Symbolic observability coefficients for univariate and multivariate analysis

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In practical problems, the observability of a system not only depends on the choice of observable(s) but also on the space which is reconstructed. In fact starting from a given set of observables, the reconstructed space is not unique, since the dimension can be varied and, in the case of multivariate measurement functions, there are various ways to combine the measured observables. Using a graphical approach recently introduced, we analytically compute symbolic observability coefficients which allow to choose from the system equations the best observable, in the case of scalar reconstructions, and the best way to combine the observables in the case of multivariate reconstructions. It is shown how the proposed coefficients are also helpful for analysis in higher dimension.

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I. INTRODUCTION

A system is fully observable from a variable when it is possible to recover all the dynamical variables of the system. Although control theory gives a yes-or-no answer to the question *Is the system observable?*, it is possible to quantify the observability using real numbers, and thus to provide an answer that takes into account the time spent by the system in the neighborhood of singularities in phase space, such are regions where the system is not observable [1]. Indeed, if such regions are not visited frequently or if the system passes by such regions quickly, one may expect that the analysis of the underlying dynamics will not be severely affected.

The quality with which the dynamics can be reconstructed from a measured variable depends on the choice of the observable [1]. Because of this, it is important that any analysis of observability should take the reconstructed space into account. The observability has to do with the coordinate transformation between the original phase space and the reconstructed one. *When observability is quantified, it should always be related to the chosen observables and the way in which the working space is reconstructed* [2]. In control theory, the observability is always investigated in terms of a "yes or no" quality, that is, the system is either observable or not for a chosen measuring function. Nevertheless, practical studies in nonlinear dynamical systems theory reveal that results of control and synchronization techniques [2] global modeling or even statistical analysis deeply depend on both the choice of the observable and the way in which the reconstructed space is built [1,3,5–8]. More often than not systems under investigation are (mathematically) observable and therefore a "yes or no" answer is inadequate to interpret the observed results. For instance, mathematically speaking, the Rössler system is not observable from the x-variable but, this variable is a quite reliable observable in practice. Such a conclusion does not hold true for the z variable from which a global model is almost impossible to obtain unless very specific structure selection procedures are used [14,15]. In practical problems, observability helps understand results in variable-dependent analyses [3], it aids in choosing the best way of coupling two systems [2], to mention a few applications.

Computing the observability for nonlinear systems may require a high analytical effort since it requires computing Lie derivatives. Generally, it is useful to rank the observables according to their respective observability—assuming that the reconstruction space has the same dimension of the original one. This can be done by computing some real coefficients [4] and averaging the results along a trajectory [2].

Recently, a simple graphical procedure was introduced to rank the observables according to the respective observability without any numerical computations [5]. This paper addresses the problem of computing a measure of the observability using an algebraic algorithm. On the other hand, the graphical approach is extended to the multivariate case as well as for those cases in which the dimension of the reconstruction space is greater than that of the original phase space. Finally, in this paper it is shown that observability is related to the possibility of rewriting the systems in a polynomial form using only the chosen observable.

II. OBSERVABILITY, DIFFERENTIAL EMBEDDINGS, AND JERK SYSTEMS

Let us start with a nonlinear system

$$\dot{x}_i = f_i(x_1,x_2,x_3), \quad i=1,2,3, \tag{1}$$

described in a three-dimensional phase space for the sake of simplicity and where $x_i \in \mathbb{R}$ are the dynamical variables. Assume that the observable is variable x_i. It is thus possible to reconstruct the phase space from the time series $\{x_i(t)\}$ using derivative coordinates $(X=x_i, Y=\dot{x}_i, Z=\ddot{x}_i)$. The coordinate transformation between the original phase space $\mathbb{R}^3(x_1,x_2,x_3)$ and the differentiable embedding $\mathbb{R}^3(X,Y,Z)$ is defined according to

$$\Phi_i \left| \begin{array}{l} X = x_i \\ Y = \dot{x}_i = f_i \\ Z = \ddot{x}_i = \dfrac{\partial f_i}{\partial x_1} f_1 + \dfrac{\partial f_i}{\partial x_2} f_2 + \dfrac{\partial f_i}{\partial x_3} f_3 . \end{array} \right. \tag{2}$$

It is seen that variables X, Y, and Z are in fact Lie derivatives of the observable of order zero, one and two, respectively. It has been shown that the observability matrix $\mathcal{O}_i$ of a nonlinear system observed using the ith variable is exactly the Jacobian matrix of the map Φ_i [2]. The system is therefore fully observable when the determinant $\text{Det}(\mathcal{J}_{\Phi_i})$ never vanishes, that is, when map Φ_i defines a global diffeomorphism (Φ_i must also be injective, a property observed in most of the cases).

When $\text{Det}(\mathcal{J}_{\Phi_i})$ never vanishes, the map Φ_i can be inverted everywhere and the system can be always rewritten under the form of a jerk system

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} = F_i(X,Y,Z) &= \frac{\partial Z}{\partial x_1} f_1 + \frac{\partial Z}{\partial x_2} f_2 + \frac{\partial Z}{\partial x_3} f_3, \end{aligned} \tag{3}$$

where the model function $F_i(X,Y,Z)$ is free of singularities and subscript i designates the measured variable. Otherwise, a jerk system may be obtained but with some singularities. When a system can be rewritten as a jerk system without any singularity, this means that there is a global diffeomorphism between the original phase space and the reconstructed differential space [9] and, consequently, that the system is fully observable. Conversely, when the system is fully observable, the system can be rewritten as a jerk system.

When a singularity occurs, that is, $\text{Det}(\mathcal{J}_{\Phi_i})=0$ at some location in the reconstructed space, the system is not fully observable. A direct consequence is that, even if the original system is polynomial, it can no longer be rewritten as a polynomial jerk system although it could be written as, say, a rational jerk system. For instance, in the case of the Rössler system, a rational jerk system can be obtained from x or z variable although the corresponding coordinate transformations involve one singularity each [14].

