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Anomalous diffusive behavior of a harmonic oscillator driven by a Mittag-Leffler noise

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The diffusive behavior of a harmonic oscillator driven by a Mittag-Leffler noise is studied. Using the Laplace analysis we derive exact expressions for the relaxation functions of the particle in terms of generalized Mittag-Leffler functions and its derivatives from a generalized Langevin equation. Our results show that the oscillator displays an anomalous diffusive behavior. In the strictly asymptotic limit, the dynamics of the harmonic oscillator corresponds to an oscillator driven by a noise with a pure power-law autocorrelation function. However, at short and intermediate times the dynamics has qualitative difference due to the presence of the characteristic time of the noise.

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I. INTRODUCTION

The study of anomalous diffusion in complex or disordered media has achieved a substantial progress during the last years [1–9]. Anomalous diffusion in physical and biological systems can be formulated in the framework of the generalized Langevin equation (GLE) [1,2,8,10–17]. If one considers the dynamics of a harmonic oscillator with frequency ω under the influence of a random force modeled as Gaussian colored noise, the corresponding GLE is written as [11,12,18]

$$\ddot{X}(t) + \int_0^t dt' \; \gamma(t-t')\dot{X}(t') + \omega^2 X = \xi(t), \tag{1}$$

where $X(t)$ represents the position of a particle of mass $m=1$ at time t and $\gamma(t)$ is the frictional memory kernel. The random force $\xi(t)$ is zero centered and stationary Gaussian that obeys the fluctuation-dissipation theorem [19]

$$\langle \xi(t)\xi(t')\rangle = C(|t-t'|) = k_B T \gamma(|t-t'|), \tag{2}$$

where k_B is the Boltzmann constant and T is the absolute temperature of the environment.

It is now well established that the physical origin of anomalous diffusion is related to the long-time tail correlations [1–3]. Therefore, in order to model anomalous diffusion process, pure power-law correlation functions are usually employed [1,2,10,12,18,20,21], which may be written as

$$C(t) = C_\lambda \frac{|t|^{-\lambda}}{\Gamma(1-\lambda)}, \tag{3}$$

where $\Gamma(z)$ is the gamma function and C_λ is a proportionality coefficient dependent on the exponent λ but independent of time. The exponent λ can be taken as $0<\lambda<1$ or $1<\lambda<2$, which is determined by the dynamical mechanism of the physical process considered.

Viñales and Despósito introduced a noise whose correlation is proportional to a Mittag-Leffler function [22]. This correlation behaves as a power law for large times but is nonsingular at the origin due to the inclusion of a characteristic time.

The aim of this work is to investigate the effects of the Mittag-Leffler noise on the behavior of a harmonically bounded particle governed by GLE (1). This paper is organized as follows. In Sec. II we discuss some characteristics of the Mittag-Leffler noise. In Sec. III, we show the formal expressions for the relaxation functions that govern the dynamics of the particle in the case of an arbitrary noise correlation function. Analytical solutions of the GLE for a harmonically bounded particle driven by a Mittag-Leffler noise are obtained in Sec. IV. Section V is devoted to the analysis of temporal behavior of the relaxation functions and is compared with that obtained in the case of a pure power-law noise correlation function. Finally, the conclusions are presented in Sec. VI.

II. MITTAG-LEFFLER NOISE

It is well known that if the correlation function (2) is a Dirac delta function the stochastic process is Markovian and its dynamics can be directly obtained [23]. However, in a complex or viscoelastic environment, one must take into account the memory effects through a long-time tail noise to describe the effect of the environment on the particle. The non-Markovian dynamics is involved in these physical processes.

Recently, Viñales and Despósito introduced a Mittag-Leffler noise given by [22]

$$C(t) = \frac{C_\lambda}{\tau^\lambda} E_\lambda(-(|t|/\tau)^\lambda), \tag{4}$$

where τ acts as a characteristic memory time and $0<\lambda<2$. The $E_\alpha(y)$ function denotes the Mittag-Leffler function [24] defined through the following series:

$$E_\alpha(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(\alpha j+1)}, \quad \alpha>0. \tag{5}$$

Using the asymptotic behaviors of the Mittag-Leffler function [25], one can easily deduce that, for $\lambda \neq 1$, the correla-

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tion function (4) behaves as a stretched exponential for short times and as an inverse power law in the long-time regime [25,26].

