

Chaos and correspondence in classical and quantum Hamiltonian ratchets: A Heisenberg approach

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(Received 8 September 2008; published 16 June 2009)

Previous work [Gong and Brumer, Phys. Rev. Lett. **97**, 240602 (2006)] motivates this study as to how asymmetry-driven quantum ratchet effects can persist despite a corresponding fully chaotic classical phase space. A simple perspective of ratchet dynamics, based on the Heisenberg picture, is introduced. We show that ratchet effects are in principle of common origin in classical and quantum mechanics, although full chaos suppresses these effects in the former but not necessarily the latter. The relationship between ratchet effects and coherent dynamical control is noted.

DOI: [10.1103/PhysRevE.79.066207](https://doi.org/10.1103/PhysRevE.79.066207)

PACS number(s): 05.45.-a, 05.40.-a, 32.80.Qk, 05.60.Gg

I. INTRODUCTION

Originally proposed by Smoluchowski [1] and Feynman [2], and motivated by an application to biological molecular motors [3], studies of ratchet transport, that is, asymmetry-driven directed transport without external bias, are now the subject of an expanded range of theoretical interest [4,5]. While earlier investigations depended on external noise to rationalize these directional effects, recent work has shown that they can persist even in its absence [6–14], thereby raising questions about the origin of transport in isolated Hamiltonian systems. Many studies have therefore focused on the relationship between ratchet dynamics and deterministic chaos (see, for instance, [7,8,10]), relating ratchet transport to the typical questions of chaology, including, naturally, the complex relationship between quantum systems and their corresponding chaotic classical counterparts. In the context of recent cold-atom testing of Hamiltonian ratchet transport in classically chaotic systems [15–17], investigations of ratchet transport are interesting both as a method of exploring quantum and classical transport properties as well as a means of addressing general questions in quantum and classical chaos.

It has been shown [9–11] that quantum ratchet transport is possible even when the corresponding classical dynamics is completely chaotic. In such a case, the classical system displays no appreciable current. Hence, these systems show a novel qualitative divergence between quantum and classical dynamical properties, motivating this study of the relationship between quantum and classical ratchet transport.

Below we show that ratchet effects emerge, both quantum mechanically and classically, via an asymmetry-induced distortion of the spatial distribution, leading to a net effective force. Classically, full chaos diminishes this distortion, and hence suppresses ratchet effects. Quantum mechanically, by contrast, the distortion generally persists except at very small values of the effective Planck constant.

Hamiltonian ratchet dynamics is also directly related to laser-induced coherent control of directional transport

[18,19]. Symmetry-breaking schemes have been used in coherent control since its inception [20], and so studying quantum vs classical ratchet transport also lends insight into quantum control scenarios. Recent results [19] suggest that such control, once thought to be exclusively quantum mechanical, is possible classically, as well. Quantum ratchet transport in the presence of full classical chaos, by contrast, is an excellent example of controlled transport that may not be possible in classical dynamics. As such, this topic is also of interest to two related more general issues: quantum controllability of classically chaotic systems, and survival conditions for quantum control in the classical limit.

The case of quantum ratchet transport with full classical chaos discussed below further strengthens the view that quantum control of classically chaotic systems is often feasible [21,22]. Indeed, this interesting possibility has already attracted some interest, both theoretically and experimentally [23–25].

We consider here spatially periodic quantum systems with Hamiltonian $\hat{H} = \hat{H}_0(\hat{p}) + \hat{V}(\hat{q}, t)$, where $\hat{V}(\hat{q}, t)$ is a time-periodic operator representing an external potential imposed on the system, \hat{q} and \hat{p} are conjugate position and momentum operators, respectively, and operators are denoted by a circumflex. These systems display ratchet transport, that is, despite being initially distributed uniformly in space, having zero initial momentum, and being driven by a force without bias, they organize to show an increase in the current or the average momentum, denoted $\langle p \rangle$ below for both quantum and classical mechanics. The absence of a biased force means that upon averaging over all space, denoted by an overbar:

$$\overline{-\frac{\partial V(q,t)}{\partial q}} \equiv \overline{F(q,t)} = 0, \quad (1)$$

where $V(q,t)$ and $F(q,t)$ are the coordinate-space representations of the applied potential and force, respectively [26]. Significantly, this zero average, which is the standard definition of the absence of bias, is entirely independent of the

structure or state of the physical system upon which $F(q, t)$ acts. As such, as emphasized below, $\overline{F}(q, t)$ is conceptually distinct from the expectation value of a net force $\langle F(t) \rangle$ actually felt by an evolving system, which, of course, is a function of the system evolution. The significance of this distinction will become apparent in what follows.

While the discussion presented below is quite general, we continue to employ our modification of the kicked Harper paradigm [10,27,28] as an illustrative example and motivator of this study. The quantum modified Harper Hamiltonian is given by [29]

$$\hat{H} = J \cos(\hat{p}) + K \hat{V}_r(\hat{q}) \sum_n \delta(t - n), \quad (2)$$

$$\hat{V}_r(\hat{q}) = \cos(2\pi\hat{q}) + \sin(4\pi\hat{q}), \quad (3)$$

where the system potential is $\hat{V}(\hat{q}, t) = K \hat{V}_r(\hat{q}) \sum_n \delta(t - n)$, and we define $\hat{F}_r(\hat{q}) \equiv -\frac{\partial \hat{V}_r(\hat{q})}{\partial \hat{q}}$. Here t is the time, n is an integer, and J and K are system parameters. The associated q space is periodic in $[0, 1]$. All system variables here should be understood to be appropriately scaled and hence dimensionless. In particular, the scaled dimensionless Planck constant is denoted as \hbar , and hence $\hat{p} = -i\hbar \frac{\partial}{\partial \hat{q}}$. The unitary time evolution operator associated with one kick from time $t=0$ to $t=1 + \epsilon$ in Eq. (2) is given by

$$\hat{U}(1, 0) = e^{-(iJ/\hbar)\cos(\hat{p})} e^{-(iK/\hbar)\hat{V}_r(\hat{q})}, \quad (4)$$

with the cumulative time evolution operator from $t=0$ to $t=m$ given by

$$\hat{U}(m, 0) = [\hat{U}(1, 0)]^m. \quad (5)$$

We also stress that the initial quantum state used here is always assumed to be a zero-momentum eigenstate, which is time-reversal symmetric and spatially uniform. As shown in Ref. [28], ratchet transport can be sensitive to the initial state. However, our analysis below readily applies to any other initial state.

