Nonperturbative thermodynamic geometry of anyon gas

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Following our earlier work on the Ruppeiner geometry of an anyon gas [B. Mirza and H. Mohammadzadeh, Phys. Rev. E **78**, 021127 (2008)], we will derive nonperturbative thermodynamic curvature of a two-dimensional ideal anyon gas. At different values of the thermodynamic parameter space, some unique and interesting behaviors of the anyon gas are explored. A complete picture of attractive and repulsive phases of the anyon gas is given.

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I. INTRODUCTION

The geometrical structure of phase space of statistical thermodynamics was explicitly studied by Gibbs. The geometrical thermodynamics was developed by Ruppeiner and Weinhold [1,2]. They introduced two sorts of Riemannian metric structure representing thermodynamic fluctuation theory, which were related to the second derivative of entropy or internal energy. This theory represents a new qualitative tool for the study of fluctuation phenomena. The thermodynamic curvature has already been calculated for some models whose thermodynamics are exactly known, where reviews for these models can be found in [3,4]. Recently, this approach has been utilized to study the thermodynamics of black holes [5–8]. The thermodynamic curvature of the ideal classical gas is zero and it could be a criterion for statistical interaction of the system [1,9]. Janyszek and Mrugała worked out the thermodynamic curvature for ideal Fermi and Bose gases and reported that the sign of the thermodynamic curvature is always different for ideal Fermi and Bose gases. It was argued that the scalar curvature could be used to show that fermion gases were more stable than boson gases [10]. Also, phase-transition properties of van der Waals gas and some other thermodynamic models have been considered and it has been shown that the singular point of the thermodynamic curvature coincides with the critical point of the system [11,12]. Recently thermodynamic curvature of the classical limit of the anyon gas has been worked out [13]. For a two-dimensional system, the statistical distribution may interpolate between fermions and bosons and respects a fractional exclusion principle [14]. Particles with the new statistics were named "anyons" by Wilczek [15]. The thermodynamic properties of systems with fractional statistical particles or anyons have been considered and some factorizable properties of these systems were introduced by Huang [16]. It has been shown that the thermodynamic quantities of a free anyon gas may be factorized to ideal Bose and Fermi gases [17,18]. Using Huang's factorized method, we will explore the thermodynamics curvature of the anyon gas in the full physical range.

The outline of this paper is as follows. In Sec. II, the thermodynamic properties of anyons are summarized and the internal energy of the anyon gas is derived. In Sec. III, the factorizable properties of fractional statistical particles are collected and the internal energy and the particle number of the system are evaluated with respect to the internal energy and the particle number of fermion and boson gases. In Sec. IV, the metric of the parameter space of this system is obtained and, finally, the thermodynamic curvature of the anyon gas in the full physical range is evaluated and its properties are investigated.

II. THERMODYNAMIC PROPERTIES OF THE IDEAL GAS OF FRACTIONAL STATISTICAL PARTICLES

Particles with fractional statistics or anyons and their thermodynamic properties have been the subject of research by a number of authors [14,19–21]. Fractional exchange statistics arises when the many-body wave function of a system of indistinguishable particles is allowed to acquire an arbitrary phase $e^{i\pi\alpha}$ upon an adiabatic exchange process of two particles. Here, α is the so-called statistical parameter, interpolating between $\alpha=0$ (bosons) and $\alpha=1$ (fermions). Such an exchange produces a nontrivial phase only if the configuration space of the collection of particles under study possesses a multiply connected topological structure. Therefore, fractional exchange statistics is usually restricted to two spatial dimensions, $d=2$. However, fractional exchange statistics can be formalized, to some extent, also in $d=1$. A different concept of fractional statistics, namely, fractional exclusion statistics, is based on the structure of the Hilbert space, rather than the configuration space, of the particle assembly and is therefore not restricted to $d\leq 2$ [14,22–27]. The statistical distribution function of anyons has been derived by Wu using Haldane's fraction exclusion statistics [21],

$$n_i = \frac{1}{w(e^{(\epsilon_i-\mu)/kT}) + \alpha}, \tag{1}$$

where the function $w(\zeta)$ satisfies the functional equation

$$w(\zeta)^{\alpha}[1 + w(\zeta)]^{1-\alpha} = \zeta \equiv e^{(\epsilon-\mu)/kT} \tag{2}$$

and α is the fractional statistical parameter. The functional equation for $w(\zeta)$ can be solved analytically only in a few

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special cases [28]. Equation (2) yields the correct solutions for two familiar cases: bosons ($\alpha=0$) $w(\zeta)=\zeta-1$ and fermions ($\alpha=1$) $w(\zeta)=\zeta$. We can also solve Eq. (2) in the classical limit where $\exp[(\epsilon-\mu)/kT \gg 1]$,

$$w(\zeta)=\zeta+\alpha-1, \tag{3}$$

$$n_i=\frac{1}{e^{(\epsilon_i-\mu)/kT}+2\alpha-1}. \tag{4}$$

Deviation from the classical limit and a more general solution of Eq. (2) is given by the following function:

$$w(\zeta)=\zeta+\alpha-1+\frac{c_1}{\zeta}+\frac{c_2}{\zeta^2}+\frac{c_3}{\zeta^3}+\cdots, \tag{5}$$

where the constant coefficient $c_1,c_2,\ldots$ can be evaluated on the condition that at each order of ζ, $w(\zeta)$ satisfies Eq. (2). The following solutions are obtained order by order:

$$c_1=\tfrac{1}{2}\alpha(1-\alpha),$$

$$c_2=\tfrac{1}{3}\alpha(1-\alpha)(1-2\alpha),$$

$$c_3=\tfrac{1}{8}\alpha(1-\alpha)(1-3\alpha)(2-3\alpha),$$

$$\vdots$$

$$c_m=-\frac{1}{[(m+1)!]m}\prod_{i=0}^{m}(i-m\alpha). \tag{6}$$

Now, it is straightforward to obtain the internal energy and the particle number of the anyon gas in the classical limit and perturbatively with a small deviation from the classical limit by the following relation:

$$U=\sum_i n_i\epsilon_i,$$

$$N=\sum_i n_i. \tag{7}$$

In the thermodynamic limit and for two-dimensional momentum spaces of nonrelativistic free anyons with a mass m, the summation can be replaced with the following integral:

$$\sum_i \to \frac{V}{h^2}2\pi m\int_0^\infty d\epsilon. \tag{8}$$

It should be noted that for obtaining above equations a free-particle energy-impulse relation has been used [21]. For a small deviation from the classical limit, we may use the first correction in Eq. (5) and obtain the internal energy and the particle number as is presented in [13]. We can use the other correction terms and get far from the classical limit perturbatively. It is obvious that this procedure does not yield nonperturbative information. We will review a nonperturbative approach based on the factorizable properties of thermodynamic quantities of the anyon gas in the next section [16–18].

III. FACTORIZABLE THERMODYNAMIC QUANTITIES OF THE ANYON GAS

Huang showed that a system of free anyons is equivalent to a system with an α fraction of fermions and a $(1-\alpha)$ fraction of bosons, while the transmutation between the boson and the fermion is allowed. The system with boson-fermion transmutation can be regarded as the ensemble average of M systems, which are classified as fermions (there are αM) and bosons [there are $(1-\alpha)M$], and each one of both cases is equal to N-fermion (boson) gas existing in the volume V and the pressure P. Therefore, the ensemble average for the thermodynamic quantity $Q_N(\alpha)$ of the system with the boson-fermion transmutation and, thereupon, the anyon system can be factorized as [17]

$$Q_N(\alpha)=\alpha Q_N(1)+(1-\alpha)Q_N(0), \tag{9}$$

where $Q_N(\alpha)$ refers to the thermodynamic quantities of the anyon system, while $Q_N(1)$ and $Q_N(0)$ are related to the thermodynamic quantities of fermion and boson gases, respectively. We will write the above equation in the following simpler form:

$$Q_a=\alpha Q_f+(1-\alpha)Q_b. \tag{10}$$

Therefore, we can evaluate the internal energy and the particle number of the anyon gas as a composition of the internal energy and the particle number of fermion and boson gases, while the particle numbers of anyon, fermion, and boson gases are the same,

$$U_a=\alpha U_f+(1-\alpha)U_b,$$

$$N_a=\alpha N_f+(1-\alpha)N_b. \tag{11}$$

Wu also derived the relation [21]

$$\frac{\mu_a}{kT}=\alpha\frac{h^2}{2\pi Vm}\frac{N}{kT}+\ln\left[1-\exp\left(-\frac{h^2}{2\pi Vm}\frac{N}{kT}\right)\right]. \tag{12}$$

We can rewrite the above equation for the boson and the fermion cases, with $\alpha=0$ and $\alpha=1$. It should be noted that

$$N=N_a=N_f=N_b. \tag{13}$$

It can be easily shown that

$$\mu_a=\alpha\mu_f+(1-\alpha)\mu_b, \tag{14}$$

where, μ_a, μ_f, and μ_b denote the chemical potential for anyon, fermion, and boson cases, respectively, which is consistent with the factorizable property. Also one can derive

$$z_a=z_f^{\alpha}z_b^{(1-\alpha)}, \tag{15}$$

where

$$z_a=\exp(\mu_a/kT)=e^{\alpha N_a\beta/A}(1-e^{-N_a\beta/A}),$$

$$z_f=\exp(\mu_f/kT)=e^{N_f\beta/A}(1-e^{-N_f\beta/A}),$$

$$z_b=\exp(\mu_b/kT)=(1-e^{-N_b\beta/A}) \tag{16}$$

are the fugacity of anyon, fermion, and boson cases, respectively, and $A=\frac{2\pi Vm}{h^2}$ and $\beta=1/kT$.

Albeit it is impossible to solve the functional equation (2) for all values of α and in the full physical range, the factorizable properties of thermodynamic quantities make it possible to obtain the internal energy and the particle number of the anyon gas [16–18]. The statistical distribution function of fermion and boson cases are given by

$$(n_i)_f = \frac{1}{e^{(\epsilon_i-\mu)/kT}+1},$$

$$(n_i)_b = \frac{1}{e^{(\epsilon_i-\mu)/kT}-1}. \tag{17}$$

Subsequently, the internal energy and the particle number of fermion and boson gases can be evaluated as in the following:

$$U_f = \frac{2\pi Vm}{h^2}\beta^{-2}\int_0^\infty \frac{\epsilon d\epsilon}{e^{(\epsilon-\mu_f)/kT}+1} = -A\beta^{-2}\mathrm{Li}_2(-z_f),$$

$$N_f = \frac{2\pi Vm}{h^2}\beta^{-1}\int_0^\infty \frac{d\epsilon}{e^{(\epsilon-\mu_f)/kT}+1} = A\beta^{-1}\ln(1+z_f), \tag{18}$$

$$U_b = \frac{2\pi Vm}{h^2}\beta^{-2}\int_0^\infty \frac{\epsilon d\epsilon}{e^{(\epsilon-\mu_b)/kT}-1} = A\beta^{-2}\mathrm{Li}_2(z_b),$$

$$N_b = \frac{2\pi Vm}{h^2}\beta^{-1}\int_0^\infty \frac{d\epsilon}{e^{(\epsilon-\mu_b)/kT}-1} = -A\beta^{-1}\ln(1-z_b). \tag{19}$$