III. GRAPHICAL INTERPRETATION

Interactions between the dynamical variables can be defined using elements of the Jacobian matrix of the vector field $f_i(x_j)$. Variable x_j acts on variable x_i when term J_{ij} of the Jacobian matrix is nonzero. This action is positive or negative depending on the sign of element J_{ij}. These interactions can be displayed as a graph. Each variable x_i is represented by a node N_i. When the variable j is present in function f_i, thus an arrow is drawn from node N_j to node N_i. When the variable only appears in a linear term, the arrow is drawn with a solid line. For nonlinear terms, the arrow is drawn with a dashed line. Such graphs—sometimes named graph of fluences—were used by Rössler in the early 1970s [10].

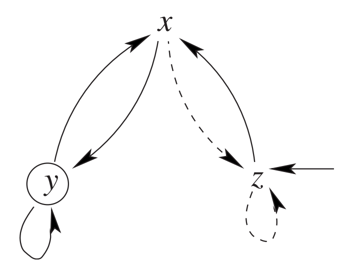

FIG. 1. Graph of interactions between dynamical variables of the Rössler system. A solid (dashed) arrow represents a (non)linear coupling. When the system is fully observable from a variable, the corresponding variable is encircled.

Let us draw the graph for the Rössler system [12]:

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c). \end{aligned} \tag{4}$$

The first equation tells us that variables y and z act linearly on the derivative of x. Thus, two arrows coming from nodes N_y and N_z will reach node N_x with a solid line. The second equation can be interpreted likewise. The third equation means that there is a constant input and that variables x and z nonlinearly act on z and, z acts linearly on z. In the graph of interactions, the nonlinear coupling is dominant over the linear one. Only a dashed line is therefore drawn from node N_z to itself. Thus there is a dashed arrow from node N_x to node N_z and one dashed arrow from node N_z to itself. The latter arrow represents the action of the variable on its own derivatives. The whole graph is shown in Fig. 1. The solid arrow not coming from a node represents the constant input b in the third equation.

When a variable is measured, it is for sure known. Taking one of its successive time derivatives corresponds to moving along the arrows (that reach this variable) in opposite direction (contrary to the arrow). For instance, assume the y variable of the Rössler system is measured. Taking its first derivative allows to reach the x variable, but not to the z variable since there is no arrow from N_z to N_y. It is necessary to take the second derivative of y to finally reach node N_z, since there is an arrow from N_z to N_x. Since all arrows involved from y to z are solid lines, that is, the z variable is seen from y through linear couplings, the system is fully observable. This can be viewed in Fig. 2 where the paths from the measured variable toward the others are displayed as successive derivatives are taken. Figure 2 is an "unfolded" version of the graph shown in Fig. 1. The path from the y variable reaches variables x and z through solid arrows. The dynamics is therefore fully observable.

Let us start now from the measurement of the x variable of the Rössler system. Taking its first derivative allows to reach both y and z through solid arrows between nodes N_y and N_x, and N_z and N_x, respectively. But at least three variables are required to fully span a three-dimensional system. The second derivative is thus computed. It allows to travel contrary to the dashed arrow between N_x and N_z (Figs. 1 and

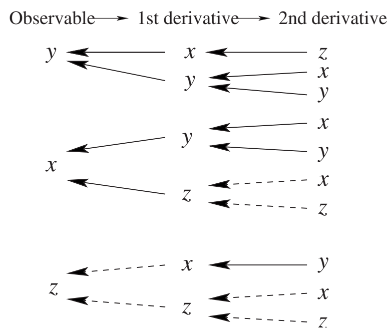

FIG. 2. Unfolded schematic view of variables reached when first and second derivative are computed. Involving a nonlinearity in the first derivative (a dashed arrow between the observable and its first derivative) induces a more serious lack of observability than when a nonlinearity occurs in the second derivative (a dashed arrow between the first and the second derivatives of the observable).

2). A nonlinearity is thus involved in the computation and a singularity will occur.

Each time a dashed arrow is visited, a nonlinearity occurs and, consequently, there is a nonconstant element in the Jacobian matrix. A singularity thus exists in the coordinate transformation between the original phase space and the reconstructed phase space. This implies that when a dashed arrow is visited contrary to its direction, the dimension of the phase space should be increased at least by one in an attempt to remove the singularity. How much should the dimension be increased in order to overcome the singularity will depend on the type (mainly the order) of the nonlinearity [11].

IV. SYMBOLIC OBSERVABILITY COEFFICIENT

From previous investigations [5] two important points have been observed: (i) any nonlinear coupling between the observable and itself does not affect the observability; and (ii) two variables that form a closed loop (with linear or nonlinear arrows) among themselves in the fluence graph do not contribute to the observability when increasing the dimension (taking two time derivatives). For instance, consider the x and z variables in Fig. 1. Starting from x and taking one time derivative leads us to z. However, taking an additional time derivative will, on the one hand, bring us back to x and, on the other, maintain us in z. Therefore, this part of the graph (the aforementioned loop) did not contribute to provide observability since no information of y was gained. Also, even when the singularity due to the dashed arrow is visited, this only means that the x information coming from the time derivative of the z variable will not be available. However, as we started from the x variable, then such info was already available. Therefore it is seen that the loop does not affect observability. Fortunately, starting from x and taking a derivative will also lead us to y, through the other path.