The properties of this model discussed below hold in general for the regime where the kicked Harper does not show dynamical localization [30]. We consider the case where $K = 2J$, although this choice is arbitrary. Since this system can be exactly mapped onto the problem of a kicked charge in a magnetic field [31], or can be related to cold-atom experiments [28,32] or to driven electrons on the Fermi surface [33], it has a realistic physical and experimental interpretation. In particular, although the unknicked part of the Hamiltonian in Eq. (2) is given by $J \cos(\hat{p})$, the underlying dispersion relation in the cold-atom and kicked-charge realizations of the kicked Harper model is still given by $E = \hat{p}^2/2$ [28,31,32]. That is, the momentum variable in our abstract model can still be directly linked to the mechanical momentum of a moving particle. Hence the current of particles can indeed be calculated via the momentum expectation value.

The quantum dynamics associated with the propagator in Eq. (4) shows unbounded [34] acceleration of the ratchet current [10]. Typical results are shown in Fig. 1, where panel (a) shows the current $\langle p \rangle$ for $K=4$ and panel (b) shows the

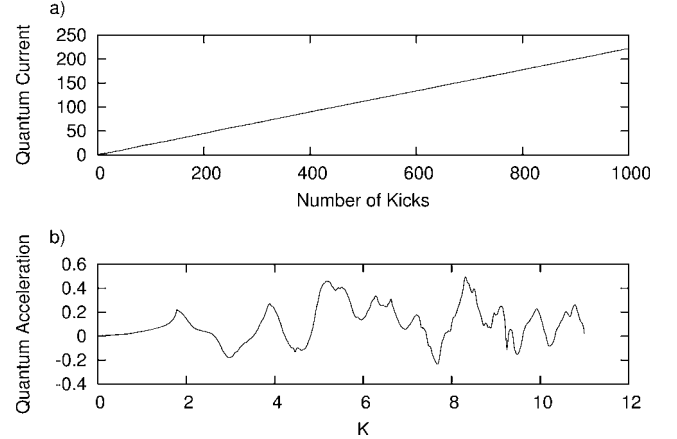


FIG. 1. (a) Time dependence of the quantum current $\langle p \rangle_Q$ of the modified kicked Harper system for $K=4$, $J=2$, and $\hbar=1$, shown here for the first 1000 kicks, with the initial state given by a momentum eigenstate with zero momentum. (b) The mean acceleration rate of the quantum current for a range of K values, with $K=2J$ and $\hbar=1$. Note that, as seen in Ref. [28], the transport direction may change with the initial condition. As explained in the text, all quantities here and those in other figures are in dimensionless units.

mean current acceleration rate as a function of K . Here the acceleration is defined approximately as $\langle p(t=1000) \rangle / 1000$.

The classical comparison with the quantum dynamics considers ensembles of trajectories that are analogous to the quantum systems discussed above: initially, the trajectories have zero momentum and are uniformly distributed in coordinate space, and are driven by a force of zero spatial mean at all times. Again, our discussion is quite general for such systems although we consider the classical analog of the modified kicked Harper system as an illustrative example, obtained by replacing the quantum operators in Eqs. (2) and (3) with their respective classical observables. Specifically, the evolution of a classical trajectory through one kick is then given by

$$\begin{aligned} p_N &= p_{N-1} + K F_r(q_{N-1}), \\ q_N &= q_{N-1} - J \sin(p_N). \end{aligned} \quad (6)$$

This system has been shown to display virtually no classical ratchet transport [10] if the system parameters are in the regime of full classical chaos.

Throughout this discussion, it will often be convenient to consider quantum and classical arguments simultaneously. We distinguish quantum and classical objects by respective subscripts Q and C , and refer to both dynamics when these subscripts are omitted.

This paper is organized as follows. Section II analyzes, from a unique perspective based on the Heisenberg picture of the dynamics, the origin of asymmetry-driven ratchet transport. Section III considers the difference and correspondence between classical and quantum ratchet transport. Section IV summarizes the conclusions of this study.

II. ASYMMETRY AND RATCHET EFFECTS

A. Heisenberg force

Classically and quantum mechanically, the rate of the ratchet current increase, here termed the acceleration, at time t is given by the expectation value of the net force at that time:

$$\frac{d\langle p \rangle}{dt} = \langle F(t) \rangle. \quad (7)$$

Evidently, $\langle p \rangle$ must remain zero if it begins at zero and $\langle F(t) \rangle = 0$ at all times. Hence, when ratchet acceleration occurs, it follows that the expectation value of the net force must be nonzero. This result calls for analysis of how ratchet acceleration is possible in the absence of a biased force.

To facilitate comparison of quantum and classical mechanics, it is convenient to cast this discussion in terms of the density-matrix formalism. The expectation value of the quantum force at time t is given by

$$\begin{aligned} \langle F(t) \rangle_Q &= \text{Tr}[\hat{\rho}_Q(t)\hat{F}_Q(\hat{q},0)] \\ &= \text{Tr}\left[\hat{\rho}_Q(0)\hat{T}\exp\left(-\frac{i}{\hbar}\int_0^t dt' \hat{L}_Q(t')\right)\hat{F}_Q(\hat{q},0)\right], \end{aligned} \quad (8)$$

where $\hat{\rho}_Q(0)$ is the (pure state) density matrix at time zero and $\hat{\rho}_Q(t)$ is the propagated density at time t [35]. Here, time evolution is mediated by the quantum Liouville operator $\hat{L}_Q \cdot = \frac{i}{\hbar}[\hat{H}, \cdot]$, the bracket $[\cdot, \cdot]$ is the commutator, and \hat{T} denotes the time-ordering operator. For the Hamiltonian in Eq. (2), $\hat{F}_Q(\hat{q},0) = -K \partial V_r(\hat{q}) / \partial \hat{q}$.

The effect of the time-ordered exponential is given in terms of the evolution operator as [36]

$$\hat{F}_{Q,H}(\hat{q},t=n) = \hat{U}^{-1}(n,0)\hat{F}_Q(\hat{q},0)\hat{U}(n,0). \quad (9)$$

Equation (8) can be rewritten as [36]

$$\langle F(t) \rangle_Q = \text{Tr}[\hat{\rho}_Q(t)\hat{F}_Q(\hat{q},0)] = \text{Tr}[\hat{\rho}_Q(0)\hat{F}_{Q,H}(\hat{q},t)], \quad (10)$$

where

$$\hat{F}_{Q,H}(\hat{q},t) \equiv \hat{T}\exp\left[-\frac{i}{\hbar}\int_0^t \hat{L}_Q(t')dt'\right]\hat{F}_Q(\hat{q},0) \quad (11)$$

defines the *Heisenberg force*, the focus of attention below.

