Discontinuous phase transitions of conserved threshold transfer process with deterministic hopping

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The deterministic conserved threshold transfer process, which is a variant of the conserved threshold transfer process modified in a way as that hopping of a particle is to be deterministic, is proposed. The critical behavior of the model is investigated in one, two, and four dimensions. It is found that the order parameter yields a discontinuous transition; i.e., the transition appears to be first ordered in all dimensions considered. The origin of such a discontinuous transition is investigated, considering clustering of active sites and accumulation of critical sites just before the steady state is reached.

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I. INTRODUCTION

Nonequilibrium, absorbing phase transition from an active phase to single or multiple absorbing states has attracted great attention over several decades [1–4]. So far, only a few universality classes have been identified; i.e., the directed percolation class [5–8] and the parity conserving class [9–12] are firmly established, but the pair-contact process with diffusion class is still controversial [13–19]. The triplet and quadruplet reaction-diffusion models were also claimed to belong to the new universality classes [20,21]. Besides these classes, a new universality class was proposed by Rossi *et al.* regarding the models with a conserved field, generated from the symmetry that an order parameter is locally coupled to a short-range nondiffusive conserved field [22]. The conserved lattice gas (CLG) model, the conserved threshold transfer process (CTTP), and the stochastic sandpile model were determined as belonging to this class [22–27].

In one dimension (1D), however, the critical behavior of the CLG model is known to be different from that of the CTTP model, i.e., universality splits [28]. In the CLG model, each lattice site is either filled with a particle or is empty, and particles with at least one particle on the nearest-neighbor sites is assumed to be active; otherwise, it is inactive. Active particles may hop to one of the nearest-neighbor empty sites. In the CTTP model, on the other hand, each site is empty, singly occupied, or doubly occupied, and doubly-occupied sites are assumed to be active. Particles on active sites may hop to randomly selected nearest-neighbor inactive sites.

While in two dimensions (2D) and higher dimensions both the CLG and CTTP models have infinite number of absorbing states and a hopping of active particles is stochastic, the two models exhibit distinct features in 1D. The CLG model in 1D has two symmetric absorbing states at criticality, having a deterministic hopping for active particles, whereas the CTTP model yields many absorbing states with stochastic hopping. Based on the studies on a checkerboard fractal and on a Sierpinski gasket, it was concluded that the universality split is attributed to the dominant deterministic hopping with respect to the CLG model, rather than to the varying types of absorbing states [29]. One can, then, raise a question of whether or not the critical behavior of the CTTP model will become similar to that of the CLG model, if a deterministic hopping is implemented on the CTTP model. In order to address the answer to this question, a variant of the CTTP model is designed in a way that the hopping is deterministic. The hopping becomes deterministic by imposing an additional constraint that the two particles on an active site must hop to different directions. A similar deterministic model may also be made in higher dimensions. On a d-dimensional hypercubic lattice, each site is allowed to be occupied by up to $2d$ particles, and the sites occupied by $2d$ particles are assumed to be active. By imposing a constraint that particles on each active site must hop to different nearest-neighbor inactive sites, the hopping becomes deterministic. This model is termed the "deterministic" CTTP model, abbreviated as the DCTTP model.

In this study, the critical behaviors of the DCTTP model in 1D, 2D, and four dimensions (4D) are studied. It is found that the order parameter falls to 0 discontinuously at the critical point in all dimensions considered and, thus, the transitions appear to be first ordered. The cause of such transitions is investigated.

