Engineering integrable nonautonomous nonlinear Schrödinger equations

Xu-Gang He,[1,2] Dun Zhao,[1,3,*] Lin Li,[4] and Hong-Gang Luo[3,5,6]
[1]*School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China*
[2]*Department of Physics, University of Connecticut, Storrs, Connecticut 06269, USA*
[3]*Center for Interdisciplinary Studies, Lanzhou University, Lanzhou 730000, China*
[4]*Department of Modern Physics, Lanzhou University, Lanzhou 73000, China*
[5]*Key Laboratory for Magnetism and Magnetic Materials of the Ministry of Education, Lanzhou University, Lanzhou 730000, China*
[6]*Key Laboratory of Frontiers in Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100080, China*
(Received 3 March 2009; revised manuscript received 1 May 2009; published 28 May 2009)

We investigate Painlevé integrability of a generalized nonautonomous one-dimensional nonlinear Schrödinger (NLS) equation with time- and space-dependent dispersion, nonlinearity, and external potentials. Through the Painlevé analysis some explicit requirements on the dispersion, nonlinearity, dissipation/gain, and the external potential as well as the constraint conditions are identified. It provides an explicit way to engineer integrable nonautonomous NLS equations at least in the sense of Painlevé integrability. Furthermore analytical solutions of this class of integrable nonautonomous NLS equations can be obtained explicitly from the solutions of the standard NLS equation by a general transformation. The result provides a significant way to control coherently the soliton dynamics in the corresponding nonlinear systems, as that in Bose-Einstein condensate experiments. We analyze explicitly the soliton dynamics under the nonlinearity management and the external potentials and discuss its application in the matter-wave dynamics. Some comparisons with the previous works have also been discussed.

DOI: 10.1103/PhysRevE.79.056610 PACS number(s): 05.45.Yv, 03.75.Lm, 03.75.Kk

I. INTRODUCTION

The soliton is a ubiquitous phenomenon in nature. Classically the soliton originates from the subtle balance between the dispersion and nonlinearity and has a particlelike nature which remains intact even after mutual collisions. The soliton dynamics plays an important role in the study of the elementary excitations in related systems. In recent years the dynamics of the matter-wave soliton has been intensively studied in Bose-Einstein condensates (BECs) [1–3]. With the development of modern technology the concept of soliton management [4] has been proposed. Essentially soliton management is control of the soliton dynamics by tuning the related control parameters. For example, we can consider the dispersion in nonlinear optics [5–9] and nonlinearity in BEC by the technique of Feshbach resonance [10]. In the one-dimensional case, such a soliton management can be described by a generalized nonautonomous nonlinear Schrödinger (NLS) equation,

$$i\frac{\partial u(x,t)}{\partial t} + f(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + g(x,t)|u(x,t)|^2 u(x,t) + V(x,t)u(x,t) + i\gamma(x,t)u(x,t) = 0. \quad (1)$$

Here $f(x,t)$ and $g(x,t)$ are the dispersion and nonlinearity management parameters, respectively. $V(x,t)$ denotes the external potential applied and $\gamma(x,t)$ is the dissipation ($\gamma>0$) or gain ($\gamma<0$). In the BEC context Eq. (1) is also known as Gross-Pitaevskii equation [11]. The general form of Eq. (1) includes many special cases discussed in the literature and its analytical solitonlike solution was recently named as nonautonomous soliton [12], which is quite different from the conventional canonical soliton concept [13].