The classical ensemble-averaged value of the force at time t is similarly given by

$$\langle F(t) \rangle_C = \int dpdq \left\{ \rho_C(0)\hat{T}\exp\left[-i\int_0^t dt' \hat{L}_C(t')\right] F_C(q,0) \right\}, \quad (12)$$

where $\rho_C(0)$ is the initial classical density distribution, $\hat{L}_C \cdot = i\{H, \cdot\}$ is the classical Liouville operator, where $\{, \cdot\}$ represents a classical Poisson bracket, and $F_C(q,0) = -K \partial V_r(q) / \partial q$. The time evolution of q , and hence of $F_C(q,0)$, is carried out via Eq. (6).

For either the quantum or the classical ensemble average $\langle F(t) \rangle$ to be nonzero, and hence induce ratchet acceleration, some system attribute needs to break the positive-negative symmetry to “choose” a direction. From Eqs. (8) and (12) it is clear that asymmetries in either the initial distribution, force, or evolution operator are essentially equivalent as the origin of bias. Since, for classical and quantum ratchets, the initial distribution and force are chosen to be symmetric, the asymmetry in the evolution operator, and hence asymmetry in dynamics induced by the Hamiltonian, must be responsible for the nonzero net current.

Specifically, consider the q representation of Eq. (10). Noting that $\hat{\rho}_Q(0)$ describes a spatially uniform state [i.e., $\hat{\rho}_Q(0) = |q\rangle\langle q|$], in normalized coordinates $\rho_Q(q,0) \equiv \langle q | \hat{\rho}_Q(0) | q \rangle = 1$, so that

$$\begin{aligned} \langle F(t) \rangle_Q &= \int dq \langle q | \hat{\rho}_Q(t) | q \rangle \langle q | \hat{F}_Q(\hat{q},0) | q \rangle \\ &= \int dq \langle q | \hat{\rho}_Q(0) | q \rangle \langle q | \hat{F}_{Q,H}(\hat{q},t) | q \rangle \end{aligned} \quad (13)$$

$$= \int dq \langle q | \hat{F}_{Q,H}(\hat{q},t) | q \rangle = \overline{\hat{F}_{Q,H}(\hat{q},t)}. \quad (14)$$

That is, the average force is dictated by the uniform spatial average over the Heisenberg force, as distinguished from the *Schrödinger force* $\hat{F}_Q(\hat{q},0) = -K \partial \hat{V}(\hat{q}) / \partial \hat{q}$. Correspondingly, since $\rho_C(0)$ is chosen to be normalized and spatially uniform, Eq. (12) indicates that

$$\langle F(t) \rangle_C = \overline{\hat{T}\exp\left[-i\int_0^t dt' \hat{L}_C(t')\right] F_C(q,0)} = \overline{F_{C,H}(q,t)}, \quad (15)$$

i.e., a spatial average over the time-evolving classical force $F_{C,H}(q,t)$, analogous to the quantum case. Since Eqs. (14) and (15) show that the expectation value of the force is given by an average over the *evolving* force, a nonzero net force as a result of an asymmetry in the dynamics becomes possible, even if the spatial average of the bare force $F(q,t)$ itself remains zero at all times.

Note that, since the force is diagonal in q in quantum mechanics, and not a function of p classically, the evolving force distribution $F_H(q,t)$ is adequately described entirely in q in both mechanics. This allows simple, direct comparisons of quantum and classical mechanics, as shown in the following section. Below, we term $F_{Q,H}(q,t) \equiv \langle q | \hat{F}_{Q,H}(\hat{q},t) | q \rangle$ the force distribution in q associated with the Heisenberg force. Similar terminology applies in classical mechanics. The diagonal element of the q representation of the distribution of the Schrödinger density $\langle q | \hat{\rho}_Q(t) | q \rangle$ is denoted $\rho_Q(q,t)$ so that $\langle F(t) \rangle_Q = \int dq \rho_Q(q,t) F_Q(q,0)$. The classical object analogous to $\rho_Q(q,t)$ is the q component of the evolving density, $\rho_C(q,t) \equiv \int dp \rho_C(p,q,t)$, where $\rho_C(p,q,t)$ is the classical evolving density. In both mechanics, the initial spatial distribution is assumed uniform. As a result, in the quantum case, for example, and in accord with Eqs. (8) and (10),

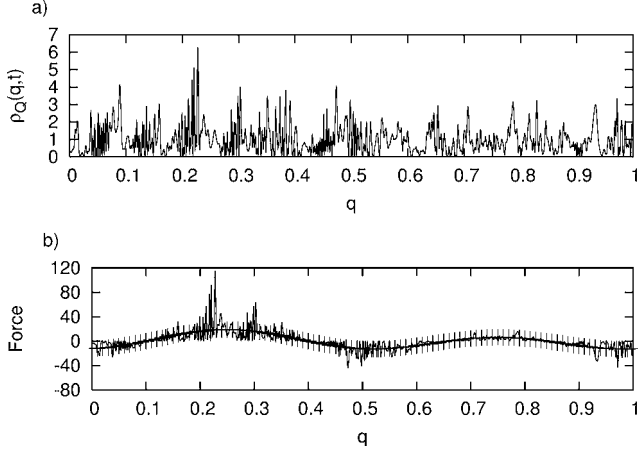


FIG. 2. (a) $\rho_Q(q,t)$ and (b) $F_{Q,H}(q,t)$ compared to $F_Q(q,t)$ (plus symbols) for the modified kicked Harper model after the first 50 kicks for $K=4$, $J=2$, and $\hbar=1$. Distortion in the density $\rho_Q(q,t)$, and hence bias in the Heisenberg force distribution function $F_{Q,H}(q,t)$, are evident.

$$\begin{aligned} F_{Q,H}(q,t) &= \langle q | \hat{F}_{Q,H}(\hat{q},t) | q \rangle = \langle q | \hat{\rho}_Q(0) \hat{F}_Q(\hat{q},t) | q \rangle \\ &= \langle q | \hat{\rho}_Q(t) | q \rangle \langle q | \hat{F}_Q(\hat{q},0) | q \rangle \\ &= \rho_Q(q,t) F_Q(q,0). \end{aligned} \quad (16)$$

That is, the evolving force distribution is given by the bare force weighted by the evolving density. The analogous result holds in classical mechanics.