Hopping of the DCTTP model is similar to the toppling of the deterministic sandpile model proposed by Bak, Tang, and Wiesenfeld (BTW) [30]. The BTW sandpile model is known as the prototypical example of the self-organized criticality (SOC), in which the system evolves spontaneously into a critical state. In the BTW model, active particles topple deterministically and are dissipated at the boundaries. A fixed energy sandpile (FES) model [24,31] was also introduced with a constraint that the total energy is preserved during the evolution process. The stochastic FES (often referred to as Manna FES) model was identified as belonging to the conserved-field universality class. However, it was found that the deterministic FES (BTW FES) model belonged to the universality class different from the stochastic sandpile model due to nonergodicity, i.e., the time averages are different from the ensemble averages [24]. More recently, it was also found that the BTW FES model with a sequential update exhibited a first-order phase transition [32]. The

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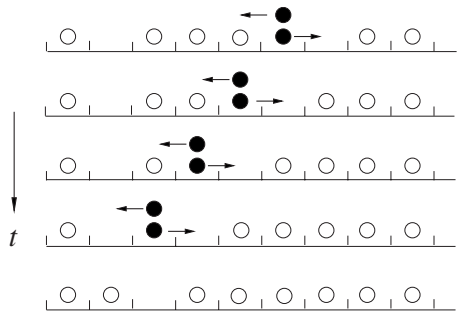

FIG. 1. Sketch of the hopping of active particles in the DCTTP model in one dimension. Arrows represent hopping of active particles (filled circles).

mechanism of such a first-order phase transition was manifested by considering the morphology of the critical clusters as well as the dynamics of active sites within the critical clusters.

Here, the mechanism of the first-order phase transition of the DCTTP model is investigated on the same framework as that of the BTW FES model, while considering the dynamics of critical sites and active sites. The critical sites are defined as those sites which will become active sites if an additional particle hops in. Near criticality, all active sites are localized and the critical sites become supersaturated. The collapse of supersaturation of the critical sites appears to be correlated with the discontinuous phase transition.

II. RESULTS AND DISCUSSIONS

Simulations for the DCTTP model are carried out on d-dimensional hypercubic lattices of side L, where $d=1$, 2, and 4 are under consideration. Initially, ρL^d particles are distributed on randomly selected sites; if the selected site is already occupied by $2d$ particles, a different site is selected. At each sequential update step, an active site is randomly selected from the list, and particles on the active site attempt to hop to the nearest-neighbor sites by following the hopping rule as defined in the model. At each hopping stage, the evolution time is increased by $\Delta t=1/N_a(t)$, with $N_a(t)$ representing the total number of active sites at time t. This procedure is continued until either the desired time step is achieved, or the system falls into an absorbing state.

A. In one dimension

In usual absorbing phase transitions, the steady-state density of active sites, $\rho_{\rm sat}$, is considered to be an order parameter and decreases toward 0 continuously as the system approaches the critical point from above, following the power-law behavior $\rho_{\rm sat}\sim(\rho-\rho_c)^{\beta}$. At $\rho=\rho_c$, the density of active sites, ρ_a, yields a power-law behavior in time, $\rho_a\sim t^{-\theta}$.

In the DCTTP model in 1D, the critical density should be $\rho_c=1$ for the following reason; for $\rho<1$, particles on an active site drift along a particular direction, depending on the local configuration of particles, by consecutive deterministic hopping, as shown in Fig. 1. All empty sites are eventually filled. Thus, all particles disperse over the system, and no more than one particle occupies each lattice site. This procedure leads the system into an absorbing state. For $\rho>1$, on the other hand, it is clear that redundant particles, after filling each site with a particle, create active sites; therefore, the system remains in an active phase continuously.

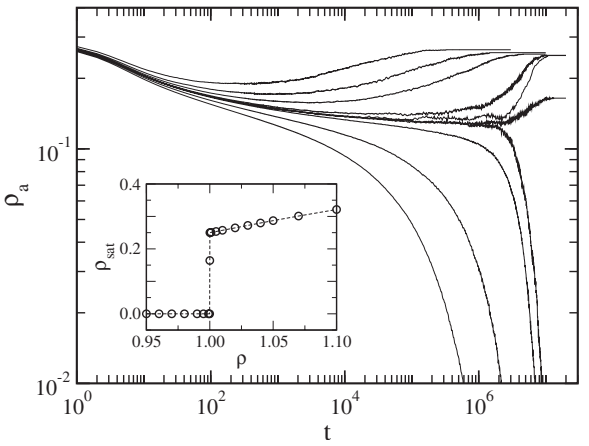

FIG. 2. Densities of active sites ρ_a, plotted against the evolution time for selected particle densities in the DCTTP model on a system of size $L=65\,536$ in one dimension. The curves falling into absorbing states are, from left to right, for $\rho=0.99$, 0.995, 0.999, 0.9999, and those which saturate are, from below to above, for $\rho=1.0$, 1.0001, 1.001, 1.005, 1.01, and 1.02. Data in the inset represent the steady-state densities of active sites against the density of particles.