Therefore, the internal energy and the particle number of the anyon gas will be

$$U_a = A\beta^{-2}[-\alpha\mathrm{Li}_2(-z_f)+(1-\alpha)\mathrm{Li}_2(z_b)],$$

$$N_a = A\beta^{-1}[\alpha\ln(1+z_f)-(1-\alpha)\ln(1-z_b)], \tag{20}$$

where $\mathrm{Li}_n(x)$ denotes the polylogarithm function. The relations between the thermodynamic quantities of the anyon gas and the composition of the thermodynamic quantities of fermion and boson gases lead to some interesting and nontrivial integral equalities that have been presented in the Appendix.

IV. THERMODYNAMIC CURVATURE OF ANYON GAS

Ruppeiner geometry is based on the entropy representation, where we denote the extended set of $n+1$ extensive variables of the system by $X=(U,N^1,\dots,V,\dots,N^r)$, while Weinhold worked in the energy representation in which the extended set of $n+1$ extensive variables of system are denoted by $Y=(S,N^1,\dots,V,\dots,N^r)$ [3]. It should be noted that we can work in any thermodynamic potential representation that is the Legendre transform of the entropy or the internal energy. The metric of this representation may be the second derivative of the thermodynamic potential with respect to intensive variables, for example, the thermodynamic potential Φ which is defined as

$$\Phi = \Phi(\{F^i\}), \tag{21}$$

where $F=(1/T,-\mu^1/T,\dots,P/T,\dots,-\mu^r/T)$. Φ is the Legendre transform of entropy with respect to the extensive parameter X^i,

$$F^i = \frac{\partial S}{\partial X^i}. \tag{22}$$

The metric in this representation is given by

$$g_{ij} = \frac{\partial^2\Phi}{\partial F^i\,\partial F^j}. \tag{23}$$

Janyszek and Mrugała used the partition function to introduce the metric geometry of the parameter space [10],

$$g_{ij} = \frac{\partial^2\ln Z}{\partial\beta^i\,\partial\beta^j}, \tag{24}$$

where $\beta^i=F^i/k$ and Z is the partition function.

According to Eqs. (18)–(20), the parameter space of ideal fermion, boson, and anyon gases are (β,γ_f), (β,γ_b), and (β,γ_a), respectively, where $\beta=1/kT$ and $\gamma_i=-\mu_i/kT$. For computing the thermodynamic metric, we select one of the extended variables as the constant system scale. We will implicitly pick V in working with the grand canonical ensemble [10]. We can evaluate the metric elements of fermion (F_{ij}), boson (B_{ij}), and anyon gases (A_{ij}) by the definition of metric in Eq. (24). The metric elements of the thermodynamic space of an ideal fermion gas are given by

$$F_{\beta\beta} = \frac{\partial^2\ln Z_f}{\partial\beta^2} = -\left(\frac{\partial U_f}{\partial\beta}\right)_{\gamma_f} = -2\beta^{-3}\mathrm{Li}_2(-z_f),$$

$$F_{\beta\gamma_f} = F_{\gamma_f\beta} = \frac{\partial^2\ln Z_f}{\partial\beta\,\partial\gamma_f} = -\left(\frac{\partial U_f}{\partial\gamma_f}\right)_\beta = \beta^{-2}\ln(1+z_f),$$

$$F_{\gamma_f\gamma_f} = \frac{\partial^2\ln Z_f}{\partial\gamma_f^2} = -\left(\frac{\partial N_f}{\partial\gamma_f}\right)_\beta = \beta^{-1}\frac{z_f}{1+z_f}. \tag{25}$$

In the same manner, the metric elements of the thermodynamic space of an ideal boson gas are given by

$$B_{\beta\beta} = \frac{\partial^2\ln Z_b}{\partial\beta^2} = -\left(\frac{\partial U_b}{\partial\beta}\right)_{\gamma_b} = 2\beta^{-3}\mathrm{Li}_2(z_b),$$

$$B_{\beta\gamma_b} = B_{\gamma_b\beta} = \frac{\partial^2\ln Z_b}{\partial\beta\,\partial\gamma_b} = -\left(\frac{\partial U_b}{\partial\gamma_b}\right)_\beta = -\beta^{-2}\ln(1-z_b),$$

$$B_{\gamma_b\gamma_b} = \frac{\partial^2\ln Z_b}{\partial\gamma_b^2} = -\left(\frac{\partial N_b}{\partial\gamma_b}\right)_\beta = \beta^{-1}\frac{z_b}{1-z_b}. \tag{26}$$

For simplicity, we have set the constant $A=1$. By using the factorizable properties, the metric elements of thermodynamic space of an ideal anyon gas can be derived as follows:

$$
\begin{aligned}
A_{\beta\beta} &= \frac{\partial^2 \ln Z_a}{\partial \beta^2} \\
&= -\left(\frac{\partial U_a}{\partial \beta}\right)_{\gamma_a} \\
&= -\frac{\partial}{\partial \beta}[\alpha U_f + (1-\alpha)U_b] \\
&= \alpha F_{\beta\beta} + (1-\alpha)B_{\beta\beta} \\
&= 2\beta^{-3}[-\alpha \mathrm{Li}_2(-z_f) + (1-\alpha)\mathrm{Li}_2(z_b)],
\end{aligned}
$$

