of cooperation when the boundary fixed point $x=0$ is stable and $x=1$ is unstable. Cooperation is apparently disadvantageous in this situation. It is worth noting that all fixed points appear alternately between stable and unstable as the first derivative of the right-hand side of the replicator dynamics at these fixed points changes alternately between negative and positive. We also suppose $F(x)=f_A-f_D(=g_D-g_A)$. As $F(0)<0$ and $F(1)<0$, the number of the interior fixed points is either zero or a positive even number, which guarantees the interior fixed points to appear alternately between stable and unstable in this case. Furthermore, the nearest fixed point to the stable equilibrium $x=0$ in the interval (0,1), $x^{\dagger}$, must be unstable, thus determining the attraction basin of the state $x=0$. With the decrease in the fixed point $x^{\dagger}$, the attraction basin of $x=0$ is reduced. Accordingly, the probability with which the population system evolves to the steady state of all defectors drops, promoting the emergence of the cooperative behavior. In addition, the nearest interior fixed point to the unstable equilibrium $x=1$, denoted by $x\ddagger$, must be stable, representing the largest possible cooperation level. This possible cooperation level declines with decreasing $x\ddagger$ and vice versa. To some extent, the fixed point $x\ddagger$ indicates how favorable the cooperative behavior is.

Therefore, we concentrate on the smallest as well as the largest interior fixed points which are two roots of the func-

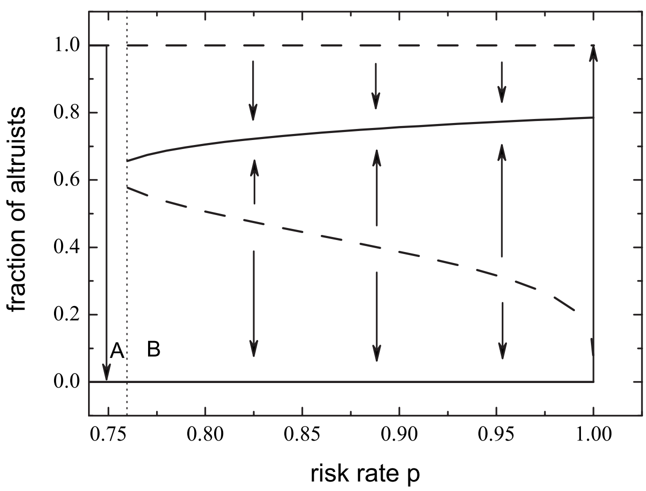

FIG. 3. The stationary cooperation level as a function of the risk rate p. The arrows indicate the stability of the fixed points (evolutionary direction of the population): solid lines represent the stable fixed points, while dashed lines represent the unstable ones. In phase $A(p<0.76)$, defectors dominate cooperators. While in phase $B(0.76\leq p<1)$, defectors and cooperators may coexist at the interior equilibrium. With a raised rate of loss p, the attraction basin of the state $x=0$ decreases and the largest possible cooperation level increases. When the risk rate $p=1$, cooperators dominate defectors. It shows that high risk rate p can enhance the emergence of cooperation and promote the cooperation level. Parameters: $W=20$, $T=40$, $H=7$, and $N=8$.

