Stochastic calculus for uncoupled continuous-time random walks

Guido Germano,[1,*] Mauro Politi,[1,2,†] Enrico Scalas,[3,‡] and René L. Schilling[4,§]

[1]*Fachbereich Chemie und WZMW, Philipps-Universität Marburg, 35032 Marburg, Germany*

[2]*Dipartimento di Fisica, Università degli Studi di Milano, Via Giovanni Celoria 16, 20133 Milano, Italy*

[3]*Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale "Amedeo Avogadro," Viale Teresa Michel 11, 15121 Alessandria, Italy*

[4]*Institut für Mathematische Stochastik, Technische Universität Dresden, 01062 Dresden, Germany*

The continuous-time random walk (CTRW) is a pure-jump stochastic process with several applications not only in physics but also in insurance, finance, and economics. A definition is given for a class of stochastic integrals driven by a CTRW, which includes the Itō and Stratonovich cases. An uncoupled CTRW with zero-mean jumps is a martingale. It is proved that, as a consequence of the martingale transform theorem, if the CTRW is a martingale, the Itō integral is a martingale too. It is shown how the definition of the stochastic integrals can be used to easily compute them by Monte Carlo simulation. The relations between a CTRW, its quadratic variation, its Stratonovich integral, and its Itō integral are highlighted by numerical calculations when the jumps in space of the CTRW have a symmetric Lévy α-stable distribution and its waiting times have a one-parameter Mittag-Leffler distribution. Remarkably, these distributions have fat tails and an unbounded quadratic variation. In the diffusive limit of vanishing scale parameters, the probability density of this kind of CTRW satisfies the space-time fractional diffusion equation (FDE) or more in general the fractional Fokker-Planck equation, which generalizes the standard diffusion equation, solved by the probability density of the Wiener process, and thus provides a phenomenologic model of anomalous diffusion. We also provide an analytic expression for the quadratic variation of the stochastic process described by the FDE and check it by Monte Carlo.

I. INTRODUCTION

A. Continuous-time random walk

The continuous-time random walk (CTRW) is a pure-jump stochastic process used as a model for standard and anomalous diffusion when the sojourn time at a site is much greater than the time needed to jump to a new position, i.e., when jumps can be considered instantaneous events. The CTRW has been introduced in physics by Montroll and Weiss [1]; other seminal papers on its application to standard and anomalous transport phenomena are due to Scher and Lax [2,3] and to Montroll and Scher [4,5]. More recently, Shlesinger [6] wrote a review that contributed to further popularize the CTRW; theoretical, numerical, and empirical studies on the CTRW have been discussed by Weiss [7], Metzler and Klafter [8,9], and some authors of the present paper [10,11].

In a CTRW, if $X(t)$ denotes the position of a diffusing particle at time t, $\xi_i = X(t_i) - X(t_{i-1})$ denotes a random jump occurring at a random time t_i, and $\tau_i = t_i - t_{i-1}$ is the waiting or sojourn or interarrival or duration time between two consecutive jumps, one has

$$X(t) \overset{\text{def}}{=} S_{N(t)} \overset{\text{def}}{=} \sum_{i=1}^{N(t)} \xi_i, \tag{1}$$

where $t_0 = 0$, $X(0) = 0$, and $N(t)$ is a counting random process that gives the number of jumps up to time t. Throughout this paper, we assume the following:

(i) the jumps ξ_i, $i = 1, 2, \ldots$, are independent and identically distributed (iid) random vectors in $\mathbb{R}^d$, $d = 1, 2, \ldots$ [12];

(ii) the waiting times τ_i, $i = 1, 2, \ldots$, are iid random variables in $\mathbb{R}_+$; and

(iii) the families $(\xi_i, i = 1, 2, \ldots)$ and $(\tau_i, i = 1, 2, \ldots)$ are independent.

The third assumption means that we consider a so-called uncoupled CTRW. The first two assumptions entail that the joint distribution of any pair (ξ_i, τ_i) does not depend on i. If, in the uncoupled case, the law of (ξ_i, τ_i) is given by a density function $\varphi(\xi, \tau)$, the independence of ξ_i and τ_i means that it can be factorized in terms of the marginal probability densities for jumps $\lambda(\xi)$ and waiting times $\psi(\tau)$: $\varphi(\xi, \tau) = \lambda(\xi)\psi(\tau)$.

Equation (1) means that a CTRW is a random sum of independent random variables. The process of the jump times,

$$t_N = \sum_{i=1}^{N} \tau_i, \quad t_0 = 0, \tag{2}$$

is a renewal point process. Therefore, a CTRW can be seen as a compound renewal process [13–15]. The existence of an uncoupled CTRW can be proved based on the corresponding theorems of existence for renewal processes and discrete-

*guido.germano@staff.uni-marburg.de;
URL: www.staff.uni-marburg.de/~germano

†mauro.politi@staff.uni-marburg.de; mauro.politi@unimi.it

‡enrico.scalas@mfn.unipmn.it; URL: www.mfn.unipmn.it/~scalas
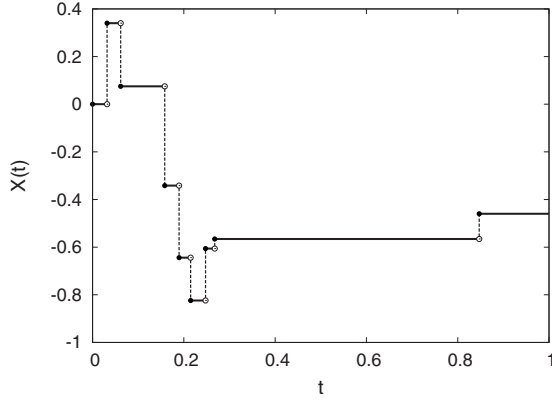§rene.schilling@tu-dresden.de; URL: www.math.tu-dresden.de/sto/schilling

FIG. 1. Realization of a CTRW with exponentially distributed waiting times ($\gamma_t = 0.1$) and normally distributed jumps ($\mu = 0$ and $\sigma = \sqrt{0.2}$). The probability density function $p_X(x)$ of $X(1)$ is plotted in Fig. 2, top left.

time random walks [16]. Càdlàg (right continuous with left limit) realizations of a CTRW can be easily and exactly generated by Monte Carlo simulation and plotted [11]. This is illustrated in Fig. 1.

An uncoupled CTRW is Markovian if and only if the waiting time distribution is exponential, i.e., $\psi(\tau) = \exp(-\tau/\gamma_t)/\gamma_t$ [17,18]. An uncoupled CTRW belongs to the class of semi-Markov processes [18–21], i.e., for any $A \subset \mathbb{R}^d$ and $t > 0$ we have

$$\mathbb{P}(S_n \in A, \tau_n \le t | S_0, \ldots, S_{n-1}, \tau_1, \ldots, \tau_{n-1})$$
$$= \mathbb{P}(S_n \in A, \tau_n \le t | S_{n-1}) \quad (3)$$

and, if we fix the position $S_{n-1} = y$ of the diffusing particle at time t_{n-1}, the probability on the right will be independent of n. In the generic coupled case, if the law of (ξ_n, τ_n) is given by a density function $\varphi(\xi, \tau)$, we can use $S_n = S_{n-1} + \xi_n$ and rewrite this as

$$\mathbb{P}(S_n \in A, \tau_n \le t | S_{n-1}) = \int_A \int_0^t \varphi(x - S_{n-1}, \tau) d\tau dx. \quad (4)$$

This can be shown as follows. Let $I_A(x)$ denote the indicator function that yields 1 if $x \in A$ and 0 otherwise. Probabilities can be replaced with expectations writing $\mathbb{P}(x \in A) = \mathbb{E}[I_A(x)]$. Moreover, one has $I_A I_B = I_{A \cap B}$. Thus, if $B = (0, t]$,

$$\mathbb{P}(S_n \in A, \tau_n \in B | S_{n-1}) = \mathbb{E}[I_A(S_n) I_B(\tau_n) | S_{n-1}]$$
$$= \mathbb{E}[I_A(S_{n-1} + \xi_n) I_B(\tau_n) | S_{n-1}]$$
$$= \int_{\mathbb{R}^d} \int_0^\infty I_A(S_{n-1} + \xi) I_B(\tau) \varphi(\xi, \tau) d\tau d\xi$$
$$= \int_{\mathbb{R}^d} \int_B I_A(S_{n-1} + \xi) \varphi(\xi, \tau) d\tau d\xi$$
$$= \int_{\mathbb{R}^d} \int_0^t I_A(x) \varphi(x - S_{n-1}, \tau) d\tau dx$$
$$= \int_A \int_0^t \varphi(x - S_{n-1}, \tau) d\tau dx. \quad (5)$$