$$
\begin{aligned}
A_{\beta\gamma_a} &= \frac{\partial^2 \ln Z_a}{\partial \gamma_a \partial \beta} \\
&= -\left(\frac{\partial N_a}{\partial \beta}\right)_{\gamma_a} \\
&= -\frac{\partial}{\partial \beta}[\alpha N_f + (1-\alpha)N_b] \\
&= \alpha F_{\beta\gamma_f} + (1-\alpha)B_{\beta\gamma_b} \\
&= \beta^{-2}[\alpha \ln(1+z_f) - (1-\alpha)\ln(1-z_b)],
\end{aligned}
$$

$$
\begin{aligned}
A_{\gamma_a\gamma_a} &= \frac{\partial^2 \ln Z_a}{\partial \gamma_a^2} \\
&= -\left(\frac{\partial N_a}{\partial \gamma_a}\right)_{\beta} \\
&= -1/\left(\frac{\partial \gamma_a}{\partial N_a}\right)_{\beta} \\
&= \frac{1}{\alpha/F_{\gamma_f\gamma_f} + (1-\alpha)/B_{\gamma_b\gamma_b}} \\
&= \beta^{-1}\frac{-z_f z_b}{2\alpha z_f z_b + \alpha z_b - \alpha z_f - z_f z_b + z_f},
\end{aligned}
\tag{27}
$$

To obtain the last equation, we use Eq. (15) and differentiate with respect to N; the particle number of system

$$
\left(\frac{\partial \mu_a}{\partial N_a}\right)_{\beta} = \alpha\left(\frac{\partial \mu_f}{\partial N_f}\right)_{\beta} + (1-\alpha)\left(\frac{\partial \mu_b}{\partial N_b}\right)_{\beta}. \tag{28}
$$

We consider a system with two thermodynamic degrees of freedom and, therefore, the dimension of the thermodynamic surface or the parameter space is equal to 2 ($D=2$). Thus, the scalar curvature is given by

$$
R = \frac{2}{\det g} R_{1212.} \tag{29}
$$

Janyszek and Mrugała demonstrated [29] that, if the metric elements are written purely as the second derivatives of a certain thermodynamic potential, the thermodynamic curvature may then be written in terms of the second and the third derivatives. The sign convention for R is arbitrary, so R may be either negative or positive for any case. Our selected sign convention is the same as that of Janyszek and Mrugała, but opposite from [3]. In two-dimensional spaces, the Ricci scalar is defined by

$$
R = \frac{2\begin{vmatrix} g_{\beta\beta} & g_{\gamma\gamma} & g_{\beta\gamma} \\ g_{\beta\beta,\beta} & g_{\gamma\gamma,\beta} & g_{\beta\gamma,\beta} \\ g_{\beta\beta,\gamma} & g_{\gamma\gamma,\gamma} & g_{\beta\gamma,\gamma} \end{vmatrix}}{\begin{vmatrix} g_{\beta\beta} & g_{\beta\gamma} \\ g_{\beta\gamma} & g_{\gamma\gamma} \end{vmatrix}^2}. \tag{30}
$$

Using the following equations for a fermion gas:

$$
\begin{aligned}
F_{\beta\beta,\beta} &= 6\beta^{-4}\mathrm{Li}_2(-z_f), \\
F_{\beta\beta,\gamma_f} &= F_{\beta\gamma_f,\beta} = -2\beta^{-3}\ln(1+z_f), \\
F_{\gamma_f\gamma_f,\beta} &= F_{\beta\gamma_f,\gamma_f} = -\beta^{-2}\frac{z_f}{1+z_f}, \\
F_{\gamma_f\gamma_f,\gamma_f} &= -\beta^{-1}\frac{z_f}{(1+z_f)^2},
\end{aligned}
\tag{31}
$$

and the following equations for the boson gas:

$$
\begin{aligned}
B_{\beta\beta,\beta} &= -6\beta^{-4}\mathrm{Li}_2(z_b), \\
B_{\beta\beta,\gamma_b} &= B_{\beta\gamma_b,\beta} = 2\beta^{-3}\ln(1-z_b), \\
B_{\gamma_b\gamma_b,\beta} &= B_{\beta\gamma_b,\gamma_b} = -\beta^{-2}\frac{z_b}{1-z_b}, \\
B_{\gamma_b\gamma_b,\gamma_b} &= -\beta^{-1}\frac{z_b}{(1-z_b)^2},
\end{aligned}
\tag{32}
$$

we can obtain the following equations for an anyon gas:

$$
\begin{aligned}
A_{\beta\beta,\beta} &= \alpha F_{\beta\beta,\beta} + (1-\alpha)B_{\alpha\alpha,\alpha} \\
&= 6\beta^{-4}[\alpha \mathrm{Li}_2(-z_f) - (1-\alpha)\mathrm{Li}_2(z_b)],
\end{aligned}
$$

$$
\begin{aligned}
A_{\beta\beta,\gamma_a} &= G_{\beta\gamma_a,\beta} \\
&= \alpha F_{\beta\beta,\gamma_f} + (1-\alpha)B_{\beta\beta,\gamma_b} \\
&= 2\beta^{-3}[-\alpha \ln(1+z_f) + (1-\alpha)\ln(1-z_b)],
\end{aligned}
$$

$$
\begin{aligned}
A_{\gamma_a\gamma_a,\beta} &= G_{\beta\gamma_a,\gamma_a} \\
&= A^2_{\gamma_a\gamma_a}\left\{\alpha\frac{F_{\beta\gamma_f,\gamma_f}}{F^2_{\gamma_f\gamma_f}} + (1-\alpha)\frac{B_{\beta\gamma_b,\gamma_b}}{B^2_{\gamma_b\gamma_b}}\right\} \\
&= -\beta^{-2}\frac{z_f z_b}{2\alpha z_f z_b + \alpha z_b - \alpha z_f - z_f z_b + z_f},
\end{aligned}
$$