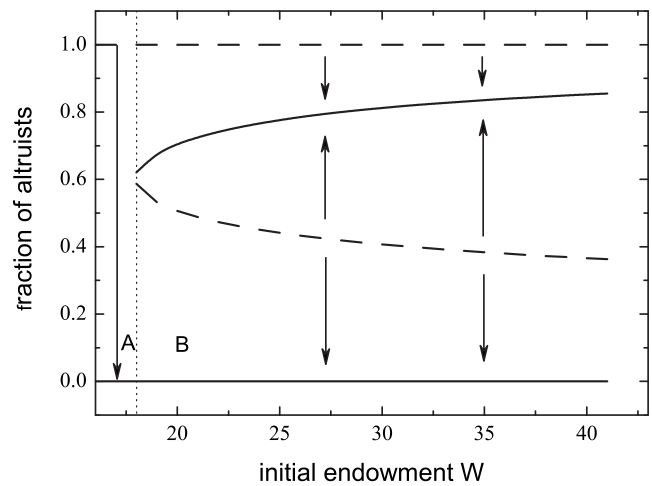

FIG. 4. The stationary cooperation level as a function of the initial endowment W. The arrows indicate the stability of the fixed points (evolutionary direction of the population): solid lines represent stable fixed points, while dashed lines represent the unstable ones. In phase A, defectors dominate cooperators. Whereas in the next phase B, defectors and cooperators may coexist at the interior equilibrium. With the increase in the initial endowment W, the attraction basin of the state $x=0$ is reduced and the largest possible cooperation level is raised. It shows that large initial endowment can enhance the emergence of cooperation and increase the cooperation level. Parameters: $T=40$, $H=7$, $N=8$, and $p=0.8$.

tion $F(x)$. As shown in Fig. 3, when the risk rate $p<0.76$ (denoted by phase A), no interior fixed point can be observed. There are only two equilibria $x=0$ and $x=1$, which are stable and unstable, respectively, resulting in no donation behavior eventually. When the risk rate $p\geq 0.76$ (phase B in Fig. 3), two interior fixed points $x^{\dagger}$ and $x\ddagger$ appear, which are unstable and stable, respectively. The transition from phase A to phase B occurs as the risk rate increases. Furthermore, when the risk rate p is raised, $x^{\dagger}$ decreases and $x\ddagger$ increases monotonously. The decrease in $x^{\dagger}$ indicates that the attraction basin of $x=0$ is reduced. Therefore the population reaches the state of all defectors with a lower probability, enhancing the emergence of cooperation. Additionally, the increase in $x\ddagger$ demonstrates that the largest possible cooperation level is improved. Hence, high rate of risk p is an alternative mechanism to enhance the emergence of cooperation and promote the cooperation level.

Increasing the initial endowment W can also advance cooperation (Fig. 4). For a fixed donation amount, the poor people think it is worthless to protect their private accounts by donating a large amount compared with the initial endowment. Instead, they would like to bear a risk, leading to no donators in the limit of small initial endowment. On the contrary, people with large initial endowment are not willing to run a risk. They come forward to donate with a high probability. The incentive for the rich people to avoid risk catalyzes the emergence of cooperation and also enhances the cooperation level. As a consequence, large initial endowment can promote the evolution of cooperation.

The effects of donation amount H, final target T, and group size N on cooperative behavior are complex, resulting in rich evolutionary dynamics. Let us first investigate the effect of donation amount. If $H<T/N$, the target cannot be reached even all individuals donate. There is only one stable fixed point, $x=0$. It shows that no one wants to cooperate in this case. For $H=T/N$, the target can be reached only if every individual donates the amount H. This situation, which is already analyzed in detail in Sec. II, is corresponding to the phase A in Fig. 5(a). The states $x=0$ as well as $x=1$ are both stable, while the unique interior fixed point x^{*} is unstable, separating the two attraction basins of all free riders and all fair sharers. For further increased H (in phase B), it is not necessary for all individuals to donate in order to complete the goal. The system even allows one individual to free ride but the target can still be reached. In this case, the fixed point $x=0$ is still stable but $x=1$ changes to be unstable. Moreover, a new stable interior fixed point $x\ddagger$ which is adjacent to $x=1$ appears. The original unstable equilibrium $x^{\dagger}$ in region A parts the two attraction basins of $x=0$ and $x\ddagger$ in region B. In addition, the state $x^{\dagger}$ and $x\ddagger$ ascends and descends, respectively, as H increases. It indicates that there is less incentive to collect the public goods with larger cost of cooperation. In the next phase C, the system allows at most two free riders under the premise of the provision of the public goods. The local dynamics in phase C is similar with that in phase B. However, in contrast with dynamics in phase B, the interior equilibrium $x^{\dagger}$ as well as $x\ddagger$ decline. This transition between phases B and C stems from the decrease in the number of donators needed to accomplish the target with increasing H. It is easier to meet the number, elevating the probability of completing the target. Accordingly, the persistence of cooperation is improved. On the other hand, since more free riders are permitted in a certain group to accomplish the goal,