Let us take two other examples to illustrate the two assertions above. *The first example* is Sprott P system (whose equations are given in Table I) [13]. When a differential space is reconstructed from variable y, the coordinate transformation Φ_{y^2} between the original phase space $\mathbb{R}^3(x,y,z)$ and the reconstructed space $\mathbb{R}^3(X,Y,Z)$ is

$$\Phi_{y^2} = \left| \begin{array}{l} X = y \\ Y = -x + by^2 \\ Z = -ay - z + 2by(-x + by^2) \end{array} \right. \tag{5}$$

and its Jacobian matrix is

$$\mathcal{J}(\Phi_{y^2}) \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2by & 0 \\ -2by & (-a-2bx+4b^2y^2) & -1 \end{bmatrix}. \tag{6}$$

Here the exponent 2 applied to the subscript y means that y was differentiated twice with respect to the time. The determinant of this matrix, $\Delta\mathcal{J}(\Phi_{y^2})=-1$, never vanishes and, consequently, Φ_{y^2} defines a global diffeomorphism, although a nonlinear term is involved. This term results from the coupling between the observable and its own derivative due to the element F_{22} in the Jacobian matrix of Sprott P system. But, as seen above, this nonlinearity has no impact on $\Delta\mathcal{J}(\Phi_{y^2})$, that is, on the observability.

A second example involves Sprott H system (Table I) [13]. When investigated from variable z, the coordinate transformation

$$\Phi_{z^2} = \left| \begin{array}{l} X = z \\ Y = x - bz \\ Z = -bx - y + b^2 z + z^2 \end{array} \right. \tag{7}$$

has a Jacobian matrix

$$\mathcal{J}(\Phi_{z^2}) \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -b \\ -b & -1 & (2z+b^2) \end{bmatrix}. \tag{8}$$

Again, although there is nonlinearity involved, the determinant $\Delta\mathcal{J}(\Phi_{z^2})=-1$ never vanishes and a global diffeomorphism exists between the original phase space and the reconstructed one. There is a nonlinear coupling between observable z and variable x forming a closed loop between node N_z and N_x in the fluence graph. As mentioned before, this type of loop does not affect the observability.

These features are the two relevant properties used to develop the observability coefficients as described below. Since graphs of interactions between dynamical variables of a given system and their derivatives become quickly complicated when the dimension either of the original phase space or those of the reconstructed phase space is increased, an algebraic procedure would avoid to draw such graphs and is therefore welcome.

A. Procedure

The first step is to encode the graph of fluences in an $m \times m$ matrix where m is the dimension of the original phase space $\mathbb{R}^m$. Each linear coupling—solid line—is associated with symbol "1" while each nonlinear coupling—dashed line—is encoded by "$\bar{1}$." When there is no coupling, symbol "0" is used. For instance the fluence matrix F_{ij} of the Rössler system is

TABLE I. Sets of equations here investigated with the Jacobian of the Δ_i coordinate transformation Φ_i between the original phase space and the phase space reconstructed from the ith variable using the derivative coordinates. When the system is fully observable, that is, when Φ_i defines a global diffeomorphism and that a jerk system can be written from this variable, $\eta=1$. Since we sometimes applied a permutation between the dynamical variables to show in a better way the similarities between the Sprott systems, we indicated (if different) in parenthesis the corresponding variable in their original presentation (and as in Ref. [9]).

	Equations	$\Delta_i=\mathrm{Det}(\mathcal{J}_{\Phi_i})$		Matrix F_{ij}
Rössler system				
	$\dot{x}=-y-z$	$\Delta_x=x-(a+c)$	$\eta_{x^2}=0.88$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ \bar{1} & 0 & \bar{1} \end{bmatrix}$
	$\dot{y}=x+ay$	$\Delta_y=1$	$\eta_{y^2}=1.0$	
	$\dot{z}=b+z(x-c)$	$\Delta_z=-z^2$	$\eta_{z^2}=0.44$	
System F				
	$\dot{x}=y+z$	$\Delta_x=-(a+b)$	$\eta_{x^2}=1.0$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ \bar{1} & 0 & 1 \end{bmatrix}$
	$\dot{y}=-x+ay$	$\Delta_y=1$	$\eta_{y^2}=1.0$	
	$\dot{z}=-bz+x^2$	$\Delta_z=4x^2$	$\eta_{z^2}=0.44$	
System H				
	$\dot{x}=-y+z^2$	$\Delta_x=-2x+2z(a+2b)$	$\eta_{x^2}=0.81$	$\begin{bmatrix} 0 & 1 & \bar{1} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
	$\dot{y}=x+ay$	$\Delta_y=-2z$	$\eta_{y^2}=0.88$	
	$\dot{z}=x-bz$	$\Delta_z=-1$	$\eta_{z^2}=1.0$	
System K				
	$\dot{x}=-ay+xz$	$\Delta_x=a(b+1)x+a^2y-axz$	$\eta_{x^2}=0.72$	$\begin{bmatrix} \bar{1} & 1 & \bar{1} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
	$\dot{y}=x+by$	$\Delta_y=-x$	$\eta_{y^2}=0.78$	
	$\dot{z}=x-z$	$\Delta_z=-a$	$\eta_{z^2}=1.0(y)$	
System O				
	$\dot{x}=y-z$	$\Delta_x=-1-y-z$	$\eta_{x^2}=0.75$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & \bar{1} & \bar{1} \end{bmatrix}$
	$\dot{y}=ax$	$\Delta_y=-a^2$	$\eta_{y^2}=1.0(x)$	
	$\dot{z}=bx+y+yz$	$\Delta_z=b^2(1+x)+by(1+z)-a(1+2z+z^2)$	$\eta_{z^2}=0.78$	
System P				
	$\dot{x}=ay+z$	$\Delta_x=-1-2aby$	$\eta_{x^2}=0.88$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & \bar{1} & 0 \\ 1 & 1 & 0 \end{bmatrix}$
	$\dot{y}=-x+by^2$	$\Delta_y=-1$	$\eta_{y^2}=1.0$	
	$\dot{z}=x+y$	$\Delta_z=1+a+2by$	$\eta_{z^2}=0.89$	
System G				
	$\dot{x}=-y+z$	$\Delta_x=(a+b)-x$	$\eta_{x^2}=0.72$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ \bar{1} & \bar{1} & 1 \end{bmatrix}$
	$\dot{y}=x+ay$	$\Delta_y=-1$	$\eta_{y^2}=1.0(x)$	
	$\dot{z}=-bz+xy$	$\Delta_z=2(x^2-y^2)+yz$	$\eta_{z^2}=0.47$	
System M				
	$\dot{x}=-z$	$\Delta_x=-1$	$\eta_{z^2}=1.0$	$\begin{bmatrix} 0 & 0 & 1 \\ \bar{1} & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
	$\dot{y}=-x^2-ay$	$\Delta_y=4x^2$	$\eta_{x^2}=0.44$	
	$\dot{z}=b+bx+y$	$\Delta_z=2x-ab$	$\eta_{z^2}=0.88$	