The density of *active sites* ρ_a is calculated for selected values of ρ close to the critical density. In Fig. 2, data for $\rho<1$ indeed decrease rapidly, and those for $\rho>1$ yield steady-state densities, indicating that the critical point is, indeed, $\rho_c=1$. However, ρ_a at ρ_c becomes nearly flat just prior to saturation, unlike usual absorbing phase transitions, in which ρ_a yields a power-law decay. Moreover, the saturated density $\rho_{\rm sat}$ does not appear to decrease continuously as ρ approaches ρ_c, as shown in the inset. The value of $\rho_{\rm sat}$ falls

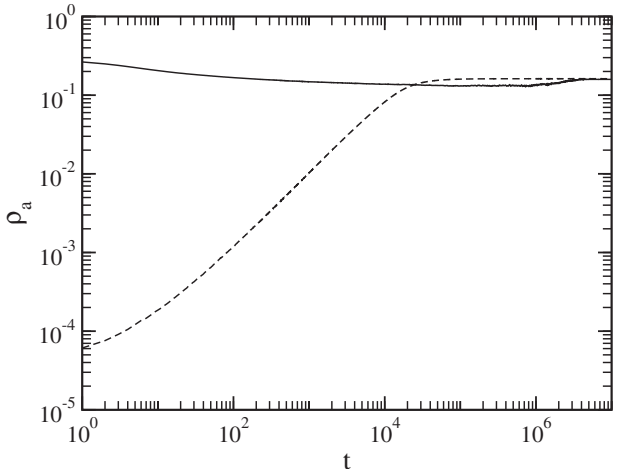

FIG. 3. Densities of active sites ρ_a obtained with a random initial state (solid curve), in comparison with those obtained with an initial state in the close vicinity of an absorbing state (dashed curve), for the DCTTP model of $\rho=1.0$ and $L=65\,536$ in one dimension.

from approximately 0.25 to 0 abruptly at ρ_c, as ρ decreases. The discontinuous transition of ρ_{sat} is similar to that observed in the BTW FES model with sequential updates and is known to be a characteristic of the first-order phase transition.

A possible way of examining whether or not such a phase transition is first-ordered is to observe the hysteresis path of an order parameter [33,34]. When density of particles decreases from a supercritical region, the order parameter decreases and falls to 0, following the path as illustrated in the inset of Fig. 2. However, when the density increases from a subcritical region, the order parameter increases by following a different path. If the system indeed yields such a hysteresis, the two different values will be observed for a given particle density, depending on the paths. If the system is in the subcritical region, it would be in the close proximity to one of the absorbing states having very small number of active sites. If, on the other hand, the system is in the supercritical region, it will have a considerable amount of active sites. Thus, the existence of hysteresis may be examined by comparing the order parameter for two different initial states, but at the same value of ρ near criticality by the two steps. In the first, the system evolves from a random distribution of particles until the system goes into the steady state. In the second, the system begins to evolve with less number of particles, i.e., with $\rho<\rho_c$, and, after the system enters an absorbing state, it is perturbed by adding a few particles to randomly selected sites so that those sites become active. This situation is similar to the spontaneous creation of active sites near criticality [33–35]. The system thereafter evolves into a steady state. In these two cases, once the system