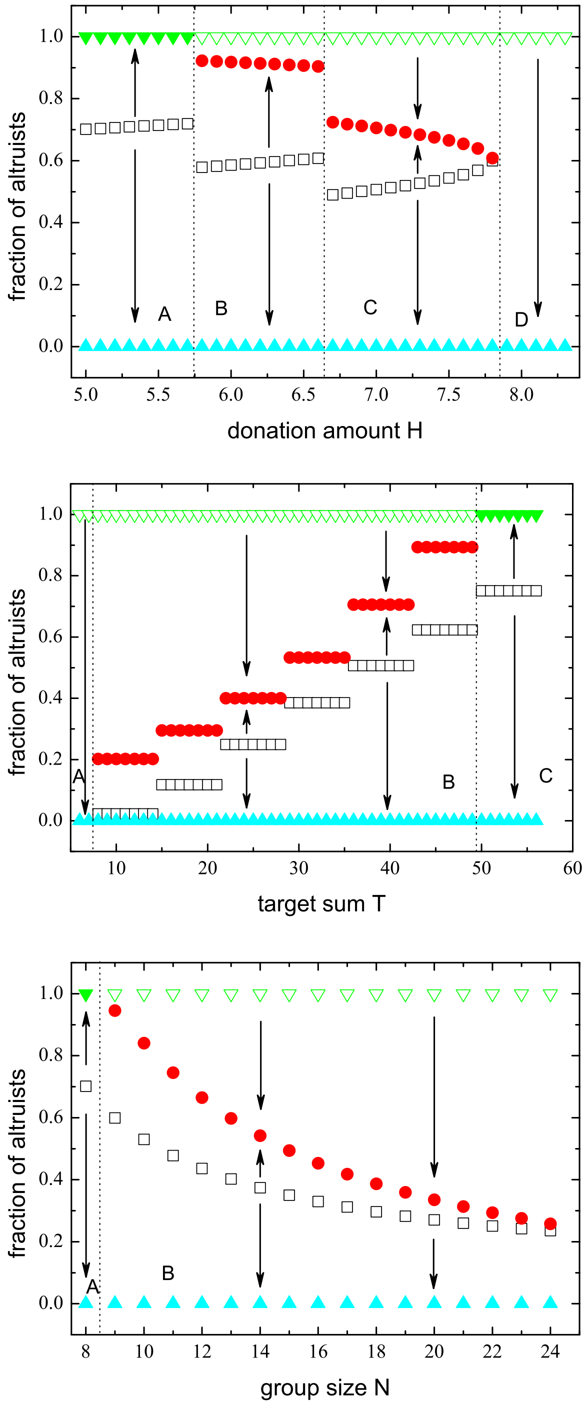

FIG. 5. (Color online) The steady cooperation level as functions of (a) the donation amount H, (b) the threshold T, and (c) the group size N. The open symbols represent the unstable fixed points, while the filled symbols represent the stable ones. Parameters: (a) $T=40$, $N=8$, $W=20$, and $p=0.8$; (b) $W=20$, $H=7$, $N=8$, and $p=0.8$; and (c) $T=40$, $W=20$, $H=5$, and $p=0.8$.

many more individuals choose to free ride, inhibiting the cooperation level. Moreover, when H exceeds everyone's upper limit for the donation amount, no one thinks it is worth to donate for the provision of the public goods, which corresponds to the only one stable equilibrium, $x=0$ [as shown in phase D in Fig. 5(a)].