TABLE I. (*Continued.*)

	Equations	$\Delta_i = \text{Det}(\mathcal{J}_{\Phi_i})$		Matrix F_{ij}
System Q				
	$\dot{x} = -z$	$\Delta_x = -2y$	$\eta_{x^2} = 0.88$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & \bar{1} & 1 \end{bmatrix}$
	$\dot{y} = x - ay$	$\Delta_y = 1$	$\eta_{y^2} = 1.0$	
	$\dot{z} = bx + y^2 + cz$	$\Delta_z = 2b(x + -2ay) - 4y^2$	$\eta_{z^2} = 0.72$	
System S				
	$\dot{x} = -x - 4z$	$\Delta_x = -2a^2 y$	$\eta_{x^2} = 0.72$	$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & \bar{1} & 0 \end{bmatrix}$
	$\dot{y} = 1 + x$	$\Delta_y = a$	$\eta_{y^2} = 1.0(z)$	
	$\dot{z} = x + y^2$	$\Delta_z = 2(1 + x + y - 2y^2)$	$\eta_{z^2} = 0.81$	

$$F_{ij} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ \bar{1} & 0 & \bar{1} \end{bmatrix}. \tag{9}$$

The relevant part of the graph of fluences is encoded in this matrix. In fact, only constant term b ending on node N_z is not taken into account. It does not play any role in the observability problem and can be omitted without major consequences.

The second step is to choose the measurement function $h: \mathbb{R}^m \mapsto \mathbb{R}$, that is, to choose the observable. The measurement function h takes the form of a row vector, called C defined as

$$s = C\boldsymbol{x} = [0 \quad 1 \quad 0] \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{10}$$

when variable y is measured. Matrix $[H_i]_1$ is a column vector defined as the sum of *rows* of C^T. For this example, this column vector is

$$[H_i]_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{11}$$

The action of matrix $[H_i]_1$ on the fluence matrix F_{ij} is achieved by a product $\otimes$ which acts as follows

$$g_1 \equiv [H_i]_1 \otimes [F_{ij}] = [[H_i]_1 \times F_{ij}] \tag{12}$$

that is, all elements of the ith row of matrix F_{ij} are multiplied by the ith element of $[H_i]_1$. Obviously,

$$[H_i] \otimes [F_{ij}] = [F_{ij}] \otimes [H_i]. \tag{13}$$

To encode the two properties observed from our previous study [5], each time the operator for derivating the observable (with respect to time) is applied, matrix $[H_i]$ has to be iterated as follows. Let us start by transposing the matrix, which in the first iteration will yield

$$[[H_i]_1 \times F_{ij}]^T. \tag{14}$$

Then, for each row i of matrix $[[H_i]_1 \times F_{ij}]^T$, the sum of *nonzero elements*

$$\sigma_i = \sum [[H_i]_1 \times F_{ij}] \tag{15}$$

defines a column vector that corresponds to the iterate of matrix $[[H_i]_1]$, designated by $[H_i]_2$. The product

$$g_2 = [H_i]_2 \otimes [F_{ij}] \tag{16}$$

is computed. In this new matrix, all elements corresponding to nonzero elements of $[[H_i]_1 \times F_{ij}]^T$ are set to 0. This is to take into account the fact that couplings forming a closed loop do not contribute to the observability problem. Then count the number p_k of remaining elements "1" and q_k of elements "$\bar{1}$." Repeat until the number of time derivatives n necessary to reconstruct the space is attained. The symbolic observability coefficient η is defined as

$$\eta_{s^n} = \frac{1}{m-1} \sum_{i=1}^{n} \frac{p_i}{(p_i + q_i)} + \frac{q_i}{(\max[p_i, 1] + q_i)^{n-i+2}}, \tag{17}$$

where m designates the dimension of the original phase space and n the number of derivatives that must be computed. This means for monovariable reconstructions that the reconstructed space has a dimension $n+1$.

Obviously, when $\Sigma_i^n q_i = 0$, the observability coefficient equals 1, that is, the dynamics is fully observable since there are no nonlinear terms involved in the coordinate transformation between the original phase space $\mathbb{R}^m$ and the reconstructed space $\mathbb{R}^{(n+1)}$.

Remark. The $\max[\cdot,\cdot]$ function used in Eq. (17) is used as a way to force a decrease in the observability coefficient due to the presence of a nonlinear term.