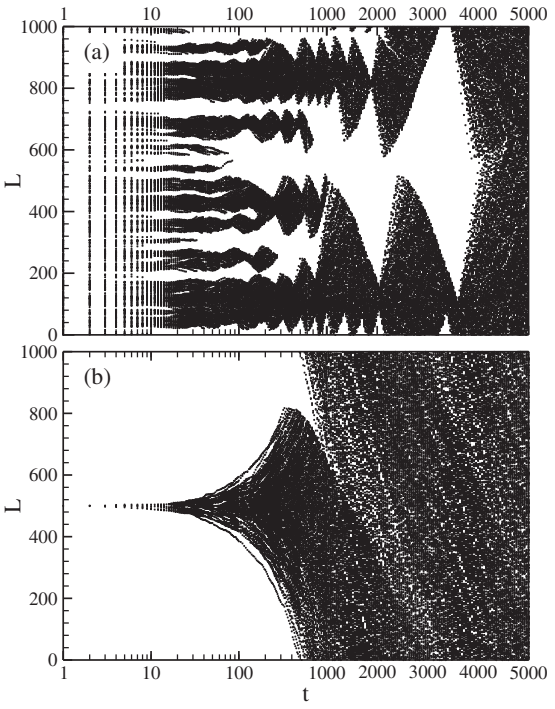

FIG. 4. Advances of active sites from (a) a random initial state and (b) a single perturbed site, both on a lattice of size $L=1000$. To better visualize the details of dynamics, the scale for $t\leq 10^3$ is set to be logarithmic and, afterward, is linear.

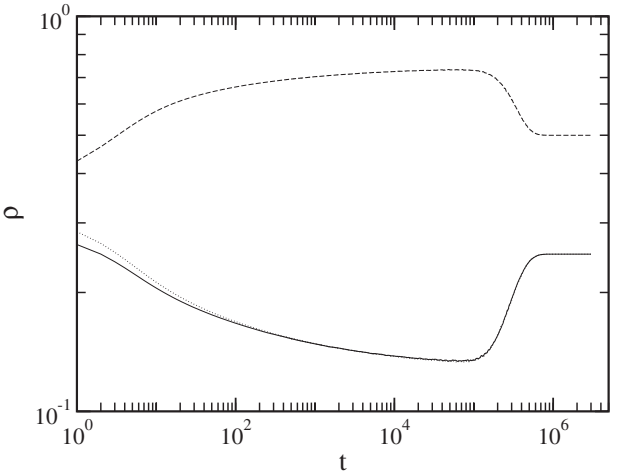

FIG. 5. Densities of active sites (solid curve), singly-occupied sites (dashed curve), and empty sites (dotted curve), plotted against the evolution time for the DCTTP model for $L=10^4$ and $\rho=1.0001$ in one dimension.

evolves into a state with the same ρ_a, the values of ρ_a should remain the same afterward if the system undergoes a second-order phase transition. We, however, find that there is a region in which the values of ρ_a are different, depending on the initial states, as shown in Fig. 3. The active site density becomes the same at $t\approx 2\times 10^4$ and, in the region of $2\times 10^4 < t < 3\times 10^6$, the values of ρ_a depend on the initial states. This indicates that the order parameter is two valued when measured in this region. If a similar test is conducted for different values of ρ near ρ_c, different values of ρ_a will be obtained, depending on the initial states. By connecting the values with the same initial states, two different paths will be obtained. The system, thus, exhibits hysteresis, which is similar to that observed with respect to the cellular automata [35].

In order to understand the difference of dynamics between the two cases, the active sites are plotted in Fig. 4 as dots in a system of size $L=1000$. It is apparent that the dynamics of active sites are remarkably different depending on the initial states. In (a), the initial distribution of particles is random and, at an early stage, active sites merge and create a number of small clusters, some of which again merge and form few larger clusters while the remainders disappear. At later stages, several large clusters drift along a certain direction until they encounter empty sites, at which point they change the direction of movement. Finally, at the time of $t\approx 4\times 10^3$, all finite clusters merge and active sites are distributed evenly throughout the system. On the other hand, in (b), active sites spread gradually from a single site to the entire system at $t\approx 10^3$ time steps. Therefore, during the time interval of $10^3<t<4\times 10^3$, the two cases yield different values of ρ_a, which are qualitatively consistent with the plot in Fig. 3. The different time intervals for which the two values of order parameter exist are due to the different system sizes. (It was, however, verified that the finite-size effect is negligible on the data in Figs. 2 and 3.)