Montroll and Weiss [1] wrote Eq. (4) as an integral equation for the probability density $p_X(x, t)$ of finding the particle in position x at time t in terms of the joint probability density $\varphi(\xi, \tau)$ of the jumps ξ and waiting times τ,

$$p_X(x, t) = \delta(x) \Psi(t) + \int_{\mathbb{R}^d} \int_0^t \varphi(\xi, \tau) p_X(x - \xi, t - \tau) d\tau d\xi, \quad (6)$$

where $\Psi(t) = 1 - \int_0^t \psi(\tau) d\tau$ is the complementary cumulative distribution function for the waiting times, also called survival function. This can be shown observing that

$$\mathbb{P}[X(t) \in dx | X(0) = 0] = p_X(x, t) dx \quad (7)$$

and

$$\mathbb{P}[X(t) \in dx | X(t') = x'] = \mathbb{P}[X(t - t') \in dx | X(0) = x']$$
$$= \mathbb{P}[X(t - t') - x' \in dx | X(0) = 0] = p_X(x - x', t - t') dx \quad (8)$$

because the increments in time and space are iid and hence homogeneous. Moreover, from Eq. (4),

$$\mathbb{P}(S_1 \in dx, \tau_1 \in dt | S_0 = 0) = \varphi(x, t) dx dt. \quad (9)$$

The probability in Eq. (7) can be decomposed depending on the duration of the first jump τ_1 with respect to t,

$$\mathbb{P}[X(t) \in dx | X(0) = 0] = \mathbb{P}[X(t) \in dx, \tau_1 > t | X(0) = 0]$$
$$+ \mathbb{P}[X(t) \in dx, \tau_1 \le t | X(0) = 0]. \quad (10)$$

The part without a jump before t is given by

$$\mathbb{P}[X(t) \in dx, \tau_1 > t | X(0) = 0] = \mathbb{P}(\tau_1 > t) \delta(x) dx$$
$$= \delta(x) \Psi(t) dx. \quad (11)$$

The other part is given by

$$\mathbb{P}[X(t) \in dx, \tau_1 \le t | X(0) = 0]$$
$$= \int_{\mathbb{R}^d} \int_0^t \mathbb{P}[X(t) \in dx | X(t') = x']$$
$$\times \mathbb{P}(S_1 \in dx', \tau_1 \in dt' | S_0 = 0)$$
$$\overset{\text{Eq. (9)}}{=} \int_{\mathbb{R}^d} \int_0^t \mathbb{P}[X(t) \in dx | X(t') = x'] \varphi(x', t') dt' dx'$$
$$\overset{\text{Eq. (8)}}{=} \int_{\mathbb{R}^d} \int_0^t p_X(x - x', t - t') dx \varphi(x', t') dt' dx'$$
$$= \left[\int_{\mathbb{R}^d} \int_0^t \varphi(\xi, \tau) p_X(x - \xi, t - \tau) d\tau d\xi \right] dx. \quad (12)$$

Combining Eqs. (11) and (12) yields Eq. (6). Notice that the latter just gives a one-point probability density, which is not enough to characterize a stochastic process without further assumptions.

Equation (6) can be solved in the Fourier-Laplace domain,

$$\hat{\tilde{p}}(k,s) = \frac{1}{1 - \hat{\tilde{\varphi}}(k,s)} \frac{1 - \tilde{\psi}(s)}{s}, \qquad (13)$$

where the Fourier and Laplace transforms are defined as

$$\hat{f}(k) = \mathcal{F}_x[f(x)](k) = \int_{-\infty}^{+\infty} f(x)e^{ikx}dx, \quad k \in \mathbb{R}, \quad (14)$$

$$\tilde{f}(s) = \mathcal{L}_t[f(t)](s) = \int_0^\infty f(t)e^{-st}dt, \quad s \in \mathbb{C}. \quad (15)$$

The inverse transforms to the space-time domain are possible in the uncoupled case, i.e., when $\varphi(\xi, \tau) = \lambda(\xi)\psi(\tau)$; this leads to a series expression written in terms of the probability $\mathbb{P}[N(t)=n]=p_N(n,t)$ of the counting process $N(t)$ and the n-fold convolution $\lambda^{*n}(x)$ of the marginal probability density of jumps $\lambda(\xi)$,

$$p_X(x,t) = \sum_{n=0}^{\infty} p_N(n,t)\lambda^{*n}(x). \qquad (16)$$

The method using integral transforms is described in several papers, including the original one by Montroll and Weiss [1]. However, Eq. (16) can also be derived directly by probabilistic considerations. Indeed, Eq. (1) is a random sum of iid random variables. This means that any position x can be reached at time t by a finite number n of jumps. The probability of reaching position x at time t in exactly n jumps is $p_N(n,t)\lambda^{*n}(x)$. Equation (16) follows given that these events are mutually exclusive. Note that $p_N(0,t)\lambda^{*0}(x)$ coincides with the singular term $\delta(x)\Psi(t)$, meaning that the distribution function for x has a jump at position $x=0$ of height $\Psi(t)$.

A CTRW with exponential waiting times is called a compound Poisson process (CPP), as in this case

$$p_N(n,t;\gamma_t) = \exp(-t/\gamma_t)\frac{(t/\gamma_t)^n}{n!}. \qquad (17)$$

A CPP is not only a Markov but also a Lévy process. This means that it has independent and time-homogeneous (stationary) increments. In the Lévy case $p_X(x,t)$, even $p_X(x,1)$, fully characterizes the stochastic process defined by Eq. (1) [16,22,23]; this is due to the infinite divisibility and the fact that the increments are stationary and independent. For a normal CPP, i.e., a CPP with normally distributed jumps (NCPP), the n-fold convolution $\lambda^{*n}(x)$ of $N(\mu, \sigma^2)$ can be evaluated as $N(n\mu, n\sigma^2)$, leading to

$$p_X(x,t;\mu,\sigma,\gamma_t) = \exp(-t/\gamma_t)\sum_{n=0}^{\infty} \frac{(t/\gamma_t)^n}{n!} \frac{1}{\sqrt{2\pi n}\sigma}$$
$$\times \exp\left(-\frac{(x-n\mu)^2}{2n\sigma^2}\right). \qquad (18)$$

B. CTRW in physics, insurance, finance, and economics

Since the seminal paper by Montroll and Weiss [1], there has been much scientific activity on the application of the CTRW to important physical problems. One line of research investigated anomalous relaxation related to power-law tails of the waiting time distribution as well as the asymptotic behavior of the CTRW for large times [4,24–28]. As mentioned above, Metzler and Klafter [8,9] extensively reviewed these and subsequent studies. Furthermore, in their book, ben-Avraham and Havlin [29] discussed the applications to physical chemistry. Here, it is worth mentioning the recent work on the relation between the CTRW and fractional diffusion that can be traced to papers by Balakrishnan [30] and Hilfer and Anton [31] and has been thoroughly discussed in Refs. [10,11,32]. Some specific applications include, e.g., plasmas [33], microporous materials [34], and biopolymers [35,36].

The CTRW has been applied also in insurance, finance, and economics. Even if well known in the field of econophysics [10,37], these applications deserve a short summary.

In ruin theory for insurance companies, the jumps ξ_i are interpreted as claims and they are positive random variables; t_i is the instant at which the ith claim is paid [38].

In mathematical finance, if $P_A(t)$ is the price of an asset at time t and $P_A(0)$ is the price of the same asset at a previous reference time $t_0=0$, then $X(t)=\log[P_A(t)/P_A(0)]$ represents the logarithmic return (or logarithmic price) at time t. In regulated markets using a continuous double-auction trading mechanism, such as stock markets, prices vary at random times t_i when a trade takes place, and $\xi_i=X(t_i)-X(t_{i-1})$ $=\log[P_A(t_i)/P_A(t_{i-1})]$ is the tick-by-tick logarithmic return, whereas $\tau_i=t_i-t_{i-1}$ is the intertrade duration; for more details, see [10,37,39] and references contained therein.

In the theory of economic growth, ξ_i represents a growth shock, which can actually be both positive and negative, $X(t)$ is the logarithm of a firm's size or of an individual's wealth, and τ_i is the time interval between two consecutive growth shocks; see [10] and references therein.

C. Motivation for the study of stochastic integrals driven by a CTRW and link with fractional calculus

Given the wide range of applications of the CTRW overviewed in Sec. I B, it is relevant to study diffusive stochastic differential equations whose driving noise is defined in terms of a CTRW,

$$dZ = a(Z,t)dt + b(Z,t)dX. \qquad (19)$$

Here $Z(X,t)$ is the unknown random function, $a(Z,t)$ and $b(Z,t)$ are known functions of Z and time t, and dX represents the CTRW "measure" with respect to which stochastic integrals are defined. In order to give a rigorous meaning to such an expression, some constraints on the properties of the CTRW are necessary. In a recent paper, the theory has been discussed for stochastic integration on a time-homogeneous (stationary) CTRW—i.e., the already mentioned CPPs [40]. Although the theory reported there was already well known by mathematicians [41] and has been used in finance for option pricing since 1976 [42], that paper contains useful material and is written in a way that is clear and appealing for physicists. Here, inspired by Ref. [40], the theory will be further discussed and developed.