Setting $\lambda=1$, the correlation function (4) reduces to an exponential form

$$C(t)=\frac{C_1}{\tau}e^{-|t|/\tau}, \tag{6}$$

which describes a standard Ornstein-Uhlenbeck process [23]. Moreover, in the limit $\tau\to 0$ and from the limit representation of the Dirac delta [27], we get $C(t)=2C_1\delta(t)$, which corresponds to a white noise, nonretarded friction, and standard Brownian motion [23]. Note that the case $\lambda=1$ does not reproduce an algebraic noise $\sim 1/t$, which has been previously investigated in Refs. [2,28].

On the other hand, for $\lambda\neq 1$ the limit $\tau\to 0$ of the proposed correlation function (4) reproduces the power-law correlation function (3). This behavior is obtained introducing in expression (4) the asymptotic behavior at large y of the Mittag-Leffler function [25],

$$E_\alpha(-y)\sim[y\Gamma(1-\alpha)]^{-1}, \quad y>0. \tag{7}$$

It is worth pointing out that the Mittag-Leffler correlation function (4) is a well defined and no-singular function. From Eq. (4), its value at $t=0$ is given by $C(0)=C_\lambda/\tau^\lambda$, while for the power-law correlation (3) $C(0)$ diverges. Then, the introduction of the characteristic time τ enables one to avoid the singularity of the power law at the origin. Considering that the Mittag-Leffler function is the natural generalization of the exponential function [24], we can also consider the Mittag-Leffler correlation function as a generalization of the power-law correlation, and similarly, the colored noise (6) is considered as a generalization of the white noise.

III. SOLUTIONS OF THE GENERALIZED LANGEVIN EQUATION

In what follows we consider the Langevin equation (1) with the deterministic initial conditions $x_0=X(0)$ and $v_0=\dot{X}(0)$. By means of the Laplace transformation to Eq. (1), one can easily obtain a formal expression for the displacement $X(t)$ and the velocity $V(t)=\dot{X}(t)$. The displacement $X(t)$ satisfies

$$X(t)=\langle X(t)\rangle+\int_0^t dt'\ G(t-t')\xi(t'), \tag{8}$$

where

$$\langle X(t)\rangle = v_0G(t)+x_0[1-\omega^2 I(t)] \tag{9}$$

is the position mean value. The relaxation function $G(t)$ is the Laplace inversion of

$$\hat{G}(s)=\frac{1}{s^2+\hat{\gamma}(s)s+\omega^2}, \tag{10}$$

where $\hat{\gamma}(s)$ is the Laplace transform of the damping kernel and

$$I(t)=\int_0^t dt'\ G(t'). \tag{11}$$

On the other hand, the velocity $V(t)$ satisfies

$$V(t)=\langle V(t)\rangle+\int_0^t dt'\ g(t-t')\xi(t'), \tag{12}$$

where

$$\langle V(t)\rangle = v_0 g(t)-\omega^2 x_0 G(t) \tag{13}$$

is the velocity mean value and the relaxation function $g(t)$ is the derivative of $G(t)$, i.e.,

$$g(t)=G'(t). \tag{14}$$

Explicit expressions of the variances can be obtained from Eqs. (8) and (12). Taking into account the symmetry property of the correlation function and Eq. (2), yields [11,12,16,18]

$$\beta\sigma_{xx}(t)=2I(t)-G^2(t)-\omega^2I^2(t), \tag{15}$$

$$\beta\sigma_{vv}(t)=1-g^2(t)-\omega^2G^2(t), \tag{16}$$

$$\beta\sigma_{xv}(t)=G(t)\{1-g(t)-\omega^2I(t)\}, \tag{17}$$

where $\beta=1/k_BT$.