The impact of the target T is shown in Fig. 5(b). The dynamics with increasing T is similar to that with decreasing H. The phases A, B, and C in Fig. 5(b) are corresponding to D, B and C, A in Fig. 5(a), respectively. With ascending target T, both fixed points $x^{\dagger}$ as well as $x\ddagger$ are raised. The population ends up with all defectors with higher probability, and the largest possible cooperation fraction is also raised. We should point out that the population may settle at a high cooperation level (for example, $x\ddagger=0.89$ or 0.71) but is often faced with a high probability of having all defectors (the corresponding attraction basin of all defectors is $x<0.62$ or $x<0.51$). Thus, large target sum T encourages the defection and inhibits the emergence of cooperation.

Figure 5(c) depicts the effect of the group size N. In phase A, the target will be reached only if all individuals donate, which is similar to the situation in phase A of Fig. 5(a). In the next phase, increasing the size of the group permits more people to free ride as long as the target can be completed. The transition is similar to that from phase B to C in Fig. 5(a). Hence, the attraction basin of the state $x=0$ drops. Moreover, many free riders allowed induces the incentive to share the collective interest without any donation, thus reducing the largest possible cooperation level. Therefore, in this case, large group size enhances the maintenance of cooperation but reduces the largest possible cooperation level as well.

Actually, this result regarding the effect of group size does not contradict the conclusion in Sec. II as the two phenomena are based on distinct mechanisms. In the case of fair sharers and defectors, increasing the group size N decreases the cost of cooperation, but also raises the number of donators needed for achieving the final target. Consequently, it becomes harder to have all donators in a given group so that the collective target can be completed. Frequent failures of the target destroy the cooperative behavior. Hence, in Sec. II, large group size hinders the emergence of cooperation. However, in the case of altruists and defectors, the cost of cooperation is fixed, and the number of necessary donators is also constant. Similarly to the case A, the probability of successfully collecting the target for an N_1 persons group is less than an N_2 persons group ($N_1<N_2$) in the limit of small fraction of cooperators. Accordingly, increasing group size N encourages the cooperative behavior, thus enhancing the conservation of cooperation even with rare cooperators.

IV. SIMULATIONS

In order to verify the results derived from the replicator equation, we perform individual based simulations for our model. Here, we adopt asynchronous update rule. In the simulations, we consider a well-mixed population consisting of M individuals. All individuals are treated as equivalent in all respects and the evolutionary process depends on the remainder of individuals. Initially, half of the population are assigned as cooperators. Each individual is provided with a fixed endowment. At each time step, an individual i and a group of $N-1$ ($N \ll M$) individuals competing with i are randomly chosen. These N individuals play the public goods game together. The remainder for individual i, P_i, is calculated according to Eqs. (14) and (15) [or Eqs. (9) and (10)]. Similarly, another randomly chosen individual j in another randomly chosen sample of N individuals has the remainder P_j. If $P_i > P_j$, the individual j changes his strategy to that of i with a probability given by $p(j \rightarrow i) = (P_i - P_j)/\pi$, where π is the normalized factor being the possible maximum remainder difference between P_i and P_j. Analogously, individual i adopts the strategy of j with the probability $p(i \rightarrow j) = (P_j - P_i)/\pi$ if $P_j > P_i$. The numerical simulation results are in excellent agreement with the analytical results from the replicator equation (see Fig. 6).