Consider a CTRW $X(t)$ whose jumps in space ξ_i are distributed according to the symmetric Lévy α-stable law, $\alpha \in (0,2]$, whose density can be expressed as a series or, more conveniently, as the inverse Fourier transform of its characteristic function,

$$L_\alpha(\xi; \gamma_x) = \mathcal{F}_k^{-1}[\exp(-|\gamma_x k|^\alpha)](\xi). \tag{20}$$

For $\alpha = 2$ this corresponds to a Gaussian with standard deviation $\sigma = \sqrt{2}\gamma_x$. Let the waiting times τ_i of the CTRW have the probability density

$$\psi_\beta(\tau; \gamma_t) = -\frac{d}{d\tau} E_\beta(-(\tau/\gamma_t)^\beta), \tag{21}$$

where $E_\beta(z)$, $\beta \in (0,1]$, is the one-parameter Mittag-Leffler function [43–45],

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C}. \tag{22}$$

For $\beta = 1$ this corresponds to an exponential function. When $\beta < 1$ and $t \in \mathbb{R}_+$, $E_\beta(-t^\beta)$ is approximated for small values of t by a stretched exponential decay (Weibull function), $\exp(-t^\beta/\Gamma(1+\beta))$, and for large values of t by a power law, $t^{-\beta}/\Gamma(1-\beta)$.

In the diffusive limit for $X(t)$, when the scale parameters γ_x of the jumps and γ_t of the waiting times vanish satisfying the scaling relation $\gamma_x^\alpha/\gamma_t^\beta = D$, if in Eq. (19) $a=0$ and $b=1$ the probability density $p_Z(z,t) = p_X(x,t;\gamma_x,\gamma_t)$ converges to the solution of the space-time fractional diffusion equation (FDE) [46,47],

$$\frac{\partial^\beta}{\partial t^\beta} u_X(x,t;D) = D\frac{\partial^\alpha}{\partial|x|^\alpha} u_X(x,t;D),$$

$$u_X(x,0^+;D) = \delta(x), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+. \tag{23}$$

The space-fractional derivative of order $\alpha \in (0,2]$ is defined according to Riesz,

$$\frac{d^\alpha}{d|x|^\alpha} f(x) = \mathcal{F}_k^{-1}[-|k|^\alpha \hat{f}(k)](x). \tag{24}$$

The time-fractional derivative of order $\beta \in (0,1]$ is defined in the sense of Caputo,

$$\frac{d^\beta}{dt^\beta} f(t) = \mathcal{L}_s^{-1}[s^\beta \tilde{f}(s) - s^{\beta-1} f(0^+)](t). \tag{25}$$

The FDE is a generalization of the standard diffusion equation, which results for $\alpha = 2$ and $\beta = 1$; in this case the solution $u_X(x,t;D)$ of the Cauchy problem given by Eq. (23) is the one-point probability density of the Bachelier-Wiener process or Brownian motion $B(t)$,

$$u_X(x,t;D) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right), \tag{26}$$

and $X(t)$ is the NCPP introduced at the end of Sec. I A. The general solution of the FDE was worked out in the Fourier-Laplace domain,

$$\hat{\tilde{u}}_X(k,s) = \frac{s^{\beta-1}}{D|k|^\alpha + s^\beta}. \tag{27}$$

Because

$$\mathcal{L}_s^{-1}\left[\frac{s^{\beta-1}}{D|k|^\alpha + s^\beta}\right](t) = E_\beta(-D|k|^\alpha t^\beta) \tag{28}$$

defining $\kappa = kt^{\beta/\alpha}$ and the time-independent Green's function

$$G_{\alpha,\beta}(\xi;D) = \mathcal{F}_\kappa^{-1}[E_\beta(-D|\kappa|^\alpha)](\xi), \tag{29}$$

the solution of the FDE [Eq. (23)] can be expressed in the space-time domain as

$$u_X(x,t;D) = t^{-\beta/\alpha} G_{\alpha,\beta}(xt^{-\beta/\alpha};D). \tag{30}$$

These results are a consequence of a generalized central limit theorem for sequences of random variables [32]. A simpler derivation can be found in Ref. [10]. For computational details see Sec. III and Ref. [11]. If $a(x,t)$ and $b(x,t)$ are not constant, a fractional Fokker-Planck equation for $u_X(x,t;D)$ has been proposed in the diffusive limit [8,48–52] starting from a generalized master equation [50] or a CTRW [51]. For the NCPP this reduces to the standard Fokker-Planck equation [53,54].

Without taking the diffusive limit and if $a=0$ and $b=1$, the time evolution of the probability density $p_X(x,t)$ is given by the Montroll-Weiss integral equation [Eq. (6)] [1]. The uncoupled case of the latter can be presented alternatively in an integrodifferential form [55],

$$\int_0^t \Phi(t-\tau)\frac{\partial}{\partial\tau} p_X(x,\tau)d\tau = -p_X(x,t) + \int_{-\infty}^{+\infty} \lambda(x-\xi)p_X(\xi,t)d\xi, \tag{31}$$

which can be interpreted as a time evolution equation of Fokker-Planck type. It involves the time derivative of $p_X(x,t)$ and an auxiliary function $\Phi(t)$ defined through its Laplace transform as $\tilde{\Phi}(s) = \tilde{\Psi}(s)/\tilde{\psi}(s)$, so that $\Psi(t) = \int_0^t \Phi(t-\tau)\psi(\tau)d\tau$. This approach has been generalized studying scores of possible kinetic equations for non-Markovian processes [56]. What follows in Secs. II and III is valid without necessarily taking the diffusive limit. Nevertheless, the latter is important because it motivates our particular choice for the marginal distributions of jumps and waiting times and because it provides analytic expressions that can be compared to our Monte Carlo results as will be shown in Sec. III.

II. STOCHASTIC INTEGRALS

In Ref. [40], the stochastic integral is not explicitly defined for a CTRW. However, starting from the fact that sample paths of a CTRW can be represented by step functions, it is possible to give an explicit formula.

A. Definitions

Some heuristic manipulations are useful for the definition of the stochastic integral

$$J(t) = \int_0^t Y(s)dX(s), \qquad (32)$$

where $X(t)$ and $Y(t)$ are CTRWs; as before we denote by t_i and ξ_i the jump times and the jumps of $X(t)$. A particularly interesting case is when $X(t)$ and $Y(t)$ are synchronous, i.e., when their jump times t_i coincide; this is, e.g., the case if $Y(t) = G[X(t)]$ with a suitable function $G(X)$, but it is also possible that the jump times coincide and the corresponding jump sizes of $X(t)$ and $Y(t)$ at $t = t_i$ are independent.

Equation (1) defining $X(t)$ can be written in terms of Heaviside's unit step function $H_\theta(t)$, which is 0 for $t < 0$, θ for $t = 0$, and 1 for $t > 0$ [57],

$$X(t) = \sum_{i=1}^{N(t)} \xi_i H_1(t - t_i). \qquad (33)$$

The value of $\theta = H(0)$ often does not matter because $H(t)$ is mostly used as a cumulative distribution function; θ appears neither in the integral of $H(t)$, the ramp function $R(t) = tH(t)$, nor in its derivative, Dirac's function $\delta(t)$ (actually $\delta(t)$ is not a proper function but rather a distribution in the sense of Sobolev and Schwartz [58]). The usual choices for the parameter θ are 0, 1/2, and 1, corresponding to the left-continuous, symmetric, and right-continuous variants of Heaviside's step function. The symmetric variant allows us to express the step function through the sign function, $H_{1/2}(t) = (\text{sgn } t + 1)/2$, and is preferred in many applications, but picking $\theta = 1$, which makes the CTRW right continuous, appears more consistent with Eq. (1). Since the "derivative" of Heaviside's function $H(t - t_i)$ is Dirac's function $\delta(t - t_i)$, we can write

$$dX(t) = \sum_{i=1}^{N(t)} \xi_i \delta(t - t_i)dt. \qquad (34)$$

Here

$$\xi_i \overset{\text{def}}{=} \Delta X(t_i) = X(t_i^+) - X(t_i^-) \qquad (35)$$

independently of θ, with

$$X(t_i^+) \overset{\text{def}}{=} \lim_{s \to t_i^+} X(s) = \lim_{s \to t_i, s > t_i} X(s), \qquad (36)$$

$$X(t_i^-) \overset{\text{def}}{=} \lim_{s \to t_i^-} X(s) = \lim_{s \to t_i, s < t_i} X(s). \qquad (37)$$