$$A_{\gamma_a\gamma_a,\gamma_a} = A^3_{\gamma_a\gamma_a}\left\{\alpha\frac{F_{\gamma_f\gamma_f,\gamma_f}}{F^3_{\gamma_f\gamma_f}} + (1-\alpha)\frac{B_{\gamma_b\gamma_b,\gamma_b}}{B^3_{\gamma_b\gamma_b}}\right\} = -\beta^{-1}\frac{z_f z_b[\alpha z_b^2(1+z_f) + \alpha z_f^2(-1+z_b) + z_f^2(1-z_b)]}{(2\alpha z_f z_b + \alpha z_b - \alpha z_f - z_f z_b + z_f)^3}. \tag{33}$$

The third equations is obtained from Eq. (27),

$$A_{\gamma_a\gamma_a,\beta} = \left(\frac{\partial A_{\gamma_a\gamma_a}}{\partial\beta}\right)_{\gamma_a} = \frac{\partial}{\partial\beta}[\alpha/F_{\gamma_f\gamma_f} + (1-\alpha)/B_{\gamma_b\gamma_b}]^{-1} \tag{34}$$

and the last equation comes from differentiating Eq. (28) with respect to N and also by using the following equation:

$$\frac{\partial^2\gamma_a}{\partial N_a^2} = -\frac{1}{\left(\frac{\partial N_a}{\partial\gamma_a}\right)^3}\frac{\partial^2 N_a}{\partial\gamma_a^2}. \tag{35}$$

Now, we can calculate the thermodynamic curvature for the ideal fermion, boson, and anyon gases. The Ricci scalar for fermion and boson gases are given by

$$R_f = -\frac{4\beta z_b[\mathrm{Li}_2(-z_f)\ln(1+z_f) - 2z_f\mathrm{Li}_2(-z_f) - \ln^2(1+z_f) - z_f\ln^2(1+z_f)]}{[2z_f\mathrm{Li}_2(-z_f) + \ln^2(1+z_f) + z_f\ln^2(1+z_f)]^2}, \tag{36}$$

$$R_b = -\frac{4\beta z_b[\mathrm{Li}_2(z_b)\ln(1-z_b) + 2z_b\mathrm{Li}_2(z_b) - \ln^2(1-z_b) + z_b\ln^2(1-z_b)]}{[2z_b\mathrm{Li}_2(z_b) - \ln^2(1-z_b) + z_b\ln^2(1-z_b)]^2}. \tag{37}$$

The thermodynamic curvature of the ideal fermion and boson gases are depicted in Figs. 1 and 2. The Ricci scalars for fermion and boson gases are negative and positive, respectively. The thermodynamic curvature of a boson gas also has a singularity at $z_b=1$. The Bose-Einstein condensation phase transition occurs at this point in the three-dimensional space [10]. In the two-dimensional space, there is no temperature below which the ground state can be said to be macroscopically occupied in comparison to the excited states. Therefore, as is well known, no Bose-Einstein condensation occurs in the two-dimensional space [30,31]. But the singular property of the thermodynamic curvature at $z_b=1$ in the two-dimensional space has remained from the higher dimension. One can realize from Fig. 1 that the thermodynamic curvature of a fermion gas has a maximum point. As shown in [10,32], we may consider the thermodynamic curvature as a measure of the stability of the system: the bigger the value of R, the less stable is the system. This interpretation of stability measures the looseness of the system to fluctuations and does not refer to the fact that the metric is definitely positive. Therefore, the maximum point of thermodynamic curvature of the fermion gas coincides with the less stable situation. The thermodynamic curvature of the anyon gas is intricate and we will explore it in the following sections.

A. Fixed temperature

In the following sections, we are going to investigate the thermodynamic curvature of an anyon gas for an isotherm; hence, we set $\beta=1$. Therefore, the thermodynamic curvature will be a function of α and z_a (z_a is a function of z_f and z_b).

1. Thermodynamic curvature as a function of α and dual points

We select values of anyon fugacity in the classical limit and get far from that limit. The result is depicted in Fig. 3, which shows the thermodynamic curvature as a function of α for different values of fugacity. It is obvious that in the classical limit (small values of fugacity) the thermodynamic curvature has two different signs. It is positive for $\alpha<\frac{1}{2}$ while it is negative for $\alpha>\frac{1}{2}$. The sign of thermodynamic curvature changes at $\alpha=\frac{1}{2}$ and the anyon gas behaves like an ideal classical gas. Deviation from the classical limit moves the zero point of the thermodynamic curvature from $\alpha=\frac{1}{2}$ to the lower values [13]. Unique and interesting phenomena appear at $z_a\geq 1$. The thermodynamic curvature for $z_a=1$ goes to infinity at $\alpha=0$ (boson gas), where in the higher dimensions the Bose-Einstein condensation occurs. For $z_a>1$, the thermodynamic curvature has a maximum point. From Eqs. (18) and (19), one can find that the particle number of boson (fermion) gas for an isotherm is convex down (up) functions with respect to the fugacity, whereas these functions for the anyon gas for an isotherm with respect to z_a face with a mutation in curvature of the function and has an inflection point for some values of α. This point for $\alpha=0$ occurs at $z_a=1$ while for $\alpha>0$ it coincides with the maximum value of R with a specified value of $z_a>1$. This means that the convexity of these functions for any fixed α changes for a special value of z_a that coincides to the maximum point of the thermodynamic curvature. According to the stability interpretation of the value of thermodynamic curvature, these maximum points may be related to a less stable state of the