B. Example

Let us treat the case of the Rössler system from the y variable. Then $[H_i]_1 = [0 \;\, 1 \;\, 0]^T$. Its product by the fluence matrix of the Rössler system [Eq. (9)] leads to [see Eq. (12)],

$$g_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ \bar{1} & 0 & \bar{1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & \cdot & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (18)$$

The $\cdot$ means zero for the present iteration and will be replaced to its original value in the next iteration. This is to take into account the fact that the feedback of a variable on itself does not contribute to the observability problem. In this case, we have $p_1=1$ and $q_1=0$. It is pointed out that the computation of p_i and q_i must be done *before* the dots are replaced by their original values. The transpose of matrix g_1 with the $\cdot$ replaced by the original values is formed; g_i^T determines which elements will be replaced by dots in the next iteration (g_{i+1}), namely, all nonzero elements in g_i^T are replaced with dots in g_{i+1}, then p_i and q_i are computed. The sum of the elements of each of its rows is computed to produce matrix $[H_i]_2$, that is

$$g_1^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto [H_i]_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \quad (19)$$

This ends the first iteration. The second iteration follows the first one with $[H_i]_2$ in stead of $[H_i]_1$, that is

$$g_2 = [H_i]_2 \otimes \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ \bar{1} & 0 & \bar{1} \end{bmatrix} = \begin{bmatrix} 0 & \cdot & 1 \\ 1 & \cdot & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (20)$$

where elements corresponding to nonzero elements of g_1^T have been replaced by $\cdot$ to take into account of the fact that couplings forming a closed loop do not contribute. The $\cdot$ means that, in the present iteration, the value is zero. But it is replaced by its original value in the next iteration. We have now $p_2=2$ and $q_2=0$. Since $\Sigma_i^2 q_i=0$ ($\forall i$), the Rössler system is fully observable from the y variable. A result found using other approaches [1,5].

Applied to the three variables of the Rössler system, we got

$$\eta_{y^2} = 1 > \eta_{x^2} = 0.88 > \eta_{z^2} = 0.44,$$

that is, variables are ranked according to the order provided by the observability measures as in [5]. This symbolic observability coefficient is thus able to correctly rank the variables of the Rössler system and it is claimed it is quite general for nonlinear systems. The observability coefficients have been computed for the nine Sprott systems previously investigated [5] and are reported in Table I. In all cases, the observability coefficient is equal to 1 whenever the system is fully observable and the variables are well ranked as the complexity of their singularity suggest (see [1] for details).

According to Takens theorem, it is suggested that the dimension of the reconstructed space is $2d+1$ to ensure a diffeomorphism between the original phase space and the reconstructed one. In the case of the z variable of the Rössler system, it is known [14] that a three-dimensional space is insufficient to allow a successful global modeling without a strong structure selection [15]. Using the observability coefficient here introduced, it is possible to give some arguments to justify such a result. The successive graph matrices g_k corresponding to the Rössler system observed from the z variable are

$$g_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{1} & 0 & \cdot \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 & 1 & \cdot \\ 0 & 0 & 0 \\ \bar{1} & 0 & \cdot \end{bmatrix}, \quad g_3 = \begin{bmatrix} 0 & 1 & \cdot \\ \cdot & 1 & 0 \\ \cdot & 0 & \cdot \end{bmatrix},$$

and

$$g_4 = \begin{bmatrix} 0 & \cdot & \cdot \\ \cdot & \cdot & 0 \\ \cdot & 0 & \cdot \end{bmatrix}.$$

From such matrices, it is clear that: $q_1=1$, $p_1=0$; $q_2=1$, $p_2=1$; $q_3=0$, $p_3=2$. When a four-dimensional (4D) reconstructed space is considered ($n=3$), the observability coefficient is obtained from Eq. (17)

$$\eta_{z^3} = \frac{1}{2}\left[\frac{1}{2^4} + \left(\frac{1}{2} + \frac{1}{2^3}\right) + 1\right] = 0.84. \quad (21)$$

By increasing the dimension of the reconstructed space, the observability of the Rössler system from the z variable increased up to 0.84. This value is significantly greater than the observability coefficient η_{z^2}. The observability is only slightly smaller than when a 3D differential space is reconstructed using variable x. Since a global model is easily obtained from that variable, this means that the 4D space should allow successful global modeling too. Such a feature was, in fact, confirmed by a successful 4D global model rather easily obtained without any strong structure selection [14].

When a five-dimensional (5D) space reconstructed from the z variable is considered, the observability coefficient becomes

$$\eta_{z^4} = \frac{1}{2}\left[\frac{1}{2^5} + \left(\frac{1}{2} + \frac{1}{2^4}\right) + 1 + 1\right] = 1.3, \quad (22)$$

that is, significantly greater than 1. We conjecture that such an excessively large value means that it is not necessary to unfold the attractor in a 5D space and, consequently, that a 4D space is sufficient to embed the attractor, as suggested by our 4D model. Matrix g_4 also suggests that there is no gain in adding a new dimension since all of its elements are zero. Our observability coefficient allows to state about the quality of the observability even when the dimension of the reconstructed space is less than what is required by Takens' criterion.

C. Example with a cubic nonlinearity

In his attempt to list the different types of chaos, Rössler proposed the system [18]