However, inserting Eq. (34) into Eq. (32) and using the properties of Dirac's δ function, it becomes necessary to evaluate $Y(t_i)$. If t_i is a common jump time of $Y(t)$ and $X(t)$, the value of θ in the Heaviside function used to express $Y(t)$ matters again because the values $Y(t_i^-)$, corresponding to $\theta = 0$, and $Y(t_i^+)$, corresponding to $\theta = 1$, lead to different results. Although the choice $\theta = 1$ is more appropriate to represent the CTRWs $X(t)$ and $Y(t)$ through Eq. (33), for the integral it is equally correct to take $Y(t_i^-)$ or any linear interpolation between $Y(t_i^-)$ and $Y(t_i^+)$, i.e., any value of $Y(t)$ in the "infinitesimal interval" $[t_i^-, t_i^+]$. The result is a whole family of stochastic integrals depending on a parameter $a \in [0, 1]$,

$$J_a(t) \overset{\text{def}}{=} \int_0^t Y(s_a)dX(s) = \sum_{i=1}^{N(t)} Y(t_i^a)\xi_i$$

$$= \sum_{i=1}^{N(t)} [(1-a)Y(t_i^-) + aY(t_i^+)][X(t_i^+) - X(t_i^-)]. \qquad (38)$$

Notice that this expression is exact without the need for a limit: the number of jumps $N(t)$ between 0 and t is a random finite integer. Moreover, for any value of a the integral is a right-continuous function with jumps $\Delta J_a(t_i) = J_a(t_i^+) - J_a(t_i^-) = Y(t_i^a)\Delta X(t_i)$. This naive definition works nicely if the driving noise is a step function with jump times t_i and jumps $\xi_i = X(t_i^+) - X(t_i^-)$. As $X(t)$ and $Y(t)$ are right continuous, we even have $X(t_i^+) = X(t_i)$ and, if $X(t)$ and $Y(t)$ are synchronous, also $Y(t_i^-) = Y(t_{i-1})$. As soon as one wants to go beyond this situation, measurability and convergence become an issue. This observation prompted Itō to use martingale convergence theorems to tackle the convergence for a large class of integrators [59]. To do so it is necessary that $J_a(t)$ is a martingale whenever $X(t)$ is; it turns out that this is the case if and only if $a = 0$. For this aim we assume that $Y(t)$ is adapted, i.e., measurable with respect to the natural filtration generated by the driving noise, $\mathcal{F}_t = \sigma(X(s) : s \le t)$. Therefore the integrand $Y(t_i^-)$ becomes statistically independent of the increment $\xi_i = X(t_i) - X(t_i^-)$ and we end up with a stochastic integral, the Itō integral $I(t) = J_0(t)$, which is a martingale (see Sec. II B for details),

$$I(t) \overset{\text{def}}{=} \int_0^t Y(s^-)dX(s) = \sum_{i=1}^{N(t)} Y(t_i^-)\xi_i = \sum_{i=1}^{N(t)} Y(t_i^-)[X(t_i) - X(t_i^-)]. \qquad (39)$$

Evaluating $Y(t)$ at the left end point t_i^- of the infinitesimal interval $[t_i^-, t_i^+]$ makes the integrand nonanticipating and adapted. This can be seen as a causality requirement: one does not want $Y(t)$ to anticipate the future behavior of $\xi(t)$ [60]. An elementary introduction to the concept of a nonanticipating function can be found in Ref. [61]. Any adapted process with right-continuous (or left-continuous) paths is progressively measurable.

Equation (38) can be rearranged to

$$J_a(t) = J_{1/2}(t) + \left(a - \tfrac{1}{2}\right)[X, Y](t), \qquad (40)$$

where

$$[X, Y](t) \overset{\text{def}}{=} \sum_{i=1}^{N(t)} [X(t_i) - X(t_i^-)][Y(t_i) - Y(t_i^-)] \qquad (41)$$

is the covariation or cross variation of $X(s)$ and $Y(s)$ for $s \in [0, t]$. When $Y(s) = X(s)$, the quadratic variation $[X, X](t)$ is denoted simply as $[X](t)$. Thus each member of the family of stochastic integrals with $a \in [0, 1]$ can be obtained adding a compensator to the Stratonovich integral $S(t) = J_{1/2}(t)$. The latter is particularly appealing because it can be computed according to the usual rules of calculus. However, the Itō integral has the advantage of being a martingale, as proved in Sec. II B. The distinction between integrals with different

values of a disappears in the continuous limit for processes with finite variation, e.g., continuously differentiable functions, because this implies that their covariation is zero [59]. Unless stated otherwise $\int Y(s)dX(s)$ indicates the Itō integral, while the Stratonovich integral is often denoted $\int Y(s) \circ dX(s)$.

B. Martingale property of the Itō integral

Although it is easy to simulate directly the stochastic process defined in Eq. (39)—see Sec. III for numerical examples—it is not so easy to derive its properties. Each term in the sum depends on the previous ones and the nice properties of convolutions are not helpful here. However, using the *martingale transform theorem*, it is possible to obtain conditions under which $I(t)$ is a martingale.

In order to define martingales, we need a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \geq 0}$ is a filtration—i.e., an increasing family of sub-σ-algebras—representing the information available up to time t. A martingale is a stochastic process $X(t)$ for which the expected value $\mathbb{E}[|X(t)|]$ exists for $t \geq 0$ and the conditional expectation $\mathbb{E}[X(t)|\mathcal{F}_s]$ is $X(s)$ for all $t \geq s$ [59,62,63].

Let us consider the natural filtration, that is the σ-algebra generated by the CTRW itself:

$$\mathcal{F}_t = \sigma(X(s):s \leq t)$$
$$\overset{\text{def}}{=} \sigma(\xi_1, \ldots, \xi_k; \tau_1, \ldots, \tau_k:k \leq N(t)) = \mathcal{G}_{N(t)}. \quad (42)$$

Then $X(t)$ is a martingale with respect to $\mathcal{F}_t$ if and only if the mean of the jumps $\mathbb{E}[\xi_i]$ is zero. Denote by (t_i, ξ_i) the time and height of the finitely many jumps $i = N(s)+1, \ldots, N(t)$ occurring between s and $t > s$. Then

$$\mathbb{E}[X(t)|\mathcal{F}_s] = X(s) + \sum_{i=N(s)+1}^{N(t)} \mathbb{E}[\xi_i|\mathcal{F}_s]. \quad (43)$$

Using the semi-Markov property [Eq. (3)], we get for $i > N(s)$

$$\mathbb{E}[\xi_i|\mathcal{F}_s] = \mathbb{E}[\xi_i|\mathcal{G}_{N(s)}] = \mathbb{E}[\xi_i|\xi_{N(s)}] = \mathbb{E}[\xi_i] = 0, \quad (44)$$

thanks to the independence of ξ_i and $\xi_1, \ldots, \xi_{N(s)}$. Equation (43) becomes

$$\mathbb{E}[X(t)|\mathcal{F}_s] = X(s), \quad (45)$$

which shows that $(X(t))_{t \geq 0}$ is indeed a martingale with respect to its natural filtration.

Note that our argument is valid for a general uncoupled CTRW. We do not need the independence of the increments $X(t+\Delta t) - X(t)$ of the process $X(t)$ for nonoverlapping intervals. Of course, if we have independent increments, i.e., a CPP $X(t)$, the proof becomes easier.

Let us now investigate the integral defined in Eq. (39) for a martingale CTRW $X(t)$. If there is an arbitrary but finite number of jumps between s and $t > s$, one has

$$\mathbb{E}[I(t)|\mathcal{F}_s] = I(s) + \sum_{i=N(s)+1}^{N(t)} \mathbb{E}[Y(t_i^-)\xi_i|\mathcal{G}_{N(s)}]. \quad (46)$$

Now, one observes that $\xi_i = X(t_i) - X(t_{i-1})$ and that the random sum in Eq. (46) becomes

$$\sum_{i=N(s)+1}^{N(t)} \mathbb{E}[Y(t_i^-)\xi_i|\mathcal{G}_{N(s)}]$$
$$= \sum_{i=N(s)+1}^{N(t)} \mathbb{E}[Y(t_i^-)(X(t_i) - X(t_{i-1}))|\mathcal{G}_{N(s)}]. \quad (47)$$