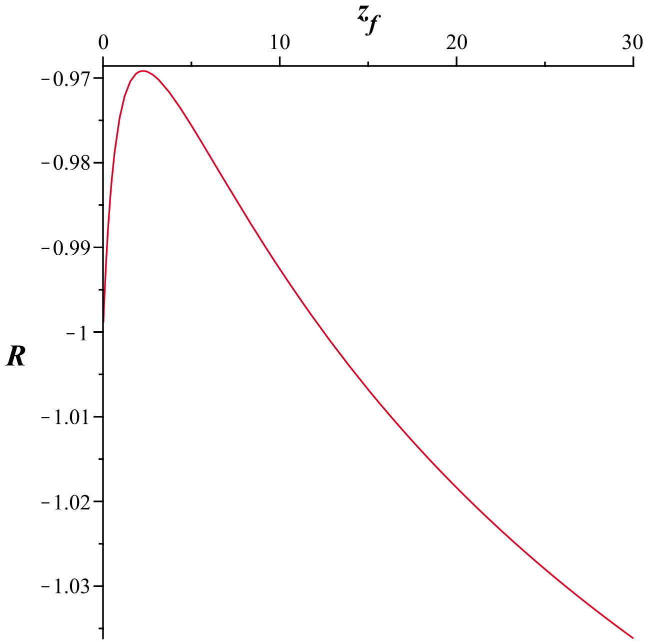

FIG. 1. (Color online) The thermodynamic curvature of a fermion gas as a function of z_f for an isotherm ($\beta=1$).

system. It is also interesting to note that we can find two different values of α with the same value for the thermodynamic curvature. At some values of z_a, we obtain two values for α with zero curvature, which indicates a duality relation between these points.

2. Thermodynamic curvature as a function of z_a

For $\alpha=\frac{1}{2}$, the full physical range of thermodynamic curvature has already been considered in [13]. In this part, we are going to evaluate the thermodynamic curvature for fixed values of α and arbitrary values of anyon fugacity. Figure 4 represents the thermodynamic curvature of the anyon gas for an isotherm for three different values of α as a function of anyon fugacity. This figure suggest that for all values of α (except $\alpha=0$), the thermodynamic curvature for large values of fugacity may go to fixed negative values.

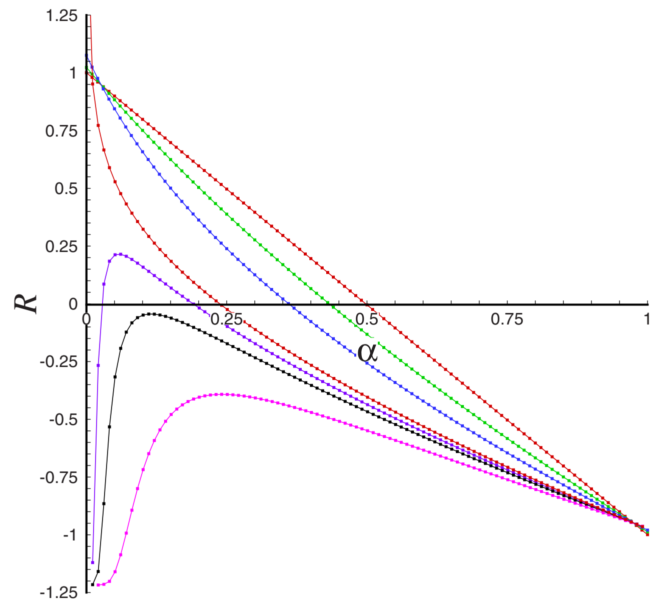

FIG. 3. (Color online) The thermodynamic curvature as a function of α for an isotherm. The values of anyon fugacity have been taken are $z_a=0.01$ [red (upper) line], 0.3 [green (light gray) line], 0.6 (blue line), 1 [red (middle) line], 1.1 (blue line), 1.2 (black line), and 1.5 [purple (lower) line].

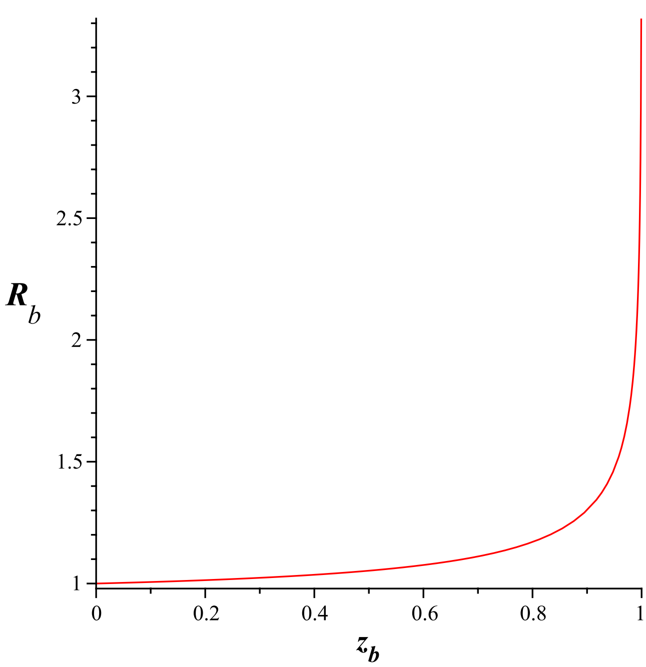

FIG. 2. (Color online) The thermodynamic curvature of a boson gas as a function of z_b for an isotherm ($\beta=1$).