$$\dot{x} = -ax - y(1 - x^2),$$

$$\dot{y} = \mu(y + 0.3x - 2z),$$

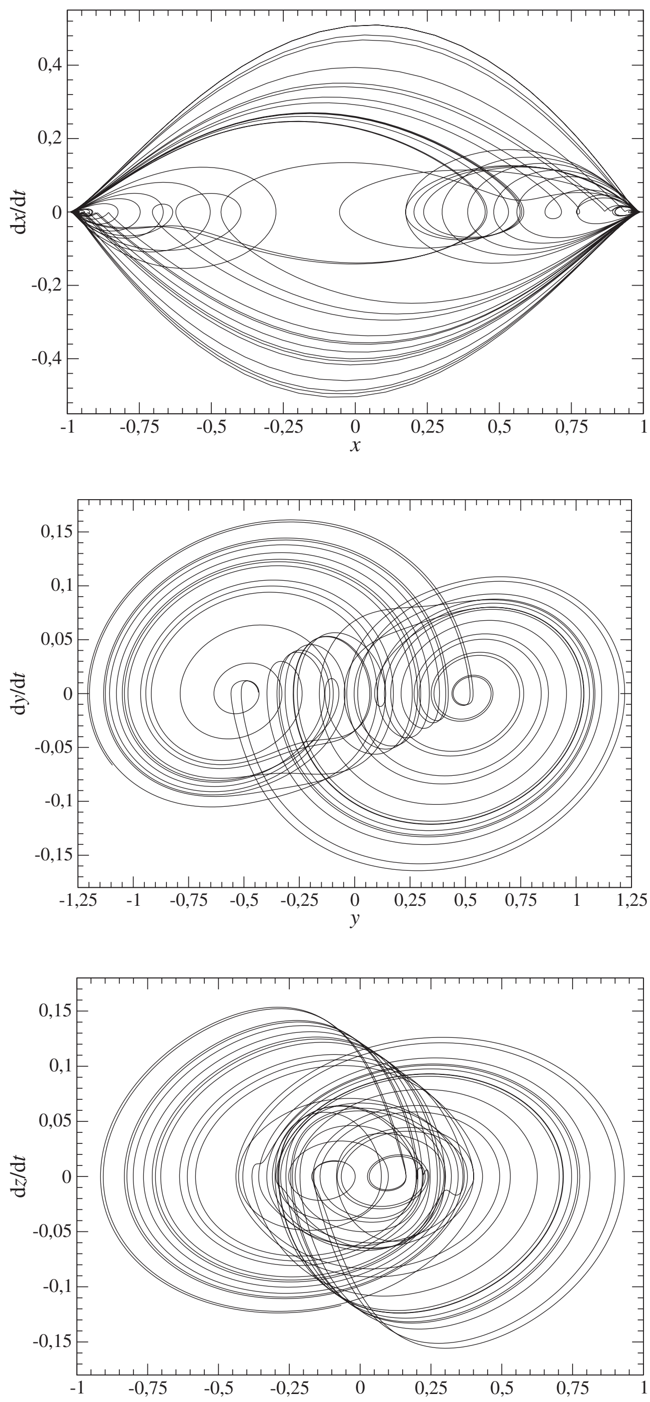

FIG. 3. Chaotic attractor solution to the cubic Rössler system.

$$\dot{z} = \mu(x + 2y - 0.5z), \tag{23}$$

with a cubic nonlinearity and that produces a double scroll attractor. Parameter values are $a=0.03$ and $\mu=0.1$ (and not 10 as in the original paper). Initial conditions are chosen such as $x_0=-1$, $y_0=0.55$, and $z_0=0.12$. Three plane projections of the differential embeddings produced by each of the variables are shown in Fig. 3. From visual inspection the embedding obtained from variable y is the best. The worst is the portrait reconstructed from variable x due to its squeezed edges [left and right in Fig. 3(a)].

The fluence matrix of this cubic Rössler system is

$$F_{ij} = \begin{bmatrix} \bar{1} & \bar{1} & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{24}$$

The observability coefficients are computed as previously described. They are: $\eta_{x^2}=0.45$, $\eta_{y^2}=0.92$, and $\eta_{z^2}=0.75$. These values lead to the order

$$y \triangleright z \triangleright x,$$

which is consistent from the phase portraits shown in Fig. 3 and normalized observability indices that can be computed.

V. CASE OF A HYPERCHAOTIC SYSTEM

A more complicated case is now investigated. It corresponds to the four-dimensional hyperchaotic Rössler system [19]. The equations are

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay + w, \\ \dot{z} &= b + xz, \\ \dot{w} &= -cz + dw. \end{aligned} \tag{25}$$

The fluence matrix of this system is

$$F_{ij} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ \bar{1} & 0 & \bar{1} & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{26}$$

When x is the measured variable, the graph matrices are

$$g_1 = \begin{bmatrix} \cdot & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \cdot & 1 & 0 & 1 \\ \cdot & 0 & \bar{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\text{and} \quad g_3 = \begin{bmatrix} 0 & \cdot & \cdot & 0 \\ 1 & \cdot & 0 & 1 \\ \bar{1} & 0 & \cdot & 0 \\ 0 & \cdot & 1 & 1 \end{bmatrix}.$$

The observability coefficient is thus

$$\eta_{x^3}\frac{1}{3}\left[1 + \left(\frac{2}{3} + \frac{1}{3^3}\right) + \left(\frac{4}{5} + \frac{1}{5^2}\right)\right] = 0.85. \tag{27}$$

The other observability coefficients are

$$\eta_{y^3} = 0.92, \quad \eta_{z^3} = 0.56, \quad \text{and} \quad \eta_{w^3} = 0.69.$$

These values lead to the order $y \triangleright x \triangleright w \triangleright z$, that is, the order obtained with the observability coefficients

$$\delta_{x^3} = 2.2 \times 10^{-4},$$

$$\delta_{y^3} = 9.0 \times 10^{-4},$$

$$\delta_{z^3} = 1.3 \times 10^{-7},$$

$$\delta_{w^3} = 2.1 \times 10^{-4}, \tag{28}$$

previously computed [5].

VI. MULTIVARIATE ANALYSIS

Sometimes it is possible to record simultaneously more than one physical variable. The available time series is thus multivariate. Multivariate time series data are available in many practical situations: for example, physiological data and economic data are often multidimensional. According to Takens theorem, multivariate time series are not required for reconstructing a space equivalent to the original phase space, since a scalar time series allows this. However, for dynamics from the real world, there may be significant gains in using all of the measurements available, in particular when the dimension of the original phase space is quite large. But in this case, a problem not seen in the monovariate case appears because, for a given dimension, there are few possible choices to reconstruct the phase space. As already shown [11], when not chosen properly, coordinates used to build the reconstructed space from multivariate measurements can lead to a dramatic failure, which was never encountered in the monovariate case.

In practical problems one is never ensured that any given scalar time series is sufficient to reconstruct the dynamics. Indeed, the Takens' theorem guarantees a diffeomorphism when the measurement function is "generic," a condition that does not always hold. On the other hand, one can expect some substantial advantages in using several different time series when available, especially if the system has a high dimension or has symmetry properties. When multivariate reconstructions are considered, an important question is "how to choose the coordinates that should span the reconstructed space." This is equivalent to answering the question "How can the measurements be used in an optimal way?" This question can be addressed using the graphical interpretation and the symbolic observability coefficients.