From an experimental point of view, the information about the observed diffusive behavior is extracted from the mean-square displacement $\rho(t)$. In the long-time measurement, $\rho(t)$ is related to the relaxation function $I(t)$ as [29]

$$\rho(\tau_L)=\lim_{t\to\infty}\langle[X(t+\tau_L)-X(t)]^2\rangle = 2k_BTI(\tau_L), \tag{18}$$

where τ_L is the so-called time lag. Alternative information about the dynamics can be extracted from the normalized velocity autocorrelation function $C_V(t)$, which is related to the relaxation function $g(t)$ as [18,29]

$$C_V(\tau_L)=\lim_{t\to\infty}\frac{\langle V(t+\tau_L)V(t)\rangle}{\langle V(t)V(t)\rangle}=g(\tau_L). \tag{19}$$

Then, the knowledge of the relaxation functions $I(t)$, $G(t)$, and $g(t)$ allows us to describe the diffusive behavior of the oscillator. In the next section we will give explicit expressions for the relaxation functions in the case of a Mittag-Leffler noise (4) assuming that $\lambda\neq 1$.

IV. ANALYTICAL RELAXATION FUNCTIONS FOR A MITTAG-LEFFLER NOISE

From relation (2), the memory kernel $\gamma(t)$ corresponding to the Mittag-Leffler noise (4) can be written as

$$\gamma(t)=\frac{\gamma_\lambda}{\tau^\lambda}E_\lambda(-(|t|/\tau)^\lambda), \tag{20}$$

where $\gamma_\lambda=C_\lambda/k_BT$. Taking into account that the Laplace transform of the memory kernel reads [25]

$$\hat{\gamma}(s) = \frac{\gamma_\lambda s^{\lambda-1}}{1 + s^\lambda \tau^\lambda}, \tag{21}$$

the relaxation function $I(t)$ can be written as the Laplace inversion of

$$\hat{I}(s) = \frac{\hat{G}(s)}{s} = \widehat{I}_0(s) + \widehat{I}_1(s), \tag{22}$$

where

$$\widehat{I}_0(s) = \frac{s^{-1}}{\tau^\lambda s^{2+\lambda} + s^2 + \bar{\gamma}_\lambda s^\lambda + \omega^2}, \tag{23}$$

$$\widehat{I}_1(s) = \tau^\lambda s^\lambda \widehat{I}_0(s), \tag{24}$$

and $\bar{\gamma}_\lambda$ is defined as

$$\bar{\gamma}_\lambda = \gamma_\lambda + \omega^2 \tau^\lambda. \tag{25}$$

Following the approach given in Ref. [30] we get

$$I_0(t) = \left(\frac{t}{\tau}\right)^\lambda \sum_{n=0}^{\infty} \frac{\left(\frac{-\omega^2 t^{2+\lambda}}{\tau^\lambda}\right)^n}{n!} \sum_{m=0}^{\infty} \frac{\left(-\frac{\bar{\gamma}_\lambda t^2}{\tau^\lambda}\right)^m}{m!} t^2 \times E^{(n+m)}_{\lambda,3+2n+\lambda+(2-\lambda)m}(-(t/\tau)^\lambda), \tag{26}$$

$$I_1(t) = \sum_{n=0}^{\infty} \frac{\left(\frac{-\omega^2 t^{2+\lambda}}{\tau^\lambda}\right)^n}{n!} \sum_{m=0}^{\infty} \frac{\left(-\frac{\bar{\gamma}_\lambda t^2}{\tau^\lambda}\right)^m}{m!} t^2 E^{(n+m)}_{\lambda,3+2n+(2-\lambda)m}(-(t/\tau)^\lambda), \tag{27}$$

where $E_{\alpha,\beta}(y)$ is the generalized Mittag-Leffler function [25] defined by the series expansion

$$E_{\alpha,\beta}(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(\alpha j + \beta)}, \quad \alpha > 0, \quad \beta > 0, \tag{28}$$

and $E^{(k)}_{\alpha,\beta}(y)$ is the derivative of the Mittag-Leffler function

$$E^{(k)}_{\alpha,\beta}(y) = \frac{d^k}{dy^k} E_{\alpha,\beta}(y) = \sum_{j=0}^{\infty} \frac{(j+k)!\, y^j}{j!\,\Gamma[\alpha(j+k)+\beta]}. \tag{29}$$

Then, from Eq. (22),

$$I(t) = I_0(t) + I_1(t), \tag{30}$$

where $I_0(t)$ and $I_1(t)$ are given by Eqs. (26) and (27), respectively.