As the first extension of the canonical soliton concept in 1976, Chen and Liu [14] found that the soliton can be accelerated in a linearly inhomogeneous plasma. At the same time the analytical soliton solutions for the Korteweg–de Vries equation with varying nonlinearity and dispersion were also found by Calogero and Degasperis [15]. In 1993 Konotop *et al.* [16,17] discussed the soliton dynamics of the discrete NLS equation with varying coefficients. Recently the generalized NLS equation with varying dispersion, nonlinearity, and dissipation or gain has been extensively investigated in the literature [12,18–23] and some useful techniques have been explored. On one hand, based on the generalized inverse scattering transformation [14], the Lax pair analysis was used in discussing the integrability condition of the systems under study [12,18,19,23]. Another available method used widely in the literature is the similarity transformation [20,24,25], in which transformation parameters can be determined by a set of differential equations. In some special cases the set of differential equations can be solved analytically and, as a result, analytical solutions of the systems under study were obtained. However, it was pointed out in Ref. [24] that in general cases such a set of differential equations is difficult to solve analytically. Therefore a general way to obtain analytical solutions of the nonautonomous nonlinear equation is still lacking. By some explicit ways mentioned above the analytical soliton solutions of the nonautonomous NLS equation have been obtained, and they are apparently different from the canonical solitons in many aspects because both amplitudes and speeds of the solitons vary with time and space. However, under some integrability conditions [12,18,19,21], these solitonlike solutions maintain the basic properties of the canonical solitons.

*zhaod@lzu.edu.cn

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From the angle of soliton applications it is desirable to keep the basic properties of the solitons when some parameters are controllable. For example, in optical soliton transmission the basic shape of the input optical soliton should be maintained in the transmission process in order to decrease the error rate. Thus a question arises: how does one manage these parameters to make nonautonomous NLS equation (1) integrable? Then as a result, the corresponding solitonlike solutions keep the basic properties of the canonical solitons. To answer this question we employ the Painlevé analysis, a quite powerful tool to explore integrability of partial differential equations. The central results of the present paper are that (i) starting from nonautonomous NLS equation (1), the explicit requirements on the constraint relations about parameters $f(x,t)$, $g(x,t)$, $V(x,t)$, and $\gamma(x,t)$ (i.e., the integrability conditions) can be obtained by the Painlevé analysis, which provides a significant way to engineer integrable nonautonomous NLS systems; and (ii) under these conditions the analytical solutions of these integrable nonautonomous NLS equation can be obtained from the solutions of the standard NLS equation. We provide a direct way to control the soliton dynamics by tuning the control parameters.

This paper is organized as follows. In Sec. II we perform the Painlevé analysis of Eq. (1) to obtain the constraints on $f(x,t)$, $g(x,t)$, $V(x,t)$, and $\gamma(x,t)$ or, in other words, the compatibility condition under which Eq. (1) can pass the Painlevé test. In Sec. III we show a transformation which can convert Eq. (1) into the standard NLS equation under the compatibility condition. In Sec. IV we give some examples which show the way to obtain the solutions of Eq. (1) from the solutions of the standard NLS equation by a set of systematic transformations. Finally we give a brief summary in Sec. V.

II. PAINLEVÉ ANALYSIS AND COMPATIBILITY CONDITIONS

Our analysis is based on the Painlevé test for partial differential equations, i.e., the well-known Weiss-Tabor-Carnevale (WTC) test, which has been proven to be a very effective tool to investigate the integrability of partial differential equations. The remarkable connection between complete integrability and the Painlevé property was first observed by Ablowitz and Segur [26]. They found that similarity reductions of nonlinear partial differential equations solvable by an inverse scattering transform give rise to nonlinear ordinary differential equations; the only movable singularities of their solutions are poles. In 1980 Ablowitz *et al.* [27] developed an algorithm, based on the work of Kowalevski [28], for giving necessary conditions for an ordinary differential equation to have the Painlevé property. The algorithm was later extended by Weiss *et al.* [29] to be applicable directly to partial differential equations. This algorithm is now called WTC test. For details of the WTC test and its applications, one can refer to Refs. [29–31]. Although some special cases of Eq. (1) have been discussed through Painlevé analysis by many authors, it seems that a thorough analysis of Eq. (1) is still lacking. In this section we follow the idea in [31] to obtain a condition which guarantees that Eq. (1) pass the WTC test. Then under this condition in Sec. III we look for a transformation which converts Eq. (1) to the standard cubic nonlinear Schrödinger equation.