If $Y(t)$ is measurable with respect to $\mathcal{F}_t = \mathcal{G}_{N(t)}$, then $Y(t_i^-)$ is $\mathcal{G}_{N(t_i^-)}$-measurable. Since $N(t_i^-) = N(t_{i-1})$, this means that $Y(t_i^-)$ is $\mathcal{G}_{N(t_{i-1})} = \mathcal{G}_{i-1}$-measurable; this is to say that $Y(t_i^-)$ is predictable for the filtration $\mathcal{G}_i$, i.e., the value of $Y(t_i^-)$ is known at time t_{i-1}. Whenever for each i the expression $Y(t_i^-)[X(t_i) - X(t_{i-1})]$ has a finite absolute mean, e.g., if the process $Y(t_i^-)$ is bounded—we have

$$\mathbb{E}[Y(t_i^-)(X(t_i) - X(t_{i-1}))|\mathcal{G}_{N(s)}]$$
$$= \mathbb{E}[\mathbb{E}[Y(t_i^-)(X(t_i) - X(t_{i-1}))|\mathcal{G}_{i-1}]|\mathcal{G}_{N(s)}]$$
$$= \mathbb{E}[Y(t_i^-)\mathbb{E}[(X(t_i) - X(t_{i-1}))|\mathcal{G}_{i-1}]|\mathcal{G}_{N(s)}]. \quad (48)$$

In the above calculation we have used the fact that $\mathcal{G}_{N(s)}$ is contained in $\mathcal{G}_{i-1}$ as $i-1 \geq N(s)$ along with the *tower property* and the fact that we can *pull out what is known* from the conditional expectation [63]. Since $X(t)$ is a martingale, we have $\mathbb{E}[X(t_i)|\mathcal{F}_{t_{i-1}}] = X(t_{i-1})$ which means that

$$\mathbb{E}[Y(t_i^-)[X(t_i) - X(t_{i-1})]|\mathcal{G}_{N(s)}] = 0. \quad (49)$$

Consequently, each term in the random sum vanishes and $\mathbb{E}[I(t)|\mathcal{F}_s] = I(s)$. Summing up, if $X(t)$ is a martingale with respect to $\mathcal{F}_t$ and if the integrand is bounded and predictable, one has that $I(t)$ is also a martingale with respect to $\mathcal{F}_t$.

III. SIMULATION

In Sec. II we have explicitly defined and rigorously characterized a martingale stochastic integral driven by an uncoupled CTRW and given in Eq. (39), as well as a more general class of stochastic integrals given by Eq. (38). A useful property of these equations is that they can be easily implemented by means of Monte Carlo simulation, as will be shown here for the case $Y(t) = X(t)$. The theory of Sec. II is the basis for the Monte Carlo solution of stochastic differential equations driven by CTRWs and discussed above in Sec. I C.

The marginal distributions of jumps and waiting times presented in Sec. I C are apparently demanding, but they can be sampled easily using one-line transformation formulas [11,64,65]. A random number ξ drawn from the symmetric Lévy α-stable probability density [Eq. (20)] can be obtained from two independent uniform random numbers $U, V \in (0, 1)$ through a transformation due to Chambers *et al.* [66] and implemented by McCulloch [67],

$$\xi = \gamma_x \left\{ \frac{-\log U \cos \Phi}{\cos[(1-\alpha)\Phi]} \right\}^{1-1/\alpha} \frac{\sin(\alpha\Phi)}{\cos \Phi}, \qquad (50)$$

where $\Phi = \pi(V - 1/2)$. For $\alpha = 2$ Eq. (50) reduces to $\xi = 2\gamma_x\sqrt{-\log U} \sin \Phi$, i.e., the Box-Muller method [68] for Gaussian deviates with standard deviation $\sigma = \sqrt{2}\gamma_x$. A random number τ drawn from the one-parameter Mittag-Leffler probability density [Eq. (21)] can be similarly obtained from two independent uniform random numbers $U, V \in (0,1)$ through a transformation proposed by Kozubowski and Rachev [69] and implemented by Germano *et al.* [70],

$$\tau = -\gamma_t \log U \left[\frac{\sin(\beta\pi)}{\tan(\beta\pi V)} - \cos(\beta\pi) \right]^{1/\beta}. \qquad (51)$$

For $\beta = 1$ Eq. (51) reduces to the transformation formula for the exponential distribution, $\tau = -\gamma_t \log U$.

Now, as outlined above, the Monte Carlo simulation of an uncoupled CTRW is straightforward. To compute the value $X(t)$, we generate a sequence of $N(t) + 1$ iid waiting times τ_i using Eq. (51) until their sum is greater than t. We discard the last waiting time and generate $N(t)$ iid jumps ξ_i using Eq. (50). Their sum is the desired value of $X(t)$. Based on Eqs. (1) and (2), this algorithm was used to generate Fig. 1. This procedure is also the basis to compute $I(t)$ according to Eq. (39) or more in general $J_a(t)$ according to Eq. (38) and the covariation $[X, Y](t)$ according to Eq. (41). Each jump $\xi_i = X(t_i) - X(t_i^-)$ is multiplied by $Y(t_i^-)$, $(1-a)Y(t_i^-) + aY(t_i)$, or $Y(t_i) - Y(t_i^-)$, and the results of these multiplications are summed to obtain $I(t)$, $J_a(t)$, and $[X, Y](t)$, respectively.

C++ code for the case $Y(t) = X(t)$, and $a = 1/2$ is shown in Table I. Furthermore, MATLAB scripts will be uploaded to the MATLAB Central File Exchange [71,72]. CPU times grow linearly with the total number of jumps and take $1-3$ μs/jump depending on α and β on a 2.2 GHz AMD Athlon 64 X2 "Toledo" Dual-Core processor with Fedora Core 7 Linux, using the Ran uniform random number generator [73] and the GNU C++ compiler (g++) Version 4.1.2 with the $-$O3 $-$static optimization options. We checked that in our simulations the empirical average of the number of jumps per run coincides with the expectation of the Mittag-Leffler counting process,

$$\mathbb{E}[N(t)] = \frac{(t/\gamma_t)^\beta}{\Gamma(\beta + 1)}. \qquad (52)$$

Figures 2 and 3 show histograms from 1×10^6 Monte Carlo realizations of $X(t)$, $I(t) = \int_0^t X(s)dX(s)$, $S(t) = \int_0^t X(s) \circ dX(s)$, and $[X](t)$, where $t = 1$ and $X(t)$ is a symmetric CTRW with jump and time scale parameters linked by the relation $\gamma_x^\alpha / \gamma_t^\beta = D = 1$. Thus the integrals in Figs. 2 and 3 give the Monte Carlo solution for $t = 1$ of the stochastic differential equation $dZ = XdX$ with initial condition $Z(0) = 0$. Since the Itō integral is a martingale starting at zero, its mean is zero. This is not true for the Stratonovich integral. The probability density of the Stratonovich integral $S(t) = X^2(t)/2$ can be worked out from the density of the stochastic process $X(t)$ by the transformation

TABLE I. Relevant lines from the central loop of our C++ program for the Monte Carlo calculation of a CTRW $X(t)$, its quadratic variation $[X](t)$, its Itō integral $I(t) = J_0(t)$, and its Stratonovich integral $S(t) = J_{1/2}(t)$. The histograms of the empirical probability densities are plotted in Figs. 2 and 3.

```
jumps = 0

// Loop over runs
for (run = 1; run <= runs; run++) {

    // Initialize and increment t, x, etc.
    t = 0, x = 0, qvar = 0, ito = 0, str = 0,
    tau = random.t();          // Eq. (51)
    while (t + tau < t_max) {
        t += tau;              // time t
        xi = random.x();       // Eq. (50)
        qvar += xi*xi;         // [X](t)
        ito += x*xi;           // I(t)
        str += (x+xi/2)*xi;    // S(t)
        x += xi;               // X(t)
        tau = random.t();      // Eq. (51)
        jumps++;               // N(t)
    }

    // Update histograms at the end of each run
    hisx.add(x);               // X(t)
    hisq.add(qvar);            // [X](t)
    hisi.add(ito);             // I(t)
    hiss.add(str);             // S(t)
}
```

$p_S(s,t) = \Sigma_i p_X(x_i(s), t)|dx_i(s)/ds|$, where the sum is over all x_i corresponding to the same s. For $s = x^2/2$ this is $x_{1,2} = \pm\sqrt{2s}$ and thus

$$p_S(s,t) = 2p_X(\sqrt{2s}, t)/\sqrt{2s}, \quad s > 0. \qquad (53)$$

In the diffusive limit the NCPP $X(t)$ approximates the Bachelier-Wiener process $B(t)$ [40], and thus the probability density of the process $X(t)$ approximates the density of $B(t)$ [Eq. (26)]. The analytic probability density for the Stratonovich integral S in the diffusive limit can be obtained inserting the probability density of the Bachelier-Wiener process into the transformation formula given by Eq. (52), yielding

$$p_S(s,t;D) = \frac{1}{\sqrt{2\pi Dts}}\exp\left(-\frac{s}{2Dt}\right), \quad s > 0. \qquad (54)$$