The relaxation functions $G(t)$ and $g(t)$ can be calculated using Eqs. (11) and (14) and the relation [30]

$$\frac{d}{dt}[t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(-\gamma t^\alpha)] = t^{\alpha k+\beta-2} E^{(k)}_{\alpha,\beta-1}(-\gamma t^\alpha). \tag{31}$$

Then, we get

$$G(t) = G_0(t) + G_1(t), \tag{32}$$

where

$$G_0(t) = \left(\frac{t}{\tau}\right)^\lambda \sum_{n=0}^{\infty} \frac{\left(\frac{-\omega^2 t^{2+\lambda}}{\tau^\lambda}\right)^n}{n!} \sum_{m=0}^{\infty} \frac{\left(-\frac{\bar{\gamma}_\lambda t^2}{\tau^\lambda}\right)^m}{m!} t \times E^{(n+m)}_{\lambda,2+2n+\lambda+(2-\lambda)m}(-(t/\tau)^\lambda), \tag{33}$$

$$G_1(t) = \sum_{n=0}^{\infty} \frac{\left(\frac{-\omega^2 t^{2+\lambda}}{\tau^\lambda}\right)^n}{n!} \sum_{m=0}^{\infty} \frac{\left(-\frac{\bar{\gamma}_\lambda t^2}{\tau^\lambda}\right)^m}{m!} t E^{(n+m)}_{\lambda,2+2n+(2-\lambda)m}(-(t/\tau)^\lambda), \tag{34}$$

and

$$g(t) = g_0(t) + g_1(t), \tag{35}$$

where

$$g_0(t) = \left(\frac{t}{\tau}\right)^\lambda \sum_{n=0}^{\infty} \frac{\left(\frac{-\omega^2 t^{2+\lambda}}{\tau^\lambda}\right)^n}{n!} \sum_{m=0}^{\infty} \frac{\left(-\frac{\bar{\gamma}_\lambda t^2}{\tau^\lambda}\right)^m}{m!} \times E^{(n+m)}_{\lambda,1+2n+\lambda+(2-\lambda)m}(-(t/\tau)^\lambda), \tag{36}$$

$$g_1(t) = \sum_{n=0}^{\infty} \frac{\left(\frac{-\omega^2 t^{2+\lambda}}{\tau^\lambda}\right)^n}{n!} \sum_{m=0}^{\infty} \frac{\left(-\frac{\bar{\gamma}_\lambda t^2}{\tau^\lambda}\right)^m}{m!} E^{(n+m)}_{\lambda,1+2n+(2-\lambda)m}(-(t/\tau)^\lambda). \tag{37}$$

It is worth mentioning that expressions (30), (32), and (35) fully determine the temporal evolution of the mean values (9) and (13), variances (15)–(17), mean-square displacement (18), and velocity autocorrelation function (19).

Notice that in the limit $\omega \to 0$ only the terms with $n=0$ in Eqs. (26) and (27) survive. Then, Eq. (25) reduces to $\bar{\gamma}_\lambda = \gamma_\lambda$, and the expression of the relaxation function $I(t)$ for the free particle case [22] is recovered.

On the other hand, in the limit $\tau \to 0$ the function $I_1(t)$ vanishes and the behavior of $I_0(t)$ can be achieved introducing the asymptotic behaviors of the generalized Mittag-Leffler function [30]

$$E_{\alpha,\beta}(-y) \sim \frac{1}{y\Gamma(\beta-\alpha)}, \quad y > 0 \tag{38}$$

and its derivative

$$E^{(k)}_{\alpha,\beta}(-y) \sim \frac{k!}{y^{k+1}} \frac{1}{\Gamma(\beta-\alpha)} \tag{39}$$

in Eq. (26). Then, after some algebra we obtain

$$I(t) = \lim_{\tau \to 0} I_0(t) = \sum_{n=0}^{\infty} \frac{(-\omega^2 t^2)^n}{n!} t^2 E^{(n)}_{2-\lambda,3+\lambda n}(-\gamma_\lambda t^{2-\lambda}), \tag{40}$$

where we have used $\bar{\gamma}_\lambda \to \gamma_\lambda$ for $\tau \to 0$, according to Eq. (25). The expression in series given in Eq. (40) coincides with the expression for the relaxation integral function $I(t)$ corresponding to a pure power-law correlation function, previously obtained in Ref. [18].