In order to perform conveniently the Painlevé analysis we first complexify Eq. (1), which becomes

$$i\frac{\partial u(x,t)}{\partial t} + f(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + g(x,t)v(x,t)u(x,t)^2 + V(x,t)u(x,t) + i\gamma(x,t)u(x,t) = 0, \tag{2}$$

$$-i\frac{\partial v(x,t)}{\partial t} + f(x,t)\frac{\partial^2 v(x,t)}{\partial x^2} + g(x,t)u(x,t)v(x,t)^2 + V(x,t)v(x,t) - i\gamma(x,t)u(x,t) = 0, \tag{3}$$

where $u(x,t)$ and $v(x,t)$ are treated as independent complex functions of variables x and t, and the functions $f(x,t)$, $g(x,t)$, $V(x,t)$, and $\gamma(x,t)$ are analytic on the noncharacteristic singularity manifolds $\varphi(x,t)=0$. Due to the ansatz of Kruskal [32] noncharacteristic singularity manifold can take the following form:

$$\varphi(x,t) = x + \phi(t). \tag{4}$$

Then the solutions of Eqs. (2) and (3) can be expanded on the noncharacteristic singularity manifold as

$$u(x,t) = [x+\phi(t)]^{-p}\sum_{j=0}^{\infty} u_j(t)[x+\phi(t)]^j, \tag{5}$$

$$v(x,t) = [x+\phi(t)]^{-q}\sum_{j=0}^{\infty} v_j(t)[x+\phi(t)]^j, \tag{6}$$

where $u_0 \neq 0$, $v_0 \neq 0$.

Furthermore we also expand $f(x,t)$, $g(x,t)$, $V(x,t)$, and $\gamma(x,t)$ on the same singularity manifold as follows:

$$f(x,t) = \sum_{i=0}^{\infty} f_i(t)[x+\phi(t)]^i,$$

$$g(x,t) = \sum_{i=0}^{\infty} g_i(t)[x+\phi(t)]^i,$$

$$V(x,t) = \sum_{i=0}^{\infty} V_i(t)[x+\phi(t)]^i,$$

$$\gamma(x,t) = \sum_{i=0}^{\infty} \gamma_i(t)[x+\phi(t)]^i, \tag{7}$$

where

$$f_i(t) = \frac{1}{i!}\left.\frac{\partial^i f(x,t)}{\partial x^i}\right|_{x=-\phi(t)},$$

and similarly for $g(x,t)$, $V(x,t)$, and $\gamma(x,t)$.

Substituting expressions (5)–(7) into Eqs. (2) and (3) and collecting the same powers of $\phi(t)$, one can obtain (i) the values of p and q, and the equations about the first terms derived from the leading-order analysis; and (ii) the general recursion relations of u_j and v_j $(j\geq 1)$ as shown below.

A. Leading-order terms and recursion relation

Substituting Eqs. (5)–(7) into Eqs. (2) and (3) by the standard procedure for leading-order analysis, we get $p=q=1$ and

$$2f_0(t)+g_0(t)u_0(t)v_0(t)=0. \tag{8}$$

The recursion relations are

$$A(j)\begin{pmatrix}u_j\\ v_j\end{pmatrix}\equiv\begin{pmatrix}Q_j & g_0u_0^2\\ g_0v_0^2 & Q_j\end{pmatrix}\begin{pmatrix}u_j\\ v_j\end{pmatrix}=\begin{pmatrix}F_j\\ G_j\end{pmatrix}, \tag{9}$$

where

$$Q_j=(j-1)(j-2)f_0+2g_0u_0v_0, \tag{10}$$

$$\begin{aligned}F_j=&-i[u_{j-2,t}+(j-2)u_{j-1}\phi_t]-g_ju_0^2v_0-\sum_{k=1}^{j}(j-k-1)\\&\times(j-k-2)f_ku_{j-k}-g_0v_0\sum_{m=1}^{j-1}u_{j-m}u_m\\&-g_0\sum_{m=1}^{j-1}\sum_{k=0}^{m}v_{j-m}u_{m-k}u_k-\sum_{m=1}^{j-1}\sum_{k=0}^{m}\sum_{l=0}^{k}g_{j-m}v_{m-k}u_{k-l}u_l\\&-\sum_{m=0}^{j-2}V_{j-m-2}u_m-i\sum_{m=0}^{j-2}\gamma_{j-m-2}u_m.\end{aligned} \tag{11}$$

$i=\sqrt{-1}$ and G_j has a similar expression which can be obtained from F_j by first interchanging u_j and v_j and then taking its complex conjugate. In addition we use the notation that once an index is less than zero, the expression itself is zero. Hereafter the subscript "t" denotes time derivative of the related functions.