Given that, in either mechanics, $F_H(q,t) = \rho(q,t)F(q,0)$, with a uniform initial distribution $\rho(0)$ and unbiased force $F(q,0)$, a net nonzero $F_H(q,t)$ requires that $\rho(q,t)$ weights $F(q,0)$ so as to break the directional symmetry. Minimally, the system evolution must be such that each point q_i in the q space does not in general have a complement q_j such that both $\rho(q_i,t) = \rho(q_j,t)$ and $F(q_i,0) = -F(q_j,0)$. This is the simplest asymmetry condition on the dynamics necessary for the generation of a ratchet current. The modified Harper Hamiltonian [Eq. (2)] clearly satisfies this condition.

This Heisenberg approach thus gives a simple picture of ratchet current generation. The origin of a current arising from a net force can be understood as either (a) a distortion in the density $\rho(q,t)$, which will weight the bare force $F(q,0)$ nonuniformly giving rise to a nonzero average, or (b) as a distortion in the evolving force $F_H(q,t)$ itself, whose average $\langle F(t) \rangle$ is nonzero due to this distortion, even if the bare force has zero mean. The advantage of using the evolving force picture is that it resolves the intuitive puzzle of how directional transport in the momentum space emerges in the absence of a biased force. Whether the force itself, that is, the bare force, is biased or not is irrelevant. Rather, the intrinsic asymmetry in the dynamics permits the evolving force $F_H(q,t)$ to develop a nonzero mean, and hence a nonzero ratchet acceleration rate.

Computationally, the Heisenberg picture is easily applied to the modified kicked Harper model to examine $F_{Q,H}(q,t)$.

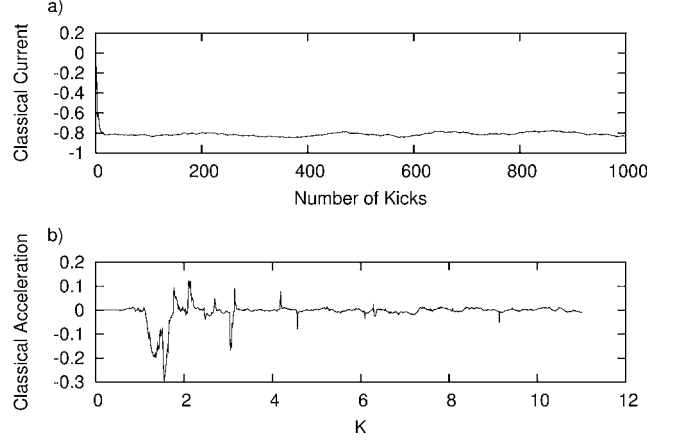


FIG. 3. (a) Time dependence of the classical current $\langle p \rangle_C$ of the modified kicked Harper system for $K=3$ and $J=1.5$, shown here for the first 1000 kicks. (b) The mean acceleration rate of the classical current for a range of K values, with $K=2J$.

For example, Fig. 2 shows $\rho_Q(q,t)$ and $F_{Q,H}(q,t)$ for parameters associated with an appreciable and unbounded ratchet current acceleration. Despite starting with a flat distribution in q , $\rho_Q(q,t)$ in Fig. 2(a) is now clearly unevenly distributed. Accordingly, the distribution of the Heisenberg force $F_{Q,H}(q,t)$ shown in Fig. 2(b) is strongly biased compared to the symmetric bare force distribution (plus symbols).

B. Two roles of the force

Implicitly, we have considered the force in two capacities: acting on the structure of the ensemble and thereby producing a nonzero net Heisenberg force, and the net force itself, acting within an ensemble average to generate ratchet acceleration, i.e., $\langle F(t) \rangle = \frac{d\langle p \rangle}{dt}$. To further elucidate how this relates to ratchet transport, consider any δ -kicked quantum ratchet model with an arbitrary kicking potential operator $K\hat{V}_r(\hat{q})$ and kinetic-energy operator $J\hat{T}(\hat{p})$. The evolution of this type of system is mediated by a propagator \hat{U} such as Eq. (4), such that a Heisenberg observable $\hat{O}_{Q,H}$ after N kicks is given by $\hat{O}_{Q,H}(N) = (\hat{U}^{-1})^N \hat{O} \hat{U}^N$. Consider the current $\langle p(1) \rangle_Q$ after the first kick:

$$\begin{aligned} \langle p(1) \rangle_Q &= \text{Tr}[\hat{\rho}_Q(0) \hat{U}^{-1} \hat{p} \hat{U}] \\ &= \text{Tr}[\hat{\rho}_Q(0) e^{(iK/\hbar)\hat{V}_r(\hat{q})} e^{(iJ/\hbar)\hat{T}(\hat{p})} \hat{p} e^{(-iJ/\hbar)\hat{T}(\hat{p})} e^{(-iK/\hbar)\hat{V}_r(\hat{q})}]. \end{aligned} \quad (17)$$

Using $\hat{p} = -i\hbar \frac{\partial}{\partial \hat{q}}$ and that the initial state is assumed uniform in q , one obtains

$$\begin{aligned} \langle p(1) \rangle_Q &= -K \text{Tr} \left[\hat{\rho}_Q(0) e^{(iK/\hbar)\hat{V}_r(\hat{q})} e^{(-iK/\hbar)\hat{V}_r(\hat{q})} \frac{\partial \hat{V}_r(\hat{q})}{\partial \hat{q}} \right] \\ &\quad + \text{Tr}[\hat{\rho}_Q(0) \hat{p}] = -K \frac{\partial \hat{V}_r(\hat{q})}{\partial \hat{q}} + 0 = 0. \end{aligned} \quad (18)$$

This illustrates the distinction between the force's role in distorting its own distribution and its role in inducing a current. That is, although no current develops after the first kick, subsequent kicks produce current. Therefore, even

though the net force remains zero for the first kick, that kick distorts the system so that it will subsequently experience a net force.