Likewise, one can verify that in the limit $\tau \to 0$ the relaxation functions $G(t)$ and $g(t)$ are also the same to that in the case of a pure power-law correlation function given by

$$G(t) = \sum_{n=0}^{\infty} \frac{(-\omega^2 t^2)^n}{n!} t E^{(n)}_{2-\lambda,2+\lambda n}(-\gamma_\lambda t^{2-\lambda}), \tag{41}$$

$$g(t) = \sum_{n=0}^{\infty} \frac{(-\omega^2 t^2)^n}{n!} E^{(n)}_{2-\lambda,1+\lambda n}(-\gamma_\lambda t^{2-\lambda}). \tag{42}$$

V. TEMPORAL BEHAVIOR OF THE RELAXATION FUNCTIONS

The analytical expressions (30), (32), and (35) are the main result of this work. In the following we will analyze the time behavior of the relaxation functions for different regimes.

The short-time behavior ($t \ll \tau$) of the relaxation functions can be obtained using the series expansions (28) and (29). Then

$$I(t) \approx \frac{t^2}{2} - \left(\frac{\gamma_\lambda}{\tau^\lambda} + \omega^2\right)\frac{t^4}{24}, \tag{43}$$

$$G(t) \approx t - \left(\frac{\gamma_\lambda}{\tau^\lambda} + \omega^2\right)\frac{t^3}{6}, \tag{44}$$

$$g(t) \approx 1 - \left(\frac{\gamma_\lambda}{\tau^\lambda} + \omega^2\right)\frac{t^2}{2}, \tag{45}$$

which are the expected for a harmonic oscillator driven by a noise with a finite correlation at the origin [11,12,31].

Now we get an expression for the function $I(t)$ for times bigger than the characteristic time τ of the noise, i.e., $t \gg \tau$. For this purpose we introduce approximation (39) in Eqs. (26) and (27). After some algebra we get

$$I_0(t) \approx \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\omega^2 t^2)^n t^2 E^{(n)}_{2-\lambda,3+\lambda n}(-\bar{\gamma}_\lambda t^{2-\lambda}) \tag{46}$$

and

$$I_1(t) \approx \left(\frac{t}{\tau}\right)^{-\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\omega^2 t^2)^n t^2 E^{(n)}_{2-\lambda,3-\lambda+\lambda n}(-\bar{\gamma}_\lambda t^{2-\lambda}). \tag{47}$$

Let us analyze the behaviors of the relaxation functions $I(t)$, $G(t)$, and $g(t)$ for $\bar{\gamma}_\lambda t^{2-\lambda} \gg 1$. Introducing approximation (39) in Eqs. (46) and (47), after some calculations and using Eq. (5) one gets

$$I(t) \approx \frac{1}{\omega^2}\left\{1 - \nu E_\lambda\left(-\nu \frac{\omega^2}{\gamma_\lambda} t^\lambda\right)\right\}. \tag{48}$$

Then, from Eqs. (11) and (14) we get

$$G(t) \approx -\frac{\nu}{\omega^2}\frac{d}{dt} E_\lambda\left(-\nu \frac{\omega^2}{\gamma_\lambda} t^\lambda\right) \tag{49}$$

and

$$g(t) \approx -\frac{\nu}{\omega^2}\frac{d^2}{d^2 t} E_\lambda\left(-\nu \frac{\omega^2}{\gamma_\lambda} t^\lambda\right), \tag{50}$$

where we introduced the dimensionless factor

$$\nu = \frac{\gamma_\lambda}{\bar{\gamma}_\lambda}, \quad 0 < \nu \le 1. \tag{51}$$

The relaxation functions (48)–(50) have the same functional form of that obtained in the pure power-law case [18] but with the presence of the scale factor ν. In the limit $\tau \to 0$, $\nu = 1$ and one recovers the expressions corresponding to a pure power-law noise [18].

It is worth pointing out that these expressions are the same to those that can be obtained directly discarding the inertial term s^2 in Eq. (10). Then, Eqs. (48)–(50) represent the solutions in the high friction limit.