B. Compatibility conditions

The recursion relations above determine the unknown expansion coefficients uniquely unless the determinant of the matrix in Eq. (9) is zero. Those values of j at which the determinant is equal to zero are called the resonances, and the conditions which ensure Eq. (9) to have solutions at the resonances are named compatibility conditions. From Eqs. (9)–(11) it is found that resonances only occur at

$$j=-1,0,3,4.$$

The resonance of $j=-1$ corresponds to the arbitrariness of the singular manifold $\phi(x,t)$. For $j=0$ the recursion relation holds automatically. It is just relation (8).

From recursion relations (9) with condition (8) the compatibility conditions for the remaining resonances are

$$j=3:\ v_0(t)F_3-u_0(t)G_3=0, \tag{12}$$

$$j=4:\ v_0(t)F_4+u_0(t)G_4=0. \tag{13}$$

To show explicitly compatibility conditions (12) and (13), u_j and v_j $(j=1,2)$ should be calculated by using the recursion relation (9), from which u_1, v_1, u_2, and v_2 can be uniquely determined by u_0, v_0, f_0, g_0, f_1, g_1, f_2, g_2, γ_0, ϕ, and their derivatives. When one substitutes u_1, v_1, u_2, v_2, and Eq. (8) into Eq. (12), it is found that Eq. (12) is equivalent to

$$\begin{aligned}&(g_0f_0f_1+4f_0^2g_1)g_{0,t}-(2g_0g_1f_0+3g_0^2f_1)f_{0,t}+2g_0^2f_0f_{1,t}\\&-2f_0^2g_0g_{1,t}-2(2f_0\gamma_0g_1+g_0\gamma_0f_1-3f_0g_0\gamma_1)g_0f_0\\&+(6f_2f_0g_0^2-2g_0g_1f_0f_1-4f_1^2g_0)\phi_t(t)=0.\end{aligned} \tag{14}$$

By the arbitrariness of $\phi(t)$, one has

$$\begin{aligned}&(f_0f_1g_0+4f_0^2g_1)g_{0,t}-(2f_0g_0g_1+3f_1g_0^2)f_{0,t}+2f_0f_{1,t}g_0^2\\&-2f_0^2g_0g_{1,t}-2(2f_0g_1\gamma_0+g_0f_1\gamma_0-3f_0g_0\gamma_1)g_0f_0=0,\end{aligned} \tag{15}$$

$$6f_2f_0g_0^2-2g_0g_1f_0f_1-4f_1^2g_0^2=0. \tag{16}$$

Under this condition at $j=3$ we get from Eq. (12) that

$$u_3=\frac{F_3-g_0u_0^2v_0}{g_0u_0v_0},$$

where v_3 is arbitrary.

Repeating the above process the compatibility condition at resonance $j=4$ is equivalent to

$$H(\cdot)=0, \tag{17}$$

where $H(\cdot)$ is determined by u_0, v_0, f_0, g_0, f_1, g_1, f_2, g_2, f_3, g_3, f_4, g_4, γ_0, γ_1, ϕ, and their derivatives. Again by the arbitrariness of ϕ and u_0, Eq. (17) is equivalent to a set of algebraic and differential equations from which we conclude that

$$g_1(t)=f_1(t)=0. \tag{18}$$

According to the definition of $f_i(t)$ we have

$$f_1(t)=\left.\frac{\partial f(x,t)}{\partial x}\right|_{x=-\phi(t)}=0, \tag{19}$$

but $\phi(t)$ is arbitrary and Eq. (19) just means that $f(x,t)$ is independent of x. So we obtain