A. Example with the Lorenz system

The Lorenz system [16],

$$\dot{x} = \sigma(y - x),$$

$$\dot{y} = Rx - y - xz,$$

$$\dot{z} = -bz + xy, \tag{29}$$

is known for presenting a rotation symmetry around the z axis. This can be expressed using a 3×3 matrix encoding the coordinate transformation from $\mathbb{R}^3(x,y,z)$ to $\mathbb{R}^3(x,y,z)$ under which the attractor is globally left invariant. This matrix is characteristic of the symmetry involved. In the case of the Lorenz system, this matrix is

$$\gamma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{30}$$

The Lorenz system $\boldsymbol{f}(\boldsymbol{x})$ thus satisfies

$$\gamma \cdot \dot{\boldsymbol{x}} = \boldsymbol{f}(\gamma \cdot \boldsymbol{x}). \tag{31}$$

The dynamical system $\boldsymbol{f}$ is said to be *equivariant* under the rotation symmetry defined by the γ matrix. The attractor solution to this vector field is left globally invariant under the rotation symmetry [17].

What is important for reconstructing the original phase space with the right symmetry property is that variables x and y are mapped into their respective opposite under the action of γ while variable z is left unchanged. As a consequence, when a space is only reconstructed from x or y variable, the attractor presents an inversion symmetry and no longer a rotation symmetry. In order to reconstruct the right symmetry property, it is necessary to record two time series, either x or y, *and* z.

The fluence matrix of the Lorenz system is

$$F_{ij} = \begin{bmatrix} 1 & 1 & 0 \\ \bar{1} & 1 & \bar{1} \\ \bar{1} & \bar{1} & 1 \end{bmatrix}. \tag{32}$$

The three symbolic observability coefficients estimated using Eq. (17) are

$$\eta_{x^2} = 0.89 > \eta_{y^2} = 0.46 > \eta_{z^2} = 0.35.$$

Also, it was recently proposed to estimate observability coefficients by a time series approach [20], useful in the case of experimental data. This is based on the exponent, k, of a nonlinear fit to a set of autocorrelation functions [21]. The degree of observability was quantified by $\frac{1}{k}$, and for the Lorenz system it was found that [20]

$$\frac{1}{k_x} = 667 > \frac{1}{k_y} = 371 > \frac{1}{k_z} = 233,$$

that is, variables are ranked in the same order than those obtained with the observability coefficients presented in this paper. Due to its symmetry, the Lorenz system was always a special case and it is known that the observability coefficient, local by definition, is averaged over the attractor. The symmetry, a global property, is therefore not taken into account [1].

As mentioned before, to retrieve the right symmetry property, it is necessary to simultaneously measure either x or y, *and* z. When both variables x and z are measured, the reconstructed space can be formed as $(x,\dot{x},z)$ or as $(x,z,\dot{z})$. It has been shown that only in the first case a global diffeomorphism exists [11]. Also, it was shown that if y is recorded in place of x, none of the reconstruction spaces $(y,\dot{y},z)$ or $(y,z,\dot{z})$ provide global diffeomorphisms [11]. In what follows we would like to confirm such conclusions using the procedure put forward in this paper.

In order to build $(x,\dot{x},z)$ the first derivative of x has to be computed to unfold the x variable in a 2D plane, z being the third coordinate. This means that for reconstructing a three-dimensional space, the single derivative that will be computed *must* give access to variable y. In this case, the useful fluence 2×2 submatrix is [see Eq. (32)]

$$\tilde{F}_{ij} = \begin{bmatrix} 1 & 1 \\ \bar{1} & 1 \end{bmatrix}, \tag{33}$$

where elements related to z were removed because they are measured in addition to the single measurement x. From this 2D subspace $\mathbb{R}^2(x,y)$, variable x is measured, so

$$[\tilde{H}_i]_1 = [1 \quad 0]^T \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{34}$$

leading to

$$g_1 = \begin{bmatrix} \cdot & 1 \\ 0 & 0 \end{bmatrix}. \tag{35}$$

We have thus $p_1=1$ and $q_1=0$. The observability coefficient is thus $\eta_{x^1,z}=1$. It is therefore easily confirmed that the Lorenz system is fully observable from x and z measurement in the reconstructed space $\mathbb{R}^3(x,\dot{x},z)$.

If we choose to use the space $\mathbb{R}^3(x,z,\dot{z})$, first a reflexion symmetry is obtained. This is already a major obstacle to a right reconstruction since such symmetry can only lead to disconnected components (there is no transition allowed from one wing to the other) [17]. Since the first derivative of z will be used to unfold the dynamics, it is assumed that z is the "monovariable" measurement and x is the additional observable. Hence the elements related to x are removed from the fluence matrix [Eq. (32)] to yield

$$\tilde{F}_{ij} = \begin{bmatrix} 1 & \bar{1} \\ \bar{1} & 1 \end{bmatrix}. \tag{36}$$

Matrix $[\tilde{H}_i]$ is thus

$$[\tilde{H}_i]_1 = [0 \quad 1]^T \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{37}$$

leading to

$$g_1 = \begin{bmatrix} 0 & 0 \\ \bar{1} & \cdot \end{bmatrix}. \tag{38}$$

for which $p_1=0$ and $q_1=1$. The observability coefficient is thus

$$\eta_{x,z^1} = \frac{1}{1}\left[\frac{1}{2^2}\right] = 0.25, \tag{39}$$

that is, the set of coordinate $(x,z,\dot{z})$ does not provide an optimal reconstructed space. This value suggests that the dimension of the space has to be increased, for instance by including $\dot{x}$.