The strictly asymptotic behavior of the relaxation functions $I(t)$, $G(t)$, and $g(t)$ can be obtained by introducing the asymptotic behavior (7) of the Mittag-Leffler function in Eqs. (48)–(50). Then, for $\nu \frac{\omega^2}{\gamma_\lambda} t^\lambda \gg 1$ the relaxation functions can be written as

$$I(t) \approx \frac{1}{\omega^2} - \frac{\gamma_\lambda}{\omega^4}\frac{\sin(\lambda\pi)}{\pi}\frac{\Gamma(\lambda)}{t^\lambda}, \tag{52}$$

$$G(t) \approx \frac{\gamma_\lambda}{\omega^4}\frac{\sin(\lambda\pi)}{\pi}\frac{\Gamma(\lambda+1)}{t^{\lambda+1}}, \tag{53}$$

$$g(t) \approx -\frac{\gamma_\lambda}{\omega^4}\frac{\sin(\lambda\pi)}{\pi}\frac{\Gamma(\lambda+2)}{t^{\lambda+2}}. \tag{54}$$

As expected, the relaxation functions (52)–(54) behave as a power law in the long-time limit. These results are in agreement with those obtained in Refs. [18,32] due to the fact that the Mittag-Leffler noise decays as a power law for

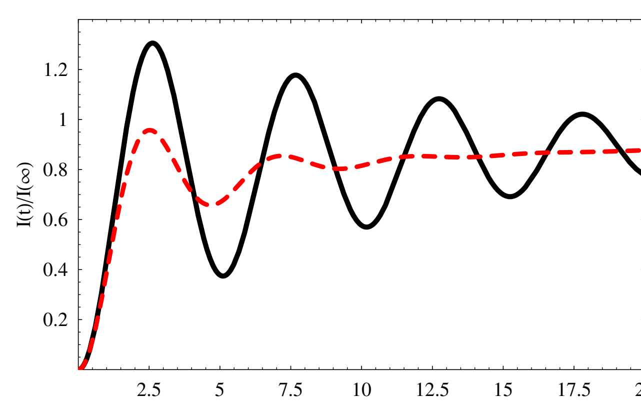

FIG. 1. (Color online) Relaxation function $I(t)$ vs time t, for $\lambda = 1/2$, $\gamma_\lambda = 1$, $\tau = 1$, and $\omega = 1$. The solid line corresponds to the exact expression (30) (for a Mittag-Leffler noise); the dashed line corresponds to the exact expression (40) (for a pure power-law noise).

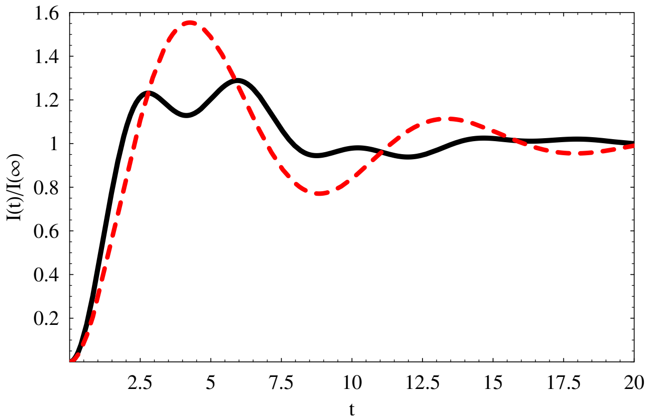

FIG. 2. (Color online) Relaxation function $I(t)$ vs time t, for $\lambda=3/2$, $\gamma_\lambda=1$, $\tau=1$, and $\omega=1$. The solid line corresponds to the exact expression (30) (for a Mittag-Leffler noise); the dashed line corresponds to the exact expression (40) (for a pure power-law noise).

very large times. In the same way, substitution of these asymptotic expansions into Eqs. (15)–(17) give the long-time behavior of the variances of the process, which again coincide with those obtained in Refs. [18,32].