The densities of both empty sites, singly-occupied (critical) sites, and active sites, measured at a density slightly

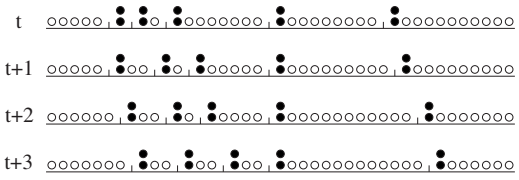

FIG. 6. Part of the typical configuration of particles in the DCTTP model in a steady state. The number of active sites remains unchanged but the active sites drift by means of consecutive hopping. The closed circles represent active particles and the open circles inactive particles.

above ρ_c, are plotted in Fig. 5. Initially, the densities of active and empty sites differ only slightly, and they conjoin at $t \approx 10^2$ and thereafter decrease together. On the other hand, the density of singly-occupied critical sites increases and reaches a maximum value of approximately 0.73 at the time when ρ_a is minimized. Afterward, a small number of active particles appear to reorganize the distribution of particles and lead the system into a steady state, in which half of the lattice sites are singly-occupied. It appears that the accumulation of the critical sites and their collapse might be the origin of the first-order phase transition in the DCTTP model. The typical configuration of particles at a steady state is illustrated in Fig. 6, in which it can readily be observed that the densities of active particles and singly-occupied inactive sites remain nearly constant during the evolution process.

B. In two dimensions

Based on the DCTTP model considered, it was found that the well-known power-law behavior of $\rho_a(t)$ at ρ_c is no longer valid and, accordingly, the power-law determination of ρ_c, described in Ref. [36] for the CLG model, is not applicable. Thus, we determine ρ_c from the plot of $\rho_a(t)$, depending on whether or not data yield saturation. Plotted in Fig. 7 are the data of ρ_a for selected densities, each averaged

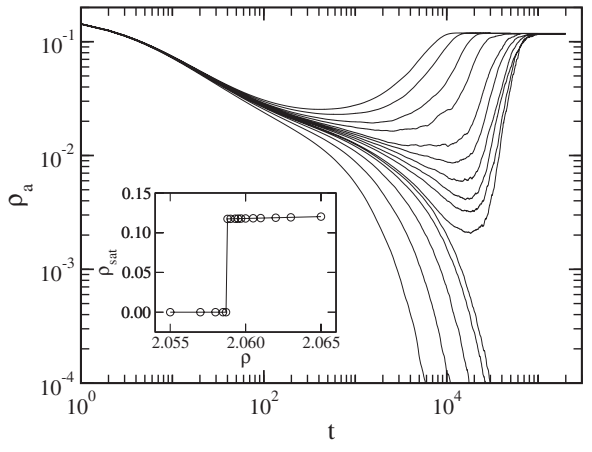

FIG. 7. Density of active sites ρ_a for the DCTTP model on a square lattice of size $L=2000$. The decaying curves are for $\rho=2.055$, 2.057, 2.058, 2.0585, 2.0587 and the saturating curves for $\rho=2.0588$, 2.059, 2.0593, 2.0595, 2.0597, 2.06, 2.0605, 2.061, 2.062, 2.063. The inset shows the order parameter against the particle density.