V. CONCLUSIONS

In summary, we have studied the effects of collective risk as well as other factors on the evolution of cooperation in threshold public goods games. This simple model presents the characteristics of some long-standing collective-risk social dilemmas in human societies. Our theoretical model is largely different from the previously most studied public goods games in the sense that provision of public goods in previous models is based on the incentive to gain more from the goods in return, whereas our model focuses on the situation in which individuals voluntarily contribute to avoid the unfavorable loss rather than to earn profit. The evolutionary dynamics obeys a simple rule: if the payout for a cooperator is higher than that for a defector, defection prevails, whereas cooperation is favored otherwise. For practical purpose, we distinguish two simplifying cases to discuss, one with fair sharers and defectors, the other with altruists and defectors. In the former case, we found that if the rate of loss p is less than $T/(NW)$, the cost of cooperation surpasses the expected loss of defection. In this situation, there is no incentive to cooperate as the payout for a cooperator exceeds that for a defector, resulting in the extinction of cooperation. The population evolves to the steady state of all defectors. If $p > T/(NW)$, the cost of cooperation is inferior to the expected loss of defection. The payout for a cooperator is possible to surmount that for a defector depending on the initial abundance of cooperators, and thus the cooperation can be maintained. The two strategies, donating the fair share and defecting, are bistable. Reaching the steady state of all cooperators is more likely as the risk rate p and the initial endowment W increase or as the final target T decreases. For the group size, large N suppresses the emergence of social cooperation. In the latter case, the result shows that the state $x=1$ is always unstable except in the situation of $p=1$ and $N-1 < T/H < N$ as the provision of public goods does not require all participants to donate. For a low risk rate, only defectors exist eventually as no one is willing to avoid such a small loss with a large cost to itself. With large p, in contrast, the social cooperation can emerge if the initial frequency of cooperators overtops some specific value. Furthermore, with an increasing risk rate p, the probability to the steady state of all defectors drops and the largest fraction of cooperators (can also be referred to as largest possible cooperation level) is promoted. In addition, we also found the conservation of cooperation can be significantly enhanced by large initial endowment, small target sum, small cost of cooperation (under the condition that the number of necessary donators to fulfill the collective target is unchanged), and large group size which plays an opposite role as compared to the former case because of different mechanisms. Moreover, the largest possible cooperation level is raised by large initial endowment, while it declines with smaller final goal, larger donation

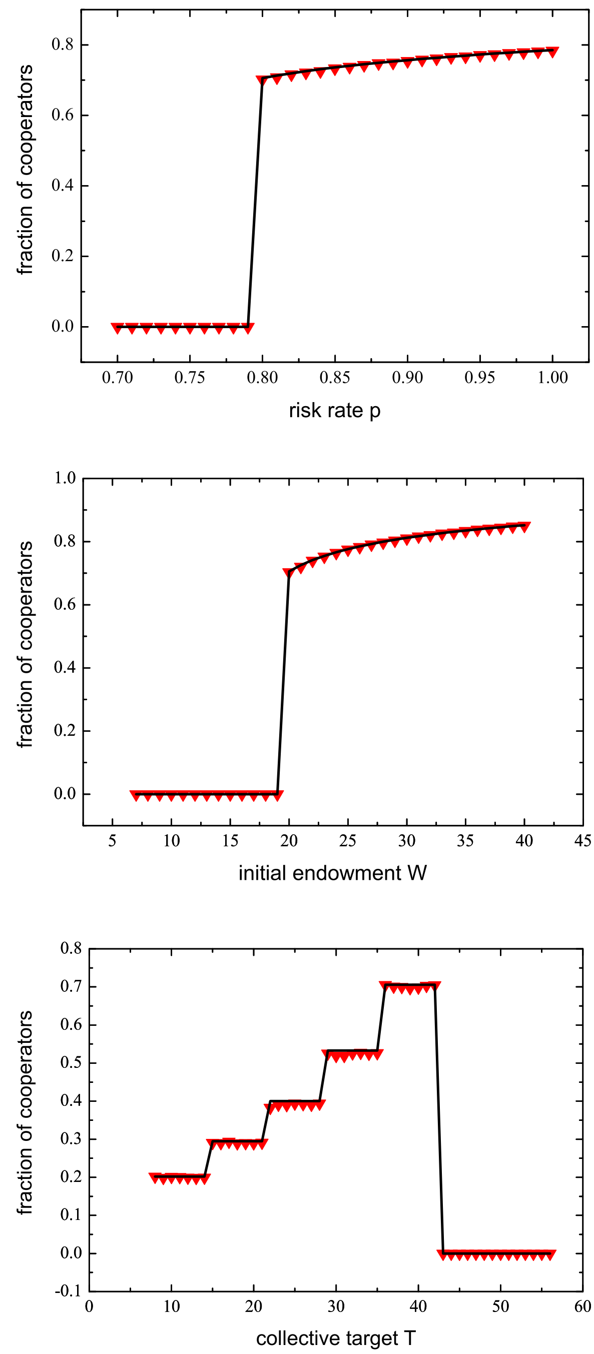

FIG. 6. (Color online) Comparisons of the simulation results with the analytical results. We apply the same parameters used in Figs. 3, 4, and 5(a), respectively. We set $M=1000$ and update for 20 000 time steps. The line represents the analytical results derived from the replicator dynamics and the symbols represent the simulation results. Each simulation data point is an average over ten realizations. It is found that in each case, the two results are in excellent agreement.