In Figs. 1 and 2 we have plotted the relaxation function $I(t)$ (30) obtained with a Mittag-Leffler noise together with the corresponding expression (40) for a pure power law. Although both functions coincide with Eq. (52) in the strictly asymptotic limit, it can be seen that in the displayed range of short and intermediate times they exhibit considerable differences. In particular, the relaxation function $I(t)$ given by Eq. (30) exhibits more oscillations with respect to expression (40), and minima and maxima are located in different positions.

Similar behavior can be seen in Figs. 3 and 4 where we have compared the relaxation function $g(t)$ (35) corresponding to a Mittag-Leffler noise with function (42) obtained with a pure power law. From these two figures, one can realize that the function $g(t)$ (35) also shows more oscillations than the function $g(t)$ given by Eq. (42). Moreover, the first one exhibits more zero crossings, which represent transitions between a positive velocity correlation and velocity anticorrelations.

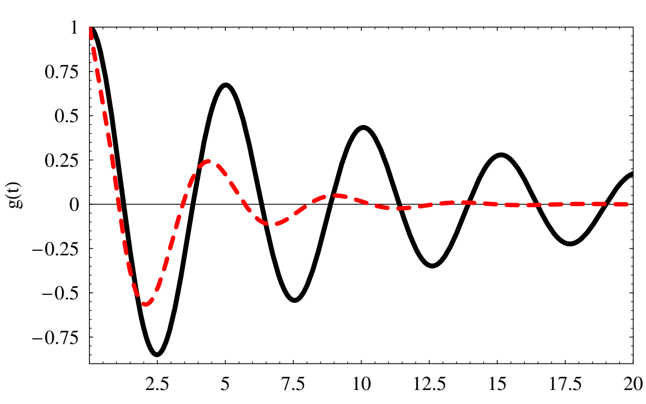

FIG. 3. (Color online) Relaxation function $g(t)$ vs time t, for $\lambda=1/2$, $\gamma_\lambda=1$, $\tau=1$, and $\omega=1$. The solid line corresponds to the exact expression (35) (for a Mittag-Leffler noise); the dashed line corresponds to the exact expression (42) (for a pure power-law noise).

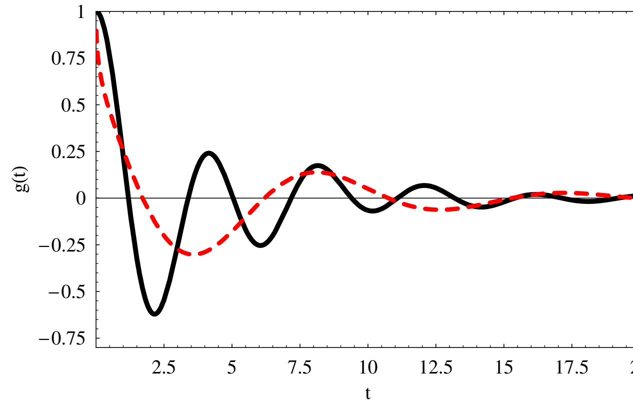

FIG. 4. (Color online) Relaxation function $g(t)$ vs time t, for $\lambda=3/2$, $\gamma_\lambda=1$, $\tau=1$, and $\omega=1$. The solid line corresponds to the exact expression (35) (for a Mittag-Leffler noise); the dashed line corresponds to the exact expression (42) (for a pure power-law noise).

VI. CONCLUSIONS

In this work we have presented an analytically resoluble model for the dynamics of a classical harmonic oscillator in a complex environment, which is valid for all time range. We have shown that an anomalous diffusion process can be generated by a Mittag-Leffler noise deriving exact expressions for the relaxation functions of the oscillator in terms of the generalized Mittag-Leffler function and its derivatives. Moreover, in the appropriate limits the results for a harmonic oscillator driven by a power-law noise are recovered. However, differences in relation to the usually employed pure power-law noise appear in the interval of short and intermediate times. For times shorter than the characteristic time of the noise, the relaxation functions include a correction due to the presence of the characteristic time τ. In the range of intermediate times, the relaxation functions have a similar functional form to that previously obtained for a pure power-law noise [18], but with the inclusion of a scaling dimensionless parameter. Finally, in the strictly asymptotic limit, we recover the anomalous behavior of a harmonically bounded particle driven by a power-law noise, which is in agreement with the previous results given in Refs. [18,32].

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