According to Eq. (40) here $I(t) = S(t) - [X](t)/2$; if the dependence of S and $[X]$ is small, the probability density of the Itō integral is approximated by the convolution of the probability density of the Stratonovich integral with that of the quadratic variation mirrored around zero and scaled to half its width,
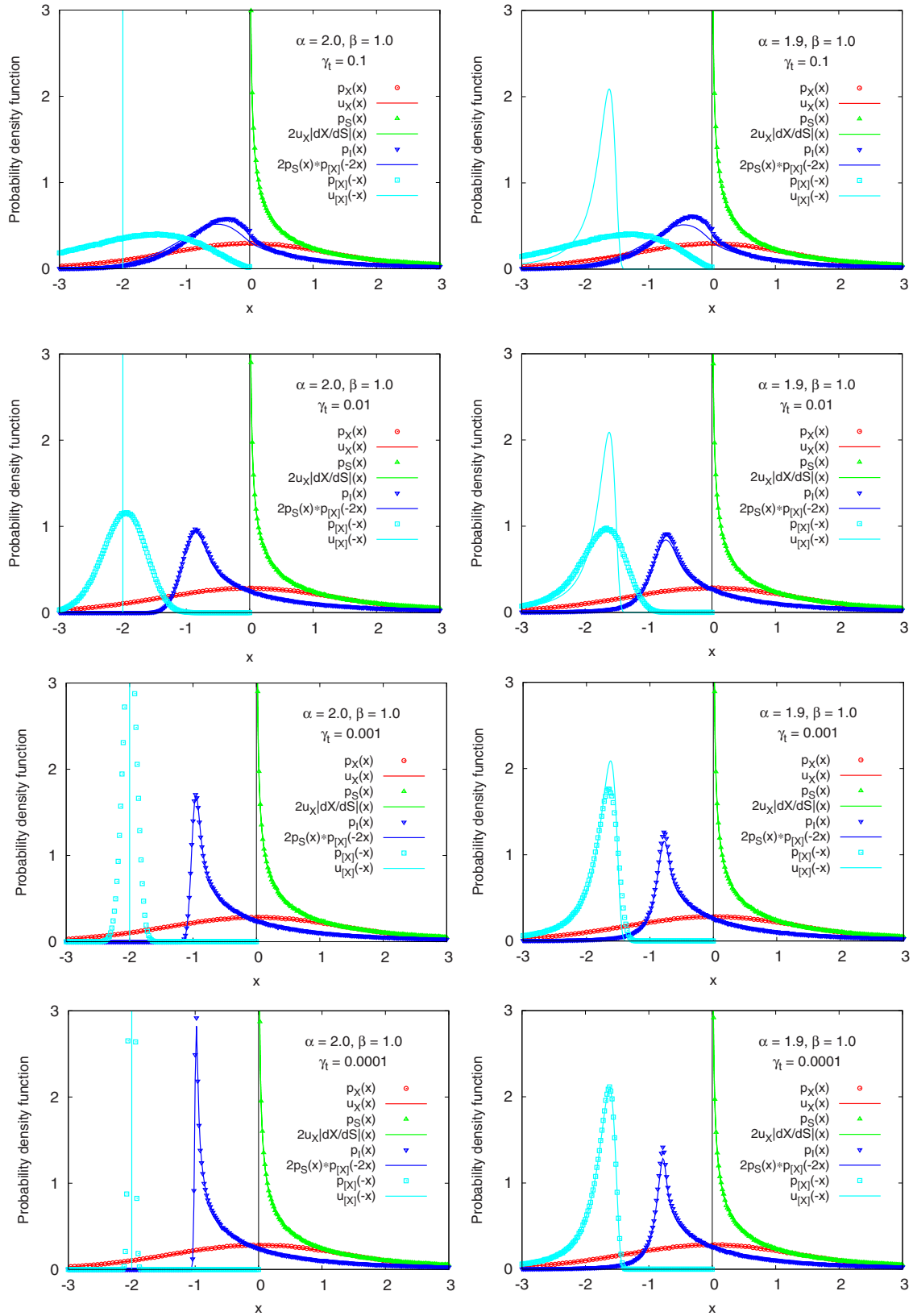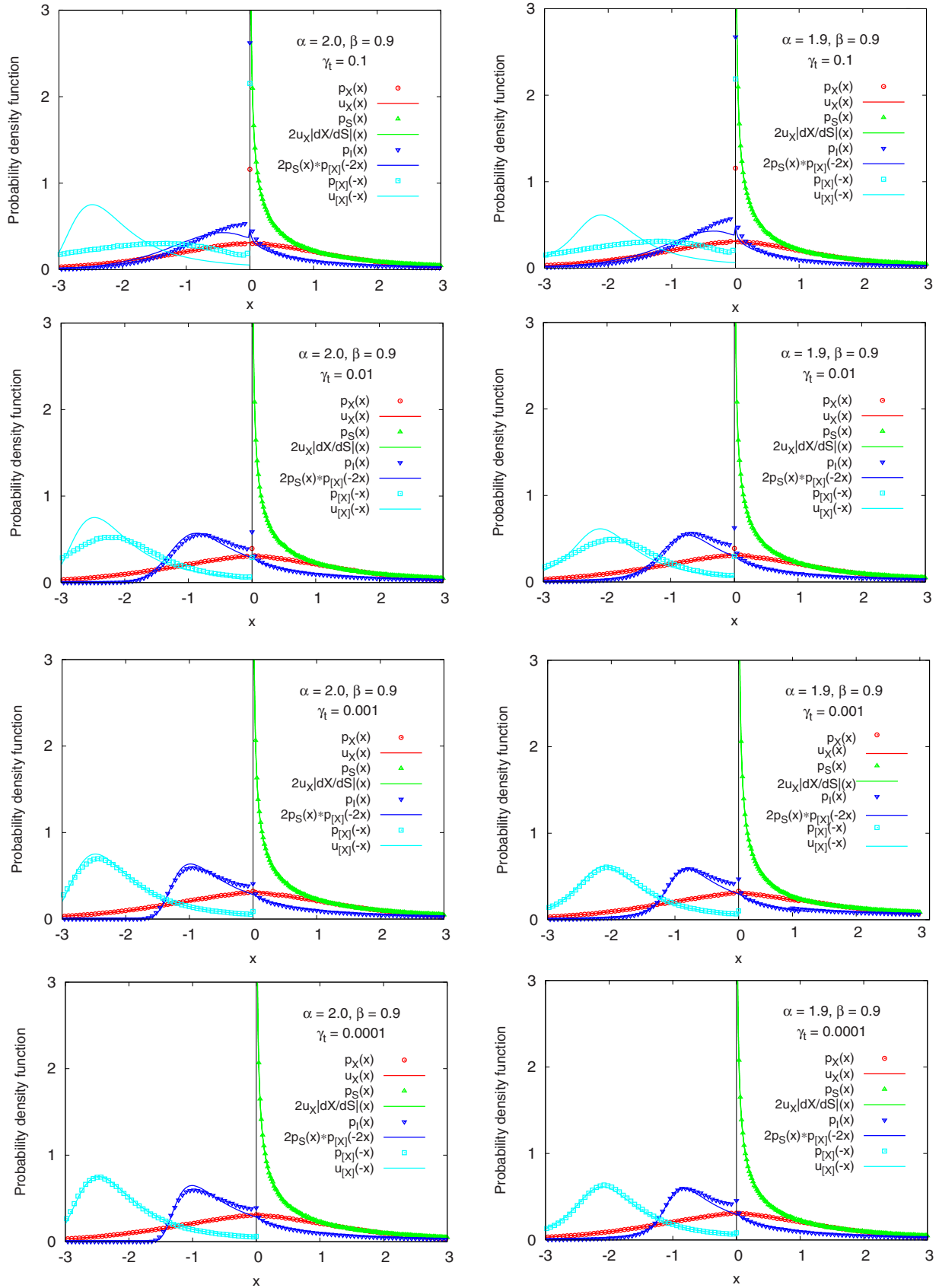
FIG. 2. (Color online) Convergence of the empirical probability densities p from 1×10^6 Monte Carlo runs (points) to the analytic probability densities u (lines) in the diffusive limit for a CTRW $X(t)$, its Stratonovich integral $S(t)$, its Itō integral $I(t)$, and its quadratic variation $[X](t)$, with $t = 1$ and different choices of the index parameters α, β and of the scale parameters γ_x, γ_t, where $\gamma_x^\alpha / \gamma_t^\beta = D = 1$.