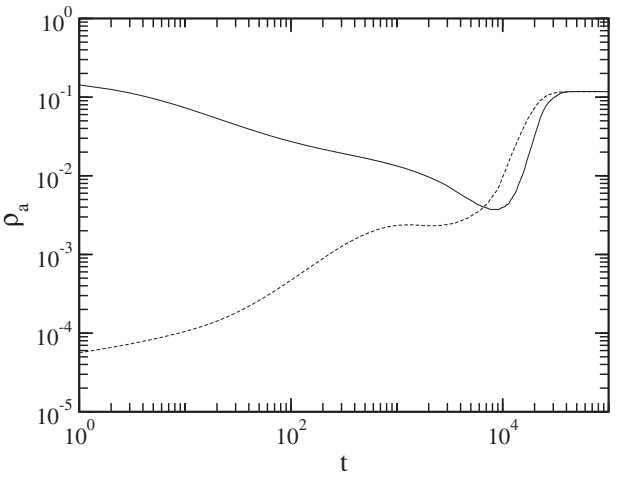

FIG. 8. Density of active sites ρ_a obtained from the random initial state (solid curve) and the perturbed initial state (dotted curve), plotted against the evolution time, in the DCTTP model for $\rho=2.059$ on a square lattice of size $L=1000$.

over 4000 or more trials. Based on the figure, it is clear that the critical density lies between $\rho=2.0587$ and 2.0588. Once the critical point is determined, the steady-state density is calculated in the supercritical region using the saturated values of surviving samples. Since some samples, even in the supercritical region, fall into absorbing states due to fluctuations in a finite-size system, the all-sample average does not provide correct steady-state density in the close vicinity of the critical point [32], which contrasts with the 1D case in which all samples remain in the supercritical region. This was confirmed numerically by comparing ρ_{sat}, calculated in a larger system of $L=2000$, with those in systems of $L=1000$ and 500. It was found that the values from the surviving-sample averages were consistent with one another. By extending this observation, it is expected that the surviving-sample averages of the steady-state density of active particles would be at a value consistent with the order parameter in the thermodynamic limit. The steady-state density, calculated in this way, is shown in the inset of Fig. 7; plotted data again show a discontinuous jump at the critical density.

The possibility of existence of the hysteresis is examined in a way similar to that of 1D, by comparing the density of active sites regarding the two systems with different initial states. Figure 8 shows the region of $6\times10^3<t<4\times10^4$, in which two values of ρ_a are measured. If one selects the relaxation time within this region and measures the density of active particles as an order parameter, it would be different, depending on the initial states. Consecutive measurements for various densities will result in two different paths and, therefore, the hysteresis will be observed.

In order to understand the dynamics with respect to how the system goes into the steady state, the densities of empty sites, singly-occupied sites, doubly-occupied sites, and critical sites, as well as the density of active sites, are calculated at a particle density slightly above ρ_c. Again, the density of critical sites is accumulated up to about 40% of the lattice sites at the time when the density of active sites is minimal, as shown in Fig. 9. A small number of active particles again

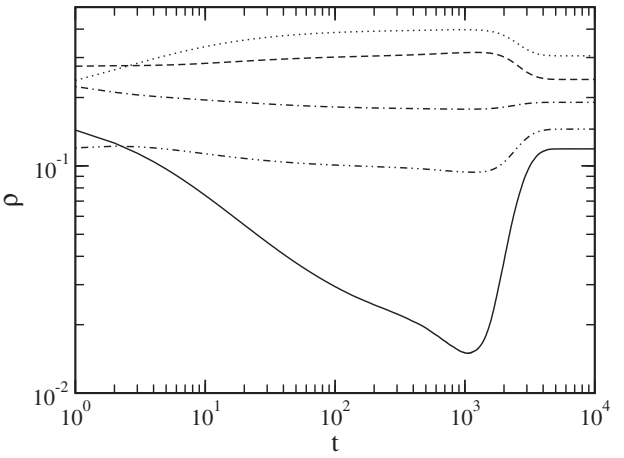

FIG. 9. Densities of sites regarding various occupations against the evolution time for $\rho=2.062$ and $L=200$ in the DCTTP model on a square lattice. The solid curve represents the density of active sites, the dotted curve the density of critical sites, the dashed curve the density of doubly-occupied sites, the dotted-dashed curve the density of singly-occupied sites and the dotted-dotted-dashed curve the density of empty sites.

appear to lead the density of critical sites to collapse and then lead the system to a steady state.