More generally, for N kicks,

$$\begin{aligned} \langle p(N) \rangle_Q &= \text{Tr}[\hat{\rho}_Q(0)(\hat{U}^{-1})^N \hat{p} \hat{U}^N] = \text{Tr}[\hat{\rho}_Q(0)(e^{(iK/\hbar)\hat{V}_r(\hat{q})} e^{(iJ/\hbar)\hat{T}(\hat{p})})^N \hat{p} (e^{(-iK/\hbar)\hat{V}_r(\hat{q})} e^{(-iJ/\hbar)\hat{T}(\hat{p})})^N] \\ &= -K \sum_{j=0}^{N-1} \text{Tr} \left[\hat{\rho}_Q(0)(\hat{U}^{-1})^j \frac{\partial \hat{V}_r(\hat{q})}{\partial \hat{q}} \hat{U}^j \right] = K \sum_{j=0}^{N-1} \langle F_r(j) \rangle_Q. \end{aligned} \quad (19)$$

It follows that the change in $\langle p \rangle$ on each step is

$$\Delta \langle p \rangle_Q \equiv \langle p(N) \rangle_Q - \langle p(N-1) \rangle_Q = K \langle F_r(N-1) \rangle_Q, \quad (20)$$

showing that the change in momentum induced at every kick is a result of the net force from the previous kick. This makes clear the general case: the force first acts on an ensemble to generate a distortion, and then a net ratchet force can develop. Exactly the same arguments apply in classical mechanics.

Thus far, this discussion has supported the view [19] that symmetry-breaking induced transport can be achieved both classically and quantum mechanically, both arising via a distortion originating from an asymmetry in the dynamics. However, despite the existence of this analogous ratchet transport mechanism in the classical modified Harper model, classical ratchet transport behaves very differently in the regime of classically chaotic motion, where the classical current quickly saturates at a value close to zero. This is clear in Fig. 3: panel (a) shows the saturating current $\langle p \rangle_C$ for a typical chaotic case, and panel (b) shows the classical mean acceleration rates for a range of parameters. When K is greater than approximately 3.7, the classical dynamics develops full chaos and the mean acceleration rate is generally negligible. (The occasional isolated nonzero mean acceleration rates seen above $K \approx 3.7$ are likely due to some remnants of pre-chaotic structure in phase space.) Therefore, even though the relevant symmetry properties are the same classically and quantum mechanically, some other important distinction must exist.

III. EVOLUTION OF CLASSICAL AND QUANTUM HAMILTONIAN RATCHETS

Results in Fig. 3 demonstrate that the behavior of the classical modified kicked Harper system is quite different from the quantum result, where unbounded ratchet effects persist. If indeed ratchet effects emerge by the same mechanism in quantum and classical mechanics, it remains to be explained why that mechanism generates different results for different mechanics. Specifically, ratchet effects diminish classically in the regime associated with classical chaos. We therefore examine how the onset of chaos affects classical ratchet dynamics, in a way that does not occur quantum me-

chanically. We also discuss the peculiar long-time behavior of the quantum modified kicked Harper model, as well as quantum-classical correspondence.

A. Chaos and the Heisenberg force

From a trajectory perspective, classical chaos is characterized by exponential sensitivity to initial conditions. However, the conventional interpretation of quantum mechanics does not describe individual trajectories. Hence, a comparison of quantum and classical dynamics demands comparison of quantum and classical distributions [37–44]. Although Kolmogorov-Arnold-Moser theory and finite-time limitations suggest deviations in the properties of distribution functions of typical classically chaotic systems from theoretical ideals [45], such systems are still expected to exponentially develop increasingly fine structure. Upon coarse graining on the scale of interest, the classical phase-space distribution in a fully chaotic system uniformly fills the phase space almost everywhere, with additional structure detectable only on an increasingly fine scale. Indeed, this is what is termed full chaos in most numerical studies of this type: when no structure is visible in the phase space on a preset fine scale, it is considered operationally chaotic.

Consider then how this applies to the Heisenberg force for the ratchet systems considered here, where the phase space is always bound or periodic in q . Ensemble averages are computed by integrating over the distribution. Since complete chaos implies no structure in $\rho_C(q, t)$ on the scale of interest, such averages will look like unweighted averages in q [provided that the scale on which the variable of interest varies is much larger than the scale of structure in $\rho_C(q)$ remaining in the chaotic phase space]. That is, we can essentially ignore the q component of the density when taking spatial averages. In the case of the force,

$$\begin{aligned} \langle F(t) \rangle_C &= \int dp dq \rho_C(p, q, t) F_C(q, t) \\ &\approx \int dq F_C(q, t) \int dp \rho_C(p, t) \propto \int dq F_C(q, t) = 0, \end{aligned} \quad (21)$$

where $\rho_C(p, t) = \int dq \rho_C(p, q, t)$ is the classical momentum

density distribution. Hence, for all times when the phase space is operationally chaotic, the ensemble average of the classical force is proportional to the spatial average of the bare force: i.e., zero. Chaotic dynamics here implies no spatial distortion of the system on the scale of interest, and hence no creation of a net evolving force. This is consistent with a result of the ‘‘classical sum rule’’ [7], which predicts that there will be no classical ratchet current in fully chaotic systems.

The comparison with quantum mechanics is straightforward. If the quantum q distribution $\rho_Q(q, t)$ is flat, or if the scale of structure remaining in this distribution is far smaller than that over which the bare force $F_Q(q, 0)$ varies, then by an argument analogous to the classically chaotic case, the net quantum force $\langle F(t) \rangle_Q$ will be essentially zero. As in the classical case, the spatial distortion giving rise to a net force would not be appreciable on the scale of interest.

However, a quantum ratchet system is not expected to display such behavior. The Fourier relationship between $\rho_Q(q, t)$ and $\rho_Q(p, t) \equiv \langle p | \hat{\rho}_Q(t) | p \rangle$ implies that a uniform distribution in space $\rho_Q(q, t) = 1$ corresponds to the lowest momentum state $\rho_Q(p, t) = \delta_{p,0}$. Once the system is driven by a force, other momentum states will of course be populated. Correspondingly, $\rho_Q(q, t) = \sum_{k,k'} c_k c_{k'}^* e^{-\frac{i}{\hbar}(p_k - p_{k'})q}$, where the c_k are constants and p_k are momenta. This density is not flat. For fixed p_k and $p_{k'}$, a sufficiently large \hbar can always be found so that the $e^{-\frac{i}{\hbar}(p_k - p_{k'})q}$ terms oscillate sufficiently slowly, giving $\rho_Q(q, t)$ structure in q on the scale of interest. Therefore, sufficiently far into the quantum regime, driven quantum systems are expected to retain coarse structure in q space; there is a limit to the fineness of scale in quantum mechanics [46]. Consequently, the net force is not in general expected to reduce to the average bare force.

This provides a qualitative explanation for the difference in behavior between quantum and classical dynamics in the regime of full classical chaos. This perspective also accounts for the difference in controllability between classical and quantum mechanics. That is, asymmetry-driven transport control is in principle possible in both. Since it relies on a distortion of the system distribution function, distributions without structure on the relevant scale show diminished control. Classically, control is therefore lost to chaos, whereas it can survive in quantum mechanics.