amount, and larger group size. In addition, we found excellent agreement between our theoretical results and the individual based simulations. Our present work demonstrates that high-risk rate is an alternative mechanism to enhance the emergence of social cooperation. We hope that our work might offer some insight into promoting social cooperation in collective-risk situations, where individuals want to avoid the unfavorable loss rather than to gain profits.

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APPENDIX: THE FIRST DERIVATIVE AT THE REAL ROOTS OF A POLYNOMIAL CHANGES SIGNS ALTERNATELY BETWEEN NEGATIVE AND POSITIVE

Let $G(x)=x(1-x)F(x)$. In what follows, we demonstrate that the first derivative at the real roots of $G(x)$ changes signs alternately between negative and positive.

As function $G(x)$ is a polynomial which can be expressed as

$$G(x) = c\prod_{i=1}^{l}(x-x_i)\prod_{j}(x^2+a_jx+b_j), \tag{A1}$$

where x_i is the real root of $G(x)$ and l is the number of real roots. The expression is satisfied for any $i_1 \neq i_2$ and $x_{i_1} \neq x_{i_2}$. Besides, there must be $a_j^2 < 4b_j$ for any j. The first derivative of the polynomial is given by

$$G'(x) = \left(\sum_i \frac{1}{x-x_i} + \sum_j \frac{2x+a_j}{x^2+a_jx+b_j}\right)G(x). \tag{A2}$$

Consequently, we have the first derivative at the fixed point x_k as follows:

$$\begin{aligned} G'(x_k) &= c\prod\nolimits_{i\neq k}(x_k-x_i)\prod\nolimits_j(x_k^2+a_jx_k+b_j) \\ &\quad + \left(\sum\nolimits_{i\neq k}\frac{1}{x_k-x_i} + \sum_j \frac{2x_k+a_j}{x_k^2+a_jx_k+b_j}\right)G(x_k) \\ &= c\prod\nolimits_{i\neq k}(x_k-x_i)\prod\nolimits_j(x_k^2+a_jx_k+b_j). \end{aligned} \tag{A3}$$

The first derivative at the fixed point x_{k+1} is obtained similarly as

$$G'(x_{k+1}) = c\prod_{i\neq k+1}(x_{k+1}-x_i)\prod_j(x_{k+1}^2+a_jx_{k+1}+b_j). \tag{A4}$$

Multiplying $G'(x_k)$ and $G'(x_{k+1})$, we get

$$\begin{aligned} &G'(x_k)G'(x_{k+1}) = -c^2(x_k-x_{k+1})^2, \\ &\prod_{i=1}^{k-1}(x_k-x_i)(x_{k+1}-x_i)\prod_{i=k+2}^{l}(x_k-x_i)(x_{k+1}-x_i), \\ &\prod_j(x_k^2+a_jx_k+b_j)(x_{k+1}^2+a_jx_{k+1}+b_j). \end{aligned} \tag{A5}$$

Note that $x_1 < \cdots < x_k < x_{k+1} < \cdots < x_l$. Equation (A5) guarantees $G'(x_k)G'(x_{k+1}) < 0$. It shows that the signs of the first derivatives of any two contiguous fixed points are opposite.