FIG. 3. (Color online) Convergence of the empirical probability densities p from 1×10^6 Monte Carlo runs (points) to the analytic probability densities u (lines) in the diffusive limit for a CTRW $X(t)$, its Stratonovich integral $S(t)$, its Itō integral $I(t)$, and its quadratic variation $[X](t)$, with $t=1$ and different choices of the index parameters α, β and of the scale parameters γ_x, γ_t, where $\gamma_x^\alpha / \gamma_t^\beta = D = 1$.

$$p_I(x,t) \simeq 2 \int_{-\infty}^{+\infty} p_S(x + 2x',t)p_{[X]}(-2x',t)dx'. \quad (55)$$

For all choices of α and β the agreement between the analytic expressions for $X(t)$ and $S(t)$ in the diffusive limit and the empirical results from Monte Carlo simulation of the CTRWs is fair already for the largest value $\gamma_t = 0.1$: the curves cannot be distinguished by eyes at the scale of our plots. Therefore we did not evaluate the analytic probability density for $X(t)$ [Eq. (18)], available for the particular case of a NCPP only, i.e., the left column of Fig. 2. Instead the quadratic variation $[X](t)$ and consequently the Itō integral tend visibly more slowly to their diffusive limits. For a NCPP the diffusive limit of $[X](t)$ is $[B](t) = 2Dt$. In this limit $I(t) = S(t) - Dt = B^2(t)/2 - Dt$, corresponding to the well-known result that the probability density of the Itō integral is equal to the density of the Stratonovich integral shifted by $-Dt$, i.e., $p_I(x,t) = p_S(x + Dt,t)$. Though the quadratic variation of the NCPP is appreciably different from its limit $\delta(x - 2Dt)$ for any noninfinitesimal value of γ_t as shown in the left column of Fig. 2, where $Dt = 1$, for $\gamma_t = 0.01$ there is a good agreement between the Itō integrals from Monte Carlo and from Eq. (55).

The density of the quadratic variation for a CTRW can be obtained from the density of squared jumps, $\lambda_{\xi^2}(x)$, which results from a transformation of the density of jumps, $\lambda_\xi(\sqrt{x})$, similar to the one that leads from $p_X(x,t)$ to $p_S(x,t)$ [Eq. (53)], except for a factor 2,

$$\lambda_{\xi^2}(x;\gamma_x) = \lambda_\xi(\sqrt{x};\gamma_x)/\sqrt{x}. \quad (56)$$

Inserting this equation into the solution of the Montroll-Weiss equation [1] in the space-time domain, Eq. (16), gives

$$p_{[X]}(x,t;\gamma_x,\gamma_t) = \sum_{n=0}^{\infty} p_N(n,t;\gamma_t)\lambda_{\xi^2}^{*n}(x;\gamma_x), \quad (57)$$

where $x > 0$. Unfortunately even for an NCPP the n-fold convolution cannot be computed as easily as for $p_X(x,t)$ in Eq. (18). However, the characteristic function of the quadratic variation can be written as

$$\hat{p}_{[X]}(k,t;\gamma_x,\gamma_t) = \sum_{n=0}^{\infty} p_N(n,t;\gamma_t)\hat{\lambda}_{\xi^2}^n(k;\gamma_x). \quad (58)$$

In order to consider nonexponential waiting times with power-law tails and infinite first moment, for the sake of simplicity let us assume that $p_N(n,t)$ is the distribution of the Mittag-Leffler counting process [32],

$$p_N(n,t;\gamma_t) = \frac{(t/\gamma_t)^{\beta n}}{n!}E_\beta^{(n)}(-(t/\gamma_t)^\beta), \quad (59)$$

where

$$E_\beta^{(n)}(z) = \frac{d^n}{dz^n}E_\beta(z). \quad (60)$$

This choice is more general than it seems, as the Mittag-Leffler distribution for waiting times is an attractor for the thinning procedure used to obtain the diffusive limit [74].

Using the Mittag-Leffler distribution from the beginning simplifies the derivation of this limit. Then Eq. (58) becomes [10]

$$\hat{\lambda}_{[X]}(k,t;\gamma_x,\gamma_t) = E_\beta(-(t/\gamma_t)^\beta[1 - \hat{\lambda}_{\xi^2}(k;\gamma_x)]). \quad (61)$$

As the jumps ξ follow a Lévy α-stable distribution, for $x \rightarrow \infty$, $\lambda_{\xi^2}(x;\gamma_x) \sim (x/\gamma_x)^{-\alpha/2-1}$, and the sum of ξ_i^2 converges to the positive stable distribution with index $\alpha/2$, whose characteristic function is

$$\hat{\lambda}_{\xi^2}(k;\gamma_x) = \hat{L}_{\alpha/2}^+(k;\gamma_x) \equiv \exp((-i\gamma_x^2 k)^{\alpha/2}). \quad (62)$$

The scale parameter γ_x is the same as in the Lévy stable distribution [Eq. (20)]. Inserting this distribution in Eq. (61), the diffusive limit yields the following characteristic function for the quadratic variation:

$$\hat{u}_{[X]}(k,t;D) = E_\beta(-D(-ik)^{\alpha/2}t^\beta). \quad (63)$$

Now we can proceed in a similar fashion as for the solution of the FDE [Eqs. (29) and (30)]. Defining $\kappa = kt^{2\beta/\alpha}$ and

$$M_{\alpha,\beta}(\xi;D) = \mathcal{F}_\kappa^{-1}[E_\beta(-D(-i\kappa)^{\alpha/2})](\xi), \quad (64)$$

where $\xi > 0$, we obtain the quadratic variation for the diffusive limit in the space-time domain,

$$u_{[X]}(x,t;D) = t^{-2\beta/\alpha}M_{\alpha,\beta}(xt^{-2\beta/\alpha};D). \quad (65)$$

When $\alpha = 2$, $M_{2,\beta}(\xi)$ coincides with the right half of the Mainardi-Wright function [75] of real argument, which is also called M function of Wright type because its shape recalls a capital M centered in the origin. This function is defined in the complex plane for $0 < \beta < 1$ as

$$M_{2,\beta}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(-\beta n + 1 - \beta)} \quad (66)$$

and is a special case of the Wright function [76,77]

$$W_{\alpha',\beta'}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\alpha' n + \beta')}, \quad (67)$$

where $\alpha' > -1$, $\beta' \in \mathbb{R}$. When $\alpha = 2$ and $\beta = 1$ (standard diffusion case), a delta function $u_{[X]}(x,t;D) = \delta(x - 2Dt)$ is recovered, corresponding to the quadratic variation of the Bachelier-Wiener process, $[X](t) = 2Dt$. The plots in Figs. 2 and 3 display quadratic variations both from Monte Carlo and from Eq. (65). The convergence of the quadratic variation in the diffusive limit can be used to prove that the integrals of $X(t)$ as defined in Sec. II converge in the same limit.

IV. CONCLUSIONS AND OUTLOOK

This paper is based on the definition, given in Eq. (38), of a class of stochastic integrals $J_a(t)$ driven by a CTRW $X(t)$. For $a = 0$ this results in the Itō integral $I(t)$ [Eq. (39)] for $a = 1/2$ in the Stratonovich integral. If the process $X(t)$ that defines the measure used in Eq. (39) is a martingale with respect to its natural filtration, then $I(t)$ is a martingale too; this is a consequence of the martingale transform theorem. It

turns out that an uncoupled CTRW with zero-mean jumps is a martingale. The stochastic integration theory developed here is more general than the one sketched in Ref. [40], as it can be applied also to a CTRW that is neither Markovian nor Lévy. In fact, exponential waiting times are not needed to prove that $I(t)$ is a martingale if $X(t)$ is a martingale.

The theory presented in Sec. II lies at the foundation of the Monte Carlo method for integrating stochastic differential equations driven by CTRWs. As explained in Sec. I, these results are relevant for applications in physics and economics as well as in all those fields such as insurance and finance where martingale methods can help in the quantitative evaluation of risk. Equation (38) is a convenient basis for the Monte Carlo calculation of stochastic integrals. This is shown in Sec. III, where Monte Carlo realizations of CTRWs are used to effectively approximate the Itō and Stratonovich integrals driven by the Bachelier-Wiener process and, more generally, by the process whose one-point probability density function solves the space-time fractional diffusion equation.

In his paper on stochastic calculus in physics, Fox [78] reported the advice of Mark Kac not to irritate his friend George Uhlenbeck by even mentioning Doob or Itō: physicists do not need to be concerned with mathematical technicalities because they in no way affect the outcome of computations of physically measurable quantities. On the contrary, we believe that up to date mathematical methods from probability theory and stochastic calculus are beneficial to the study of the CTRW and of other random processes useful in statistical physics. We fear that progress will be slower or impossible if these methods are ignored by physicists.

Future work will deal with Monte Carlo simulations of coupled CTRWs where jumps and waiting times obey fat-tailed distributions [79,80]. There will also be a discussion of convergence based on the results collected in Ref. [81].

[1] E. Montroll and G. H. Weiss, J. Math. Phys. **6**, 167 (1965).

[2] H. Scher and M. Lax, Phys. Rev. B **7**, 4491 (1973).

[3] H. Scher and M. Lax, Phys. Rev. B **7**, 4502 (1973).

[4] E. W. Montroll and H. Scher, J. Stat. Phys. **9**, 101 (1973).

[5] H. Scher and E. Montroll, Phys. Rev. B **12**, 2455 (1975).

[6] M. F. Shlesinger, in *Encyclopedia of Applied Physics*, edited by G. L. Trigg (VCH, New York, 1996), Vol. 16, pp. 45–70.