B. Quantum long-time dynamics

To achieve stable, unbounded acceleration of the ratchet current, as observed in the modified Harper system, requires that $\langle F(t) \rangle_Q$ continually operate in the same direction, driving a current with essentially the same bias for all time. This implies that the profile of the time-evolving density $\rho_Q(q, t)$, and hence of the Heisenberg force distribution $F_{Q,H}(q, t)$, does not change appreciably in time (or that it changes in the highly unlikely way that always maintains the same bias). If the quasienergy spectrum of the system is purely discrete, this cannot be the case. Specifically, from Floquet theory we have that for any time-periodic bounded quantum system with discrete quasienergy spectrum, the density is given by $\rho_Q(q, t) = \sum_{l,l'} d_l d_{l'}^* e^{\frac{i}{\hbar}(E_l - E_{l'})t} \rho_Q(q, 0)$, where the d_l are con-

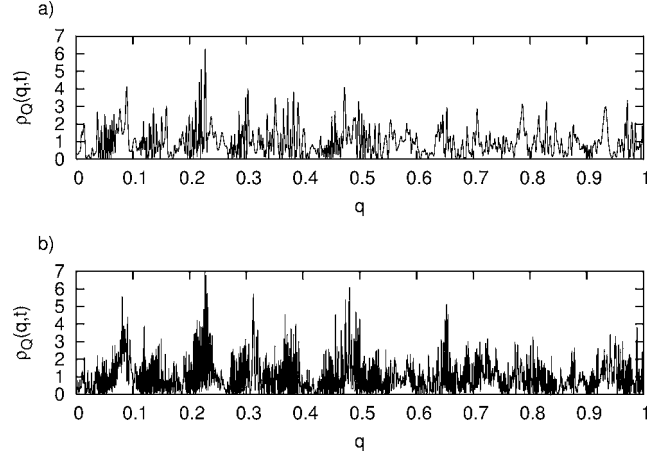


FIG. 4. $\rho_Q(q, t)$ of the modified kicked Harper system for $K=4$, $J=2$, and $\hbar=1$, after (a) the first 50 kicks, and (b) the first 200 kicks. Note that the probability distribution function in (b) oscillates more drastically than in (a) but their overall shape remains roughly the same. This is consistent throughout the parameter space.

stants and the E_l are the quasienergies [47]. Since this density is the sum of periodic functions, it is itself quasiperiodic. Therefore, ensemble averages in such systems are also quasiperiodic, and hence do not continuously increase in time [47,48]. This is true as well for the Heisenberg force $F_{Q,H}(q, t)$, which would be quasiperiodic and hence eventually reverse its direction.

For this reason, earlier quantum ratchet models without current saturation occurred for kicked-rotor systems with quantum resonance conditions [9,49], displaying a continuous quasienergy spectrum. The behavior of the modified kicked Harper model here, which apparently does not satisfy a quantum resonance condition, and for which extended computational results (not shown here) have suggested unbounded directional current, therefore requires explanation.

In fact, it can be shown that all kicked Harper systems can be exactly mapped onto the problem of a kicked charge in a magnetic field, although *only at resonance* [31]. Consequently, the quasienergy spectrum of this model is not necessarily purely discrete, the system evolution need not be quasiperiodic, and the modified kicked Harper system need not necessarily show dynamical saturation in time.

As an example, Fig. 4 shows $\rho_Q(q, t)$ after 50 and 200 kicks for typical parameters, and Fig. 5 shows $F_{Q,H}(q, t)$ compared to the bare force $F_Q(q, 0)$ for the same circumstances. Indeed, there is no appreciable change in the qualitative shape of either $\rho_Q(q, t)$ and $F_{Q,H}(q, t)$ after the first few kicks, although the very fine details of the oscillatory structure increase.

C. Quantum-classical correspondence

Given the above-mentioned quantum-classical differences, it is natural to ask how the classical results emerge from the quantum mechanics as the effective Planck constant \hbar decreases.

Before resorting to computational studies, let us first examine how the quantum dynamics may appear more classical

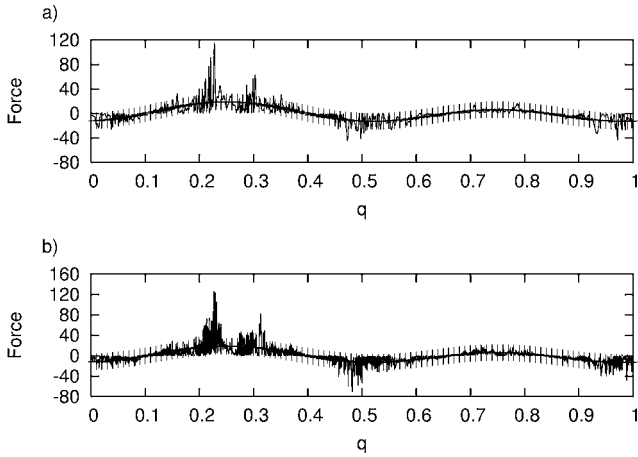


FIG. 5. $F_{Q,H}(q,t)$ of the modified kicked Harper system compared to $F_Q(q,t)$ (plus symbols) for $K=4$, $J=2$, and $\hbar=1$, after (a) the first 50 kicks, and (b) the first 200 kicks. Note that the Heisenberg force distribution function in (b) oscillates more drastically than in (a), but their overall shape remains roughly the same. This is consistent throughout the parameter space.

for small \hbar . Consider a time-evolving quantum density $\rho_Q(q,t) = \sum_{k,k'} c_k c_{k'}^* e^{-\frac{i}{\hbar}(p_k - p_{k'})q}$, where the c_k are constants and p_k are momenta. For large \hbar the interference between different momentum components induces large-scale patterns in the density. However, for sufficiently small \hbar relative to $(p_k - p_{k'})q$, the exponential factor will rapidly oscillate; the smallest scale of structure can be much finer than the scale over which the bare force changes. Hence, at a given time, and for smaller and smaller \hbar , the quantum limit on the fineness of scale diminishes. As in classical mechanics, coarse-scale structure can persist, but it no longer has to. Therefore, it becomes possible for the ensemble-averaged quantum force to either maintain an appreciable bias, as in the classically partially integrable regime, or to approach its average over a flat distribution, as in the classically chaotic regime. Qualitatively, then, the coarse-scale structure in q imposed by quantum coherence can diminish as $\hbar \rightarrow 0$.