$$f_1(t)=f_2(t)=\cdots=0$$

and

$$f(x,t)=f_0(t)=f(t). \tag{20}$$

Similarly we have

$$g(x,t)=g_0(t)=g(t),$$

$$\gamma(x,t)=\gamma_0(t)=\gamma(t). \tag{21}$$

With Eqs. (20) and (21) it is found that the compatibility condition for $j=3$ holds automatically and the compatibility condition for $j=4$ is equivalent to

$$\begin{aligned}&(4f^2gg_t-2ff_tg^2)\gamma-4f^2g^2\gamma^2-2f^2g^2\gamma_t-g^2ff_{tt}+f^2gg_{tt}\\&-2f^2g_t^2+f_t^2g^2+f_tgfg_t+4V_2f^3g^2=0,\end{aligned} \tag{22}$$

which imposes a constraint condition on $f(t)$, $g(t)$, $\gamma(t)$, and $V_2(t)$. Since

$$V_i(t) = \frac{1}{i!} \left. \frac{\partial^i V(x,t)}{\partial x^i} \right|_{x=-\phi(t)},$$

condition (22) also implies that

$$\frac{\partial^3}{\partial x^3} V(x,t) = 0.$$

Therefore we conclude that the conditions due to the Painlevé analysis for Eq. (1) are

$$f(x,t) = f(t), \quad g(x,t) = g(t), \quad \gamma(x,t) = \gamma(t),$$

$$V(x,t) = V_0(t) + V_1(t)x + V_2(t)x^2, \tag{23}$$

where $V_0(t)$ and $V_1(t)$ are arbitrary, and $f(t)$, $g(t)$, $\gamma(t)$, and $V_2(t)$ satisfy relation (22).

Condition (23) suggests that the Painlevé integrable class of Eq. (1) should have the form of

$$i\frac{\partial u(x,t)}{\partial t} + f(t)\frac{\partial^2 u(x,t)}{\partial x^2} + g(t)|u(x,t)|^2 u(x,t) + [V_0(t) + V_1(t)x + V_2(t)x^2]u(x,t) + i\gamma(t)u(x,t) = 0, \tag{24}$$

where $f(t)$, $g(t)$, $\gamma(t)$, and $V_2(t)$ satisfy relation (22) and $V_0(t)$ and $V_1(t)$ are arbitrary. In this case it is easy to check that by a transformation

$$u(x,t) = q(x,t)\exp\left[-\int \gamma(t)dt\right],$$

the term of loss/gain in Eq. (24) can be eliminated formally and so in this paper we only consider the model without loss/gain [$\gamma(t) \equiv 0$ in Eq. (24)], i.e.,

$$i\frac{\partial u(x,t)}{\partial t} + f(t)\frac{\partial^2 u(x,t)}{\partial x^2} + g(t)|u(x,t)|^2 u(x,t) + [V_0(t) + V_1(t)x + V_2(t)x^2]u(x,t) = 0. \tag{25}$$

According to Eq. (22) its constraint condition on $f(t)$, $g(t)$, and $V_2(t)$ is

$$-g^2 f f_{tt} + f^2 g g_{tt} - 2f^2 g_t^2 + g^2 f_t^2 + g f g_t f_t + 4V_2 f^3 g^2 = 0. \tag{26}$$

One notes that Eq. (25) with $V_0(t)=0$ has been studied by Lax pair method [12] and the integrability condition obtained is essentially consistent with Eqs. (23) and (26). Actually $V_0(t)$ can also be removed by means of a time-dependent phase in the wave function as shown in Eq. (45), but its presence is necessary to clarify the roles played by $V_0(t)$ in the dynamics of soliton in comparison with those played by $V_1(t)$ and $V_2(t)$ terms.