[7] G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994).

[8] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).

[9] R. Metzler and J. Klafter, J. Phys. A **37**, R161 (2004).

[10] E. Scalas, Physica A **362**, 225 (2006).

[11] D. Fulger, E. Scalas, and G. Germano, Phys. Rev. E **77**, 021122 (2008).

[12] M. M. Meerschaert and H. P. Scheffler, *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice* (Wiley, New York, 2001).

[13] D. R. Cox, *Renewal Theory* (Methuen, London, 1967).

[14] W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1971), Vol. 2.

[15] D. R. Cox and V. Isham, *Point Processes* (Chapman & Hall, London, 1979).

[16] P. Billingsley, *Probability and Measure* (Wiley, New York, 1979).

[17] P. G. Hoel, S. C. Port, and J. Stone, *Introduction to Stochastic Processes* (Houghton Mifflin, Boston, 1972).

[18] E. Çinlar, *Introduction to Stochastic Processes* (Prentice-Hall, Englewood Cliffs, 1975).

[19] O. Flomenbom and J. Klafter, Phys. Rev. Lett. **95**, 098105 (2005).

[20] O. Flomenbom and R. J. Silbey, Phys. Rev. E **76**, 041101 (2007).

[21] J. Janssen and R. Manca, *Semi-Markov Risk Models for Finance, Insurance and Reliability* (Springer, New York, 2007).

[22] J. Bertoin, *Lévy Processes* (Cambridge University Press, Cambridge, UK, 1996).

[23] K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions* (Cambridge University Press, Cambridge, UK, 1999).

[24] M. F. Shlesinger, J. Stat. Phys. **10**, 421 (1974).

[25] J. K. E. Tunaley, J. Stat. Phys. **11**, 397 (1974).

[26] J. K. E. Tunaley, J. Stat. Phys. **12**, 1 (1975).

[27] J. K. E. Tunaley, J. Stat. Phys. **14**, 461 (1976).

[28] M. F. Shlesinger, J. Klafter, and Y. M. Wong, J. Stat. Phys. **27**, 499 (1982).

[29] D. ben-Avraham and S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems* (Cambridge University Press, Cambridge, UK, 2000).

[30] V. Balakrishnan, Physica A **132**, 569 (1985).

[31] R. Hilfer and L. Anton, Phys. Rev. E **51**, R848 (1995).

[32] E. Scalas, R. Gorenflo, and F. Mainardi, Phys. Rev. E **69**, 011107 (2004).

[33] D. del-Castillo-Negrete, B. A. Carreras, and V. E. Lynch, Phys. Rev. Lett. **94**, 065003 (2005).

[34] P. Demontis and G. B. Suffritti, Phys. Rev. E **74**, 051112 (2006).

[35] J. L. A. Dubbeldam, A. Milchev, V. G. Rostiashvili, and T. A. Vilgis, Phys. Rev. E **76**, 010801(R) (2007).

[36] J. L. A. Dubbeldam, A. Milchev, V. G. Rostiashvili, and T. A. Vilgis, Europhys. Lett. **79**, 18002 (2007).

[37] J. Masoliver, M. Montero, J. Perelló, and G. H. Weiss, J. Econ. Behav. Organ. **61**, 577 (2006).

[38] P. Embrechts, C. Klüppelberg, and T. Mikosch, *Modelling Extremal Events for Insurance and Finance* (Springer, New York, 1997).

[39] A. Cartea and D. del-Castillo-Negrete, Phys. Rev. E **76**, 041105 (2007).

[40] R. Zygadło, Phys. Rev. E **68**, 046117 (2003).

[41] T. Szabados and B. Sźekely, J. Theor. Probab. **22**, 203 (2009).

[42] R. C. Merton, J. Financ. Econ. **3**, 125 (1976).

[43] R. Gorenflo, J. Loutchko, and Yu. Luchko, Fractional Calculus Appl. Anal. **5**, 491 (2002).

[44] I. Polubny and M. Kacenak, MLF.M: Mittag-Leffler function—Calculates the Mittag-Leffler function with desired accuracy, MATLAB Central File Exchange, File ID No. 8738, 2005 (www.mathworks.com/matlabcentral/fileexchange).

[45] R. Hilfer and H. J. Seybold, Integral Transforms Spec. Funct. **17**, 637 (2006).

[46] S. G. Samko, A. A. Kilbas, and O. Marichev, *Fractional Integrals and Derivatives, Theory and Applications* (Gordon and Breach, London, 1993).

[47] I. Podlubny, *Fractional Differential Equations* (Academic, San Diego, 1999).

[48] R. Metzler, J. Klafter, and I. M. Sokolov, Phys. Rev. E **58**, 1621 (1998).

[49] R. Metzler, E. Barkai, and J. Klafter, Phys. Rev. Lett. **82**, 3563 (1999).

[50] R. Metzler, E. Barkai, and J. Klafter, Europhys. Lett. **46**, 431 (1999).

[51] E. Barkai, R. Metzler, and J. Klafter, Phys. Rev. E **61**, 132 (2000).

[52] M. Magdziarz, A. Weron, and K. Weron, Phys. Rev. E **75**, 016708 (2007).

[53] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).

[54] H. Risken, *The Fokker-Planck Equation. Methods of Solution and Applications*, 2nd ed. (Springer, Berlin, 1992).

[55] F. Mainardi, M. Raberto, R. Gorenflo, and E. Scalas, Physica A **287**, 468 (2000).

[56] A. Mura, M. S. Taqqu, and F. Mainardi, Physica A **387**, 5033 (2008).

[57] R. Bracewell, *The Fourier Transform and Its Applications*, 3rd ed. (McGraw-Hill, New York, 2000).

[58] I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964).

[59] P. Protter, *Stochastic Integration and Differential Equations*, 2nd ed. (Springer, Berlin, 2004).

[60] W. Paul and J. Baschnagel, *Stochastic Processes—From Physics to Finance* (Springer, Berlin, 2000).

[61] C. W. Gardiner, *Handbook of Stochastic Methods*, 2nd ed. (Springer, Berlin, 1996).

[62] D. Williams, *Probability with Martingales* (Cambridge University Press, Cambridge, UK, 1991).

[63] R. L. Schilling, *Measures, Integrals and Martingales* (Cambridge University Press, Cambridge, UK, 2005).

[64] L. Devroye, *Non-Uniform Random Variate Generation* (Springer, New York, 1986).

[65] L. Devroye, in *Proceedings of the 1996 Winter Simulation Conference*, edited by J. M. Charnes, D. J. Morrice, D. T. Brunner, and J. J. Swain (IEEE, New York, 1996), pp. 265–272.

[66] J. M. Chambers, C. L. Mallows, and B. W. Stuck, J. Am. Stat. Assoc. **71**, 340 (1976).

[67] J. H. McCulloch, STABRND.M: Stable random number generator, MATLAB script, 1996 (www.econ.ohio-state.edu/jhm/jhm.html).

[68] G. E. P. Box and M. E. Muller, Ann. Math. Stat. **29**, 610 (1958).

[69] T. J. Kozubowski and S. T. Rachev, J. Comput. Anal. Appl. **1**, 177 (1999).

[70] G. Germano, D. Fulger, and E. Scalas, MLRND.M: Mittag-Leffler pseudorandom number generator, MATLAB Central File Exchange, File ID No. 19392, 2008.

[71] G. Germano, D. Fulger, M. Politi, and E. Scalas, CTRW.M: Continuous-time random walk with Mittag-Leffler distributed waiting times and Levy alpha-stable distributed jumps, forthcoming on the MATLAB Central File Exchange.

[72] G. Germano, M. Politi, and E. Scalas, FDESOLVER.M: Solution of the space-time-fractional diffusion equation, forthcoming on the MATLAB Central File Exchange.

[73] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes—The Art of Scientific Computing*, 3rd ed. (Cambridge University Press, Cambridge, UK, 2007).

[74] F. Mainardi, R. Gorenflo, and E. Scalas, Vietnam J. Math. **32**, 53 (2004).

[75] F. Mainardi, Yu. Luchko, and G. Pagnini, Fractional Calculus Appl. Anal. **4**, 153 (2001).

[76] A. Erdély, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 3.

[77] Yu. Luchko, Fractional Calculus Appl. Anal. **11**, 57 (2008).

[78] R. F. Fox, J. Stat. Phys. **46**, 1145 (1987).

[79] M. M. Meerschaert, D. A. Benson, H.-P. Scheffler, and P. Becker-Kern, Phys. Rev. E **66**, 060102(R) (2002).

[80] M. M. Meerschaert and E. Scalas, Physica A **370**, 114 (2006).

[81] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes (Grundlehren der mathematischen Wissenschaften, Vol. 288)*, 2nd ed. (Springer, Berlin, 2003).