Figure 6 shows the q representation of $\rho_Q(q,t)$, as well as a comparison of the quantum Heisenberg and Schrödinger force distributions, $F_{Q,H}(q,t)$ and $F_Q(q,0)$, for a typical case in a semiclassical regime, represented by $\hbar=0.0001$ (a computationally, intensive regime). The system parameters here are associated with classical chaos. The density in Fig. 6(a) shows clear, truly drastic, oscillations, with a roughly uniform oscillation amplitude. Further, it is evident from Fig. 6(b) that on this scale, the overall distribution of the Heisenberg force is similar to that of the initial Schrödinger force distribution, justifying the loss of directional effects in going from quantum to classical mechanics. Figure 7 shows the quantum ratchet current $\langle p \rangle_Q$ in the semiclassical regime of $\hbar=0.0001$, as compared with the corresponding classical current $\langle p \rangle_C$. The quantum current $\langle p \rangle_Q$ remains close to zero, and mimics the classical current $\langle p \rangle_C$ almost exactly.

IV. CONCLUSION

We sought here to explain certain general features of ratchet transport in Hamiltonian systems, and in particular to

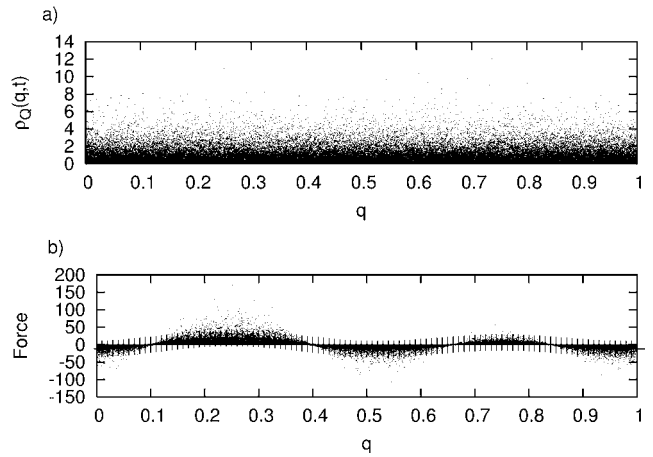


FIG. 6. (a) $\rho_Q(q,t)$ and (b) $F_{Q,H}(q,t)$ compared to $F_Q(q,t)$ (plus symbols) for the modified kicked Harper model after the first 50 kicks for $K=4$, $J=2$, and $\hbar=0.0001$.

explain the quantum vs classical behavior of the ratchet accelerator model developed in Ref. [10].

Here we have introduced, and applied, the concept of a Heisenberg or evolving force, in both quantum and classical mechanics, to ratchet transport. This showed that whether the bare force (i.e., the external force applied to the system) is unbiased is irrelevant since it is the evolving force that actually affects net transport. In both mechanics, asymmetry in the dynamical evolution can cause asymmetric spatial distortion which leads to the development of a net force and a nonzero current. Symmetry-breaking-based control of quantum and classical transport is hence of the same origin.

However, quantum and classical ratchet systems behave differently due to chaos. Classical systems fail to generate ratchet current when their phase space is fully chaotic, as the system distortion is effectively canceled, and the asymmetry that leads to directionality is lost. A completely chaotic phase space forces ensemble averages to reduce to phase space means that are independent of the detailed aspects of the

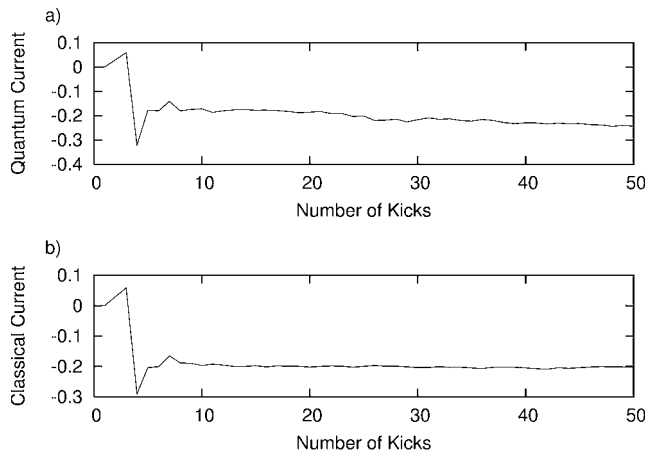


FIG. 7. Comparison of the (a) $\hbar=0.0001$ quantum current $\langle p \rangle_Q$ with (b) its classical analog $\langle p \rangle_C$ in the modified kicked Harper system for the first 50 kicks for $K=4$ and $J=2$.

dynamics. In such cases the ensemble-averaged net force remains zero for a nonbiased external force. By contrast, the equivalent effect is prevented in quantum mechanics, where coarse-scale structure is preserved. Symmetry-breaking-based quantum control of transport in classically chaotic systems is hence possible. For the same reason, quantum ratchet transport with full classical chaos becomes a strong indication of nonchaotic properties of the quantum dynamics.

The peculiar feature of the modified kicked Harper system, in which it shows unbounded linear transport for a wide parameter regime, is explained by its mapping onto a resonant system, and hence having a continuous spectrum. Its dynamics therefore is not necessarily quasiperiodic. Further, we computationally showed that if the quantum system is

sufficiently close to the classical limit, then quantum ratchet behavior smoothly approaches classical ratchet behavior.

The advantage of using the Heisenberg force to gain insight into the ratchet dynamics is expected to be generalizable to other systems.

ACKNOWLEDGMENTS

J.P. and P.B. are supported by a grant from the National Sciences and Engineering Research Council of Canada. J.G. is supported by the start-up fund, (WBS Contracts No. R-144-050-193-101 and No. R-144-050-193-133) and the NUS “YIA” fund (WBS Contract No. R-144-000-195-123), both from the National University of Singapore.