Figure 10 shows the typical morphologies for a system of $L=100$, plotted in the region of $x=y=(1,50)$, for three selected evolution times; (a) represents the density of active sites for a single trial, and (b), (c), and (d) represent the morphologies at the selected time steps of $t=1$, 300, and 10 000, respectively. In (b), active sites and critical sites are initially distributed randomly, forming many small clusters due to the random distribution of particles, as expected. In (c), i.e., at the time when ρ_a is minimum, critical sites and doubly-occupied sites are distributed dominantly over the system, and there are few isolated active sites. Thus, the critical sites are supersaturated within this region. Finally, in (d), active sites are distributed uniformly over the system. It should be noted that most of the active sites are isolated and are surrounded by, in average, each of the empty, singly-occupied, doubly-occupied, and critical sites. (On a square lattice having 300×300 lattice sites, it was found that the mean numbers of surrounding sites with respect to each active site were 0.94, 1.01, 1.03, and 1.02 for empty, singly-occupied, doubly-occupied, and critical sites, respectively.) Based on the local configurations, one can readily realize that the hopping of particles destroys each of singly-occupied, doubly-occupied, and critical sites and then recreates them and, thus, this cyclic destruction and recreation lead the system into a stationary state. The initial distributions of active sites and critical sites as well as the steady-state distributions are found to be qualitatively comparable to those of the BTW FES model with sequential updates [32].

C. In four dimensions

The critical behavior of the DCTTP model in 1D and 2D are studied and the order parameter was found to converge to 0 discontinuously as the system approached the critical

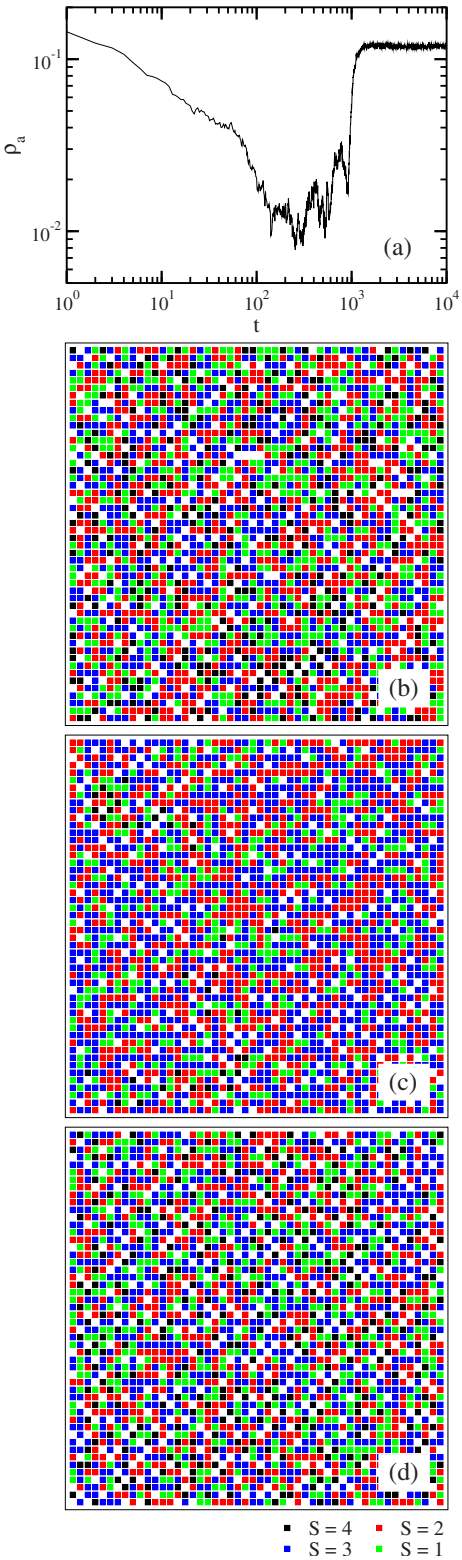

FIG. 10. (Color online) (a) Density of active sites calculated for $\rho=2.062$ and $L=100$ based on a single trial in the DCTTP model on a square lattice, and the morphologies of multiply occupied sites at the three selected relaxation times, (b) $t=1$, (c) $t=300$, and (d) $t=10\,000$, for the same trial as in (a). The black, blue (dark gray), red (medium gray), and green (light gray) symbols represent, respectively, the active sites, critical sites, doubly-occupied sites, and singly-occupied sites, as denoted at bottom.