Proceeding in a similar way with the hyperchaotic system [Eq. (25)], we found that there is a global diffeomorphism between the original phase space $\mathbb{R}^4(x,y,z,w)$ and reconstructed spaces $\mathbb{R}^4(y,\dot{y},\ddot{y},w)$, $\mathbb{R}^4(x,\dot{x},y,\dot{y})$, and $\mathbb{R}^4(x,\dot{x},w,\dot{w})$. In these three cases, the symbolic observability coefficients are found to be equal to 1. This means that when two variables are simultaneously measured from the hyperchaotic system, depending on the way the reconstructed space is built, observability is greatly increased and it should be easy, for instance, to obtain a global model.

B. Rewriting the system in term of measured variables

It was shown in previous works that when a global diffeomorphism exists between the original phase space and the reconstructed spaces, it is possible to rewrite the system in a polynomial form involving the measured variable and its successive derivatives [5]. This was done for the univariate case. Let us do similar computations for multivariate cases. We will treat explicitly two cases, one considering the Lorenz system investigated from variables x and z, and another one investigating the hyperchaotic case from variables y and w.

As shown elsewhere [2], as long as the coordinate transformation defines a global diffeomorphism and provided the original system is polynomial, the latter can be rewritten in the reconstructed space in a polynomial form. For instance, we checked that in the multivariate case with the Lorenz system which can be rewritten in the space $\mathbb{R}^3(X=x,Y=\dot{x},Z=z)$ as

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= \sigma(R-1)X - (\sigma+1)Y - \sigma XZ, \\ \dot{Z} &= -bZ + X\left(X + \frac{Y}{\sigma}\right). \end{aligned} \tag{40}$$

The case of the hyperchaotic system is slightly more complicated. As seen in the previous subsection, the 4D Rössler system can be fully observable in the following reconstructed spaces: (i) $\mathbb{R}^4(y,\dot{y},\ddot{y},w)$, (ii) $\mathbb{R}^4(x,\dot{x},y,\dot{y})$, and (iii) $\mathbb{R}^4(x,\dot{x},w,\dot{w})$. Denoting by $\mathcal{J}\Phi_{y^3,w}$ the Jacobian matrix of the coordinate transformation between the original phase space $\mathbb{R}^4(x,y,z,w)$ and the reconstructed phase space $\mathbb{R}^4(y,\dot{y},\ddot{y},w)$, in each of these three cases, we get

$$\mathrm{Det}(\mathcal{J}\Phi_{y^3,w}) = \mathrm{Det}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & a & 0 & 1 \\ a & a^2-1 & -(c+1) & a+d \\ 0 & 0 & 0 & 1 \end{bmatrix} = (c+1), \tag{41}$$

$$\mathrm{Det}(\mathcal{J}\Phi_{x^2,y^2}) = \mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & a & 0 & 1 \end{bmatrix} = 1, \tag{42}$$

and

$$\mathrm{Det}(\mathcal{J}\Phi_{x^2,w^2}) = \mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -c & d \end{bmatrix} = -c. \quad (43)$$

For these three cases, the number of "$\bar{1}$" is zero ($q_i=0, \forall i$) and the observability coefficient is equal to 1, as expected. Again, we checked that the system can be rewritten in a polynomial form. Only the most difficult case is explicitly presented. Using the coordinate $(X=y, Y=\dot{y}, Z=\ddot{y}, w)$, the 4D Rössler system [Eq. (25)] is rewritten as

$$\dot{X} = Y,$$

$$\dot{Y} = Z,$$

$$\dot{Z} = \alpha_0 + \alpha_1 X + \alpha_2 Y + \alpha_3 Z + \alpha_4 W + \alpha_5 W^2 + \alpha_6 XW + \alpha_7 YW + \alpha_8 ZW,$$

$$\dot{W} = \frac{c(X - aY + Z) + dW}{c+1}, \quad (44)$$

where

$$\alpha_0 = -b(c+1),$$

$$\alpha_1 = \frac{d(c-ad)}{c+1},$$

$$\alpha_2 = -1 + \frac{d^2 - acd}{c+1},$$

$$\alpha_3 = a + \frac{cd}{c+1},$$

$$\alpha_4 = \frac{-d^2}{c+1},$$

$$\alpha_5 = d,$$

$$\alpha_6 = ad + 1,$$

$$\alpha_7 = -(a+d),$$

$$\alpha_8 = 1. \quad (45)$$

Such a feature has also been verified for $(X=x, Y=\dot{x}, Z=y, W=\dot{y})$ and $(X=x, Y=\dot{x}, Z=w, W=\dot{w})$.

VII. CONCLUSION

From a graphical representation of couplings between variables, we were able to analytically compute symbolic observability coefficients. In every studied case, the proposed coefficients are in good agreement with all previous computations. There is one case—the Lorenz system investigated from the z-variable—which is still subject to investigation due to a problem induced by the rotation symmetry. The new procedure is appealing because it does not require the derivation of Lie derivatives neither the computation of eigenvalues of a large number of observability matrices. They are normalized, that is, they are one or greater when a global diffeomorphism is identified between the original phase space and the reconstructed space, both having the same dimension. Normalized observability coefficients based on eigevalues of observability matrices were previously introduced [20]. The computation of symbolic observability coefficients is based on the so-called "fluence matrix" of the system and on a recursive procedure for which the number of iterates depends on the dimension of the reconstructed phase space. With this observability coefficient, it is possible to investigate how increasing the dimension of the reconstructed space helps to unfold the dynamics and, consequently, increase the observability of the original phase space. The symbolic observability coefficients can be greater than one when the dimension of the reconstructed state space is too large. Thus, observability coefficients provide an upper limit (sometimes smaller than those provided by the Takens criterion) where additional coordinates are no longer required. These computations provide an additional proof that observability mostly results from couplings between dynamical variables than the dynamical regime itself.

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