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- [1] M. V. Smoluchowski, *Phys. Z.* **13**, 1069 (1912).
- [2] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1969), Vol. 1, Chap. 46.
- [3] F. Jülicher, A. Ajdari, and J. Prost, *Rev. Mod. Phys.* **69**, 1269 (1997).
- [4] P. Reimann, *Phys. Rep.* **361**, 57 (2002).
- [5] R. D. Astumian and P. Hanggi, *Phys. Today* **55** (11), 33 (2002).
- [6] S. Flach, O. Yevtushenko, and Y. Zolotaryuk, *Phys. Rev. Lett.* **84**, 2358 (2000).
- [7] H. Schanz, M. F. Otto, R. Ketzmerick, and T. Dittrich, *Phys. Rev. Lett.* **87**, 070601 (2001); H. Schanz, T. Dittrich, and R. Ketzmerick, *Phys. Rev. E* **71**, 026228 (2005).
- [8] J. Gong and P. Brumer, *Phys. Rev. E* **70**, 016202 (2004).
- [9] E. Lundh and M. Wallin, *Phys. Rev. Lett.* **94**, 110603 (2005).
- [10] J. Gong and P. Brumer, *Phys. Rev. Lett.* **97**, 240602 (2006).
- [11] D. Poletti, G. G. Carlo, and B. Li, *Phys. Rev. E* **75**, 011102 (2007).
- [12] S. Denisov, L. Morales-Molina, S. Flach, and P. Hänggi, *Phys. Rev. A* **75**, 063424 (2007).
- [13] J. Gong, D. Poletti, and P. Hänggi, *Phys. Rev. A* **75**, 033602 (2007).
- [14] A. Kenfack, J. B. Gong, and A. K. Pattanayak, *Phys. Rev. Lett.* **100**, 044104 (2008).
- [15] P. H. Jones, M. Goonasekera, D. R. Meacher, T. Jonckheere, and T. S. Monteiro, *Phys. Rev. Lett.* **98**, 073002 (2007).
- [16] M. Sadgrove, M. Horikoshi, T. Sekimura, and K. Nakagawa, *Phys. Rev. Lett.* **99**, 043002 (2007).
- [17] I. Dana, V. Ramareddy, I. Talukdar, and G. S. Summy, *Phys. Rev. Lett.* **100**, 024103 (2008).
- [18] M. Shapiro and P. Brumer, *Principles of the Quantum Control of Molecular Processes* (John Wiley, New York, 2003).
- [19] I. Franco and P. Brumer, *Phys. Rev. Lett.* **97**, 040402 (2006).
- [20] G. Kurizki, M. Shapiro, and P. Brumer, *Phys. Rev. B* **39**, 3435 (1989).
- [21] J. Gong and P. Brumer, *Phys. Rev. Lett.* **86**, 1741 (2001); *J. Chem. Phys.* **115**, 3590 (2001).
- [22] J. Gong and P. Brumer, *Annu. Rev. Phys. Chem.* **56**, 1 (2005).
- [23] T. Takami and H. Fujisaki, *Phys. Rev. E* **75**, 036219 (2007).
- [24] M. Sadgrove, M. Horikoshi, T. Sekimura, and K. Nakagawa, *Eur. Phys. J. D* **45**, 229 (2007).
- [25] Toshiya Takami and Hiroshi Fujisaki, *J. Mod. Opt.* **56**, 822 (2009).
- [26] A rocking ratchet constitutes a different case, where the absence of a biased force means that the average of the force is zero when averaging over an interval of time.
- [27] A similar system was proposed by D. L. Shepelyansky, in *Quantum Chaos—Quantum Measurement*, edited by P. Cvitanovic, I. Percival, and A. Wirzba (Kluwer Academic Publishers, Dordrecht, 1992).
- [28] For a more recent and even simpler quantum ratchet model with full classical chaos, see J. Wang and J. B. Gong, *Phys. Rev. E* **78**, 036219 (2008).
- [29] This model was originally [10] proposed with $\hat{V}_r(\hat{q}) = \cos(\hat{q}) + \sin(2\hat{q})$. The spatial period has been rescaled for convenience.
- [30] For kicked Harper models, specifying the exact parameter ranges where dynamical localization appears is subtle. Roughly speaking, one needs $K > L$ to have delocalization in the modified kicked Harper model considered here. For a more detailed study of this issue for the original kicked Harper model, see R. Artuso, F. Borgonovi, I. Guarneri, L. Rebuzzini, and G. Casati, *Phys. Rev. Lett.* **69**, 3302 (1992).
- [31] I. Dana, *Phys. Lett. A* **197**, 413 (1995).
- [32] J. Wang and J. Gong, *Phys. Rev. A* **77**, 031405(R) (2008).
- [33] A. Iomin and S. Fishman, *Phys. Rev. Lett.* **81**, 1921 (1998).
- [34] Throughout this paper we use the term unbounded acceleration to denote the results of computational studies carried out for long times (and indeed longer than those shown in this paper). However, there is no formal proof that such dynamics shows unbounded acceleration for times longer than those that were computed.
- [35] In general, an operator in the Schrödinger representation does not evolve in time as is assumed to be the case for the force in Eq. (8). If, however, the operator of interest has an *explicit* time dependence, then the time evaluation of the Heisenberg representation of the operator is obtained by treating the explicit and implicit time dependences independently, and combining the results.
- [36] D. N. Zubarev, in *Nonequilibrium Statistical Thermodynamics*, edited by R. Gray and P. J. Sheperd (Consultants Bureau, New

- York, 1974), p. 73.
- [37] R. Kosloff and S. A. Rice, *J. Chem. Phys.* **74**, 1340 (1981).
- [38] Y. Gu, *Phys. Lett. A* **149**, 95 (1990).
- [39] L. E. Ballentine, Yumin Yang, and J. P. Zibin, *Phys. Rev. A* **50**, 2854 (1994).
- [40] L. E. Ballentine and J. P. Zibin, *Phys. Rev. A* **54**, 3813 (1996).
- [41] A. K. Pattanayak and P. Brumer, *Phys. Rev. Lett.* **77**, 59 (1996).
- [42] A. K. Pattanayak and P. Brumer, *Phys. Rev. E* **56**, 5174 (1997).
- [43] J. Wilkie and P. Brumer, *Phys. Rev. A* **55**, 27 (1997); **55**, 43 (1997).
- [44] J. Gong and P. Brumer, *Phys. Rev. A* **68**, 062103 (2003).
- [45] G. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics* (Oxford University Press, Oxford, 2005).
- [46] B. Eckhardt, in *Fractals, Noise and Chaos*, edited by E. R. Pike and L. A. Lugiato (Adam Hilger, Bristol, 1987).
- [47] T. Hogg and B. A. Huberman, *Phys. Rev. Lett.* **48**, 711 (1982).
- [48] A. Peres, *Phys. Rev. Lett.* **49**, 1118 (1982).
- [49] F. M. Izrailev and D. L. Shepelyanskii, *Theor. Math. Phys.* **43**, 553 (1980).