[1] K. G. Binmore, *Playing Fair: Game Theory and the Social Contract* (MIT, Cambridge, 1994).
[2] A. M. Colman, *Game Theory and Its Applications in the Social and Biological Sciences* (Butterworth-Heinemann, Oxford, 1995).
[3] M. G. Zimmermann and V. M. Eguíluz, Phys. Rev. E **72**, 056118 (2005).
[4] C. Hauert, F. Michor, M. A. Nowak, and M. Doebeli, J. Theor. Biol. **239**, 195 (2006).
[5] W.-X. Wang, J. Ren, G. Chen, and B.-H. Wang, Phys. Rev. E **74**, 056113 (2006).
[6] J. Gómez-Gardeñes, M. Campillo, L. M. Floría, and Y. Moreno, Phys. Rev. Lett. **98**, 108103 (2007).
[7] D. Okada and P. M. Bingham, J. Theor. Biol. **253**, 261 (2008).
[8] F. C. Santos and J. M. Pacheco, Phys. Rev. Lett. **95**, 098104 (2005).
[9] G. Szabó, J. Vukov, and A. Szolnoki, Phys. Rev. E **72**, 047107 (2005).
[10] *The Handbook of Experimental Economics*, edited by J. H. Kagel and A. E. Roth (Princeton University Press, Princeton, 1995).
[11] K. Sigmund, C. Hauert, and M. A. Nowak, Proc. Natl. Acad. Sci. U.S.A. **98**, 10757 (2001).
[12] E. Fehr and S. Gachter, Nature (London) **415**, 137 (2002).
[13] C. Hauert, A. Traulsen, H. Brandt, M. A. Nowak, and K. Sigmund, Science **316**, 1905 (2007).
[14] G. Szabó and C. Hauert, Phys. Rev. Lett. **89**, 118101 (2002).
[15] J.-Y. Guan, Z.-X. Wu, and Y.-H. Wang, Phys. Rev. E **76**, 056101 (2007).
[16] F. C. Santos, M. D. Santos, and J. M. Pacheco, Nature (London) **454**, 213 (2008).
[17] C. Hauert, S. D. Monte, J. Hofbauer, and K. Sigmund, J. Theor. Biol. **218**, 187 (2002).
[18] G. Szabó and C. Hauert, Phys. Rev. E **66**, 062903 (2002).
[19] C. Hauert, M. Holmes, and M. Doebeli, Proc. R. Soc. London, Ser. B **273**, 2565 (2006).
[20] G. Szabó and G. Fáth, Phys. Rep. **446**, 97 (2007).
[21] A. J. C. van de Kragt, J. Orbell, and R. M. Dawes, Am. Polit. Sci. Rev. **77**, 112 (1983).
[22] M. Bagnoli and M. Mckee, Econom. Inquiry **29**, 351 (1991).

[23] M. Bagnoli and B. L. Lipman, Rev. Econ. Stud. **56**, 583 (1989).
[24] L. A. Bach, T. Helvik, and F. B. Christiansen, J. Theor. Biol. **238**, 426 (2006).
[25] C. B. Cadsby and E. Maynes, J. Econ. Behav. Organ. **34**, 603 (1998).
[26] M. Marks and R. Croson, J. Public Econ. **67**, 195 (1998).
[27] I. Erev and A. Rapoport, J. Conflict Resolut. **34**, 401 (1990).
[28] C. B. Cadsby and E. Maynes, J. Public Econ. **71**, 53 (1999).
[29] R. Croson and M. Marks, Exp. Econ. **2**, 239 (2000).
[30] C. B. Cadsby, R. Croson, M. Marks, and E. Maynes, Public Choice **135**, 277 (2008).
[31] M. Milinski, R. D. Sommerfeld, H. J. Krambeck, F. A. Reed, and J. Marotzke, Proc. Natl. Acad. Sci. U.S.A. **105**, 2291 (2008).
[32] P. Taylor and L. Jonker, Math. Biosci. **40**, 145 (1978).
[33] J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics* (Cambridge University Press, Cambridge, 1998).
[34] A. Traulsen, J. C. Claussen, and C. Hauert, Phys. Rev. Lett. **95**, 238701 (2005).
[35] H. Ohtsuki, M. A. Nowak, and J. M. Pacheco, Phys. Rev. Lett. **98**, 108106 (2007).