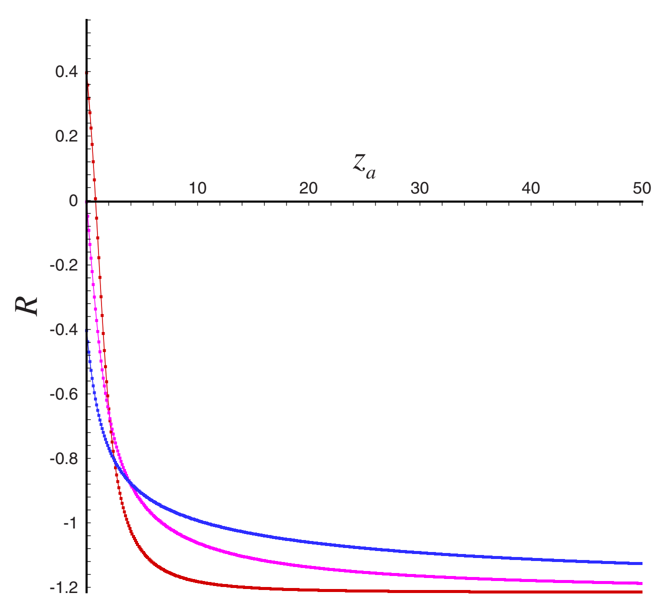

FIG. 4. (Color online) The thermodynamic curvature as a function of z_a (anyon fugacity) for $\alpha=0.3$ [red (lower) line], 0.5 [purple (middle) line], and 0.7 [blue (upper) line] and an isotherm in the full physical range.

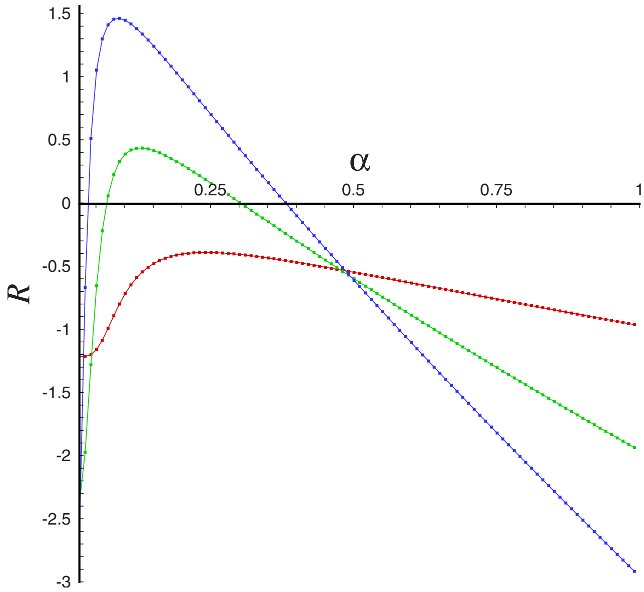

FIG. 5. (Color online) The thermodynamic curvature as a function of α for $\beta=1$ [red (lower) line], 2 [green (light gray) line], and 3 [blue (upper maximum) line] and a fixed fugacity at $z_a=1.5$.

B. Fixed fugacity

It is straightforward to obtain the thermodynamic curvature at a fixed fugacity and as a function of α. We restrict ourselves to the more interesting region $z_a>1$ and set $z_a=1.5$. The thermodynamic curvature as a function of α and for three different values of β is depicted in Fig. 5. It is shown that by increasing the value of β or at lower temperatures, the maximum value of the thermodynamic curvature increases while it also goes toward the lower values of α. This means that, at the limit of $T=0$, the thermodynamic curvature for $\alpha=0$ goes to large values. Although there is no phase transition in the two-dimensional space, we see a behavior similar to Bose-Einstein condensation. Actually, the maximum value and sharp changes in the thermodynamic curvature in the $z_a>1$ region can be interpreted as the remaining of a phase transition in a higher three-dimensional world, which is of course the familiar Bose-Einstein condensation. The dual points with $R=0$ can clearly be identified in Fig. 5.

C. Fixed particle number

We can drive the thermodynamic curvature as a function of β and N_a by substituting the fugacity from (15). It has been shown that, at $T=0$, particles of general exclusion statistics exhibit a Fermi surface [24]. This fact dictates the low-temperature thermodynamics of these particles when the particle number is conserved. Figure 6 shows the thermodynamic curvature of the anyon gas at different values of α. The particle number has been fixed for simplicity at $N_a=1$. The upper curve coincides with the thermodynamic curvature of the boson gas ($\alpha=0$) and the lower curve coincides with the fermion gas ($\alpha=1$). The other curves show the thermodynamic curvature of the intermediate values of α. It is clear that by increasing the value of β or by going toward low temperatures, the thermodynamic curvature approaches that of a fermion gas, which is consistent with the result in [24].