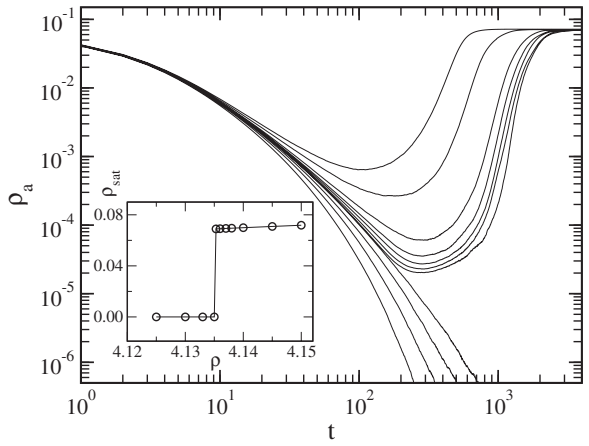

FIG. 11. Density of active sites ρ_a for selected particle densities in a system of 40^4 lattice sites for the DCTTP model in four dimensions; the lower set of curves are, from left to right, for ρ=4.125, 4.13, 4.133, 4.135, and the upper set are, from below, ρ=4.1353, 4.136, 4.137, 4.138, 4.14, 4.145, 4.15. The inset shows the order parameter for various initial densities.

point; i.e., the system undergoes a first-order phase transition. In order to determine whether or not this observation is generally true in all dimensions, the same model is studied on a 4D hypercubic lattice. The critical behavior in 4D is particularly interesting because the four-dimension represents the upper critical dimension of the CTTP model [26].

The density of active particles against the evolution time is presented in Fig. 11 with respect to the selected values of the density of particles. It appears that the critical density lies between ρ=4.135 and ρ=4.1353. The steady-state density ρ_{sat} appears to decrease slowly as ρ approaches ρ_c from above, and falls to 0 abruptly at the critical density, as shown in the inset of Fig. 11, which again yields a discontinuous transition.

Based on the results in 1D, 2D, and 4D, it appears that the order parameter of the DCTTP model undergoes a discontinuous transition in all dimensions. The deterministic hopping leads the critical sites to be supersaturated and their collapse accordingly leads the system into a steady state.

III. SUMMARY AND CONCLUDING REMARKS

The "deterministic" conserved threshold transfer process is proposed and the critical behavior of the model is studied in one, two, and four dimensions. The DCTTP model is a variant of the CTTP model, modified in a way that hopping of particles is deterministic. It was found that the order parameter displayed a discontinuity at a critical point, i.e., the transition appeared to be first-ordered in all dimensions considered. The possible existence of the hysteresis path was analyzed by the measurement of the density of active sites with respect to the two different initial states, i.e., a random initial state and a state in close proximity to one of the absorbing states. It was determined that there exists a region in which two paths yield different values for the order parameter.

The mechanism regarding the first-order phase transition was manipulated by considering the critical sites, which will become active if an additional particle moves in. It was found that the critical sites were accumulated and reached a maximum value at the time when the density of active sites is minimum. The density of critical sites then collapsed to the steady-state value, and the steady state was attained by replacing these critical sites with active sites by the hopping of particles in active sites. With respect to particles in each active site hopping deterministically to their nearest-neighbor sites, it was found that the densities of active sites and critical sites remain nearly constant as hopping creates a new active site and a critical site. If, on the other hand, the stochastic hopping is allowed by a small probability, then one can expect that the system undergoes a continuous phase transition. Indeed, the crossover from the first-order phase transition to the second-order transition is observed when a small amount of stochastic hopping is allowed. This study is still under progress.

ACKNOWLEDGMENTS

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