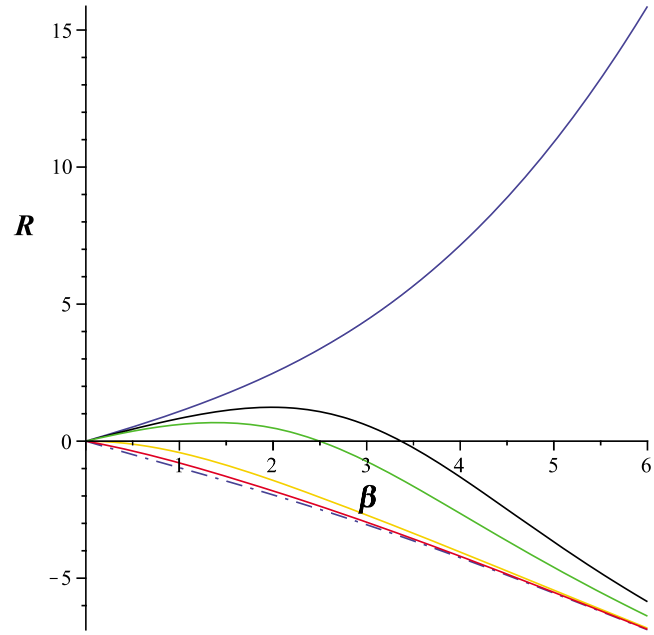

FIG. 6. (Color online) The thermodynamic curvature as a function of β while the particle number is conserved ($N_a=1$). The solid and the dashed-dotted blue lines correspond to the boson ($\alpha=0$) and the fermion ($\alpha=1$) gases, respectively. The other lines correspond to the intermediate values of fraction parameter $\alpha=0.05$ (black line), 0.1 (green line), 0.5 (orange line), and 0.8 (red line).

V. CONCLUSION

We derived the nonperturbative thermodynamic curvature of an ideal anyon gas. It is interesting that, for $z_a>1$, there is a maximum point for the thermodynamic curvature. At low temperatures and at a fixed particle number, the thermodynamic curvature approaches that of a fermion gas, which indicates that, at $T=0$, particles of general exclusion statistics exhibit a Fermi surface. It is clear from Fig. 1 that the thermodynamic curvature of a fermion gas is negative while it is positive for a boson gas; likewise, the statistical interaction for a fermion gas is repulsive but it is attractive for a boson gas. We may propose a unique interpretation for the thermodynamic curvature of the anyon gas according to its sign. It has already been shown that, in the classical limit, statistical interaction of an anyon gas can be attractive or repulsive [21,33]. The attractive case corresponds to a positive curvature and the repulsive one corresponds to a negative curvature [13]. We may identify the attractive and the repulsive parts for an ideal anyon gas from Figs. 3 and 5. This work suggests that there may be dual points where we get equivalent anyon gases but with different α's.

APPENDIX

The factorizable property of anyon thermodynamic quantities enable us to write the internal energy and the particle number of an anyon gas as a composition of the internal energy and the particle number of fermion and boson gases while the condition $N_a = N_f = N_b$ is preserved. Whereas the distribution function of the anyon gas can be solved analytically for some values of the fraction parameter α, the thermodynamic quantities can be derived for such values of the fraction parameter. For semions with $\alpha = 1/2$, the statistical distribution function is given by [21]

$$n_i = \frac{1}{\sqrt{1/4 + \exp[2(\epsilon_i - \mu_a)/kT]}} = \frac{2}{\sqrt{1 + \frac{4}{z_a^2}\exp(2\epsilon_i/kT)}}. \tag{A1}$$

So, from Eqs. (7) and (8), we get

$$U_a = A\beta^{-2}\int_0^\infty \frac{xdx}{\sqrt{1/4 + \frac{1}{z_a^2}\exp(2x)}},$$

$$N_a = A\beta^{-1}\int_0^\infty \frac{dx}{\sqrt{1/4 + \frac{1}{z_a^2}\exp(2x)}}, \tag{A2}$$

and the following equivalent relation can be obtained by using Eqs. (11) and (18):

$$U_a = A\beta^{-2}\left(\frac{1}{2}\int_0^\infty \frac{xdx}{\frac{1}{z_f}\exp(x) + 1} + \frac{1}{2}\int_0^\infty \frac{xdx}{\frac{1}{z_b}\exp(x) - 1}\right),$$

$$N_a = A\beta^{-2}\left(\frac{1}{2}\int_0^\infty \frac{dx}{\frac{1}{z_f}\exp(x) + 1} + \frac{1}{2}\int_0^\infty \frac{dx}{\frac{1}{z_b}\exp(x) - 1}\right). \tag{A3}$$

Therefore, we get the following nontrivial relations:

$$\int_0^\infty \frac{xdx}{\sqrt{1/4 + \frac{1}{z_a^2}\exp(2x)}} = \frac{1}{2}\left(\int_0^\infty \frac{xdx}{\frac{1}{z_f}\exp(x) + 1} + \int_0^\infty \frac{xdx}{\frac{1}{z_b}\exp(x) - 1}\right),$$

$$\int_0^\infty \frac{dx}{\sqrt{1/4 + \frac{1}{z_a^2}\exp(2x)}} = \frac{1}{2}\left(\int_0^\infty \frac{dx}{\frac{1}{z_f}\exp(x) + 1} + \int_0^\infty \frac{dx}{\frac{1}{z_b}\exp(x) - 1}\right), \tag{A4}$$

where the value of fugacity for anyon, fermion, and boson gases must be evaluated from Eq. (16) with the condition $N_a = N_f = N_b$. For example, if we set $z_a = 2$, $z_f = 4.828\,427\,125$, and $z_b = 0.828\,427\,1247$, the above condition is satisfied and the integral equalities (A4) are valid, which can be checked by MAPLE or MATHEMATICA. The following nontrivial equality can also be obtained by using Eq. (27), with those values of fugacity for anyon, fermion, and boson gases, which will satisfy Eq. (13),

$$\int_0^\infty \frac{8z_a\exp(2x)dx}{[z_a^2 + 4\exp(2x)]^{3/2}} = 2\left(\frac{1}{\int_0^\infty \frac{z_f\exp(x)dx}{[\exp(x) + z_f]^2}} + \frac{1}{\int_0^\infty \frac{z_b\exp(x)dx}{[\exp(x) - z_b]^2}}\right)^{-1}. \tag{A5}$$

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