We point out that relation (26) is invariant if we replace $g(t)$ with $-g(t)$. Setting

$$\frac{g_t(t)}{g(t)} - \frac{f_t(t)}{f(t)} = \theta(t),$$

Eq. (26) can be rewritten as the following Riccati equation of $\theta(t)$:

$$\theta_t - \theta^2 - \frac{f_t}{f}\theta + 4fV_2 = 0.$$

Thus all functions satisfying Eq. (26) can be presented as

$$g(t) = f(t)\exp\left[\int \theta(t)dt\right], \quad V_2(t) = -\frac{f\theta_t - f\theta^2 - f_t\theta}{4f^2},$$

with $\theta(t)$ and $f(t)$ given arbitrarily.

C. Some explicit integrable models

Below we discuss some explicit models of Eq. (25) that satisfy compatibility conditions (23) and (26) and compare them with some previous works.

(A) If $V_2(t)=0$, Eq. (26) becomes

$$g(t) = \frac{f(t)}{C_1\int f(t)dt + C_2}, \tag{27}$$

where C_1 and C_2 are constants such that the denominator in the above expression is not equal to zero at any time. This is a physically interesting case in that the soliton can be coherently compressed by the technique of Feshbach resonance. We discuss it in detail below. Note that if $V_0(t)=V_1(t)=0$, $C_1=0$, $C_2=1$, and $f(t)=\pm\frac{1}{2}$, then Eq. (25) recovers the standard NLS equation.

(B) If $f(t)=1$, Eq. (26) becomes

$$g(t)g_{t,t}(t) - 2g_t^2(t) + 4V_2(t)g^2(t) = 0. \tag{28}$$

(1) We take $V_2(t)=\lambda^2/4$, where λ is a constant. In this case $g(t)$ has the following form:

$$g(t) = \frac{2\lambda e^{\lambda t}}{C_1 e^{2\lambda t} + C_2}, \tag{29}$$

where C_1 and C_2 are constants such that the denominator in Eq. (29) is not equal to zero at any time. This condition was obtained in [33] through the Lax pair technique. In particular, the model with $C_1=2\lambda$ ($\lambda>0$) and $C_2=0$ was studied in [23].

(2) Let $V_2(t)=[1-\tanh(\lambda t)]\lambda^2/2$, where λ is a constant. From Eq. (26) one has

$$g(t) = \frac{\lambda[1+\tanh(\lambda t)]}{C_1 + C_2\{\tanh(\lambda t) + 2\ln[1-\tanh(\lambda t)]\}}, \tag{30}$$

where C_1 and C_2 are constants. If $C_1=\lambda$ and $C_2=0$, then Eq. (30) becomes

$$g(t) = 1 + \tanh(\lambda t), \tag{31}$$

which is a model studied in [34].

(C) If $g(t)=\pm 1$, Eq. (26) becomes

$$f_t(t)^2 - f(t)f_{t,t}(t) + 4f(t)^3 V_2(t) = 0. \tag{32}$$

(1) We take $V_2(t)=\pm\lambda^2/4$, where λ is a constant. In this case one has

$$f(t) = \mp \frac{1}{2\lambda^2 C_1^2}\mathrm{sech}^2\left(\frac{t+C_2}{2C_1}\right),$$

where C_1 and C_2 are constants and $C_1 \neq 0$. Setting $C_1 = 1/2\lambda$ and $C_2 = 0$ we get

$$f(t) = \mp 2\ \mathrm{sech}^2(\lambda t). \tag{33}$$

(2) We take $f(t) = \mathrm{sech}(\lambda t)$, where λ is a constant. One can solve for $V_2(t)$ that

$$V_2(t) = -\frac{\lambda^2}{4}\mathrm{sech}(\lambda t). \tag{34}$$

The list can be continued if one wishes. This list also shows sufficiently how to engineer integrable nonautonomous NLS systems (1) by making use of condition (26); some of them have been studied in the literature. In the following we explore systematically the analytical solutions of these integrable nonautonomous NLS equations.

III. TRANSFORMATION TO THE STANDARD NLS EQUATION

In this section we look for a transformation which can convert Eq. (25) into the standard NLS equation

$$i\frac{\partial}{\partial T}Q(X,T) + \varepsilon\frac{\partial^2}{\partial X^2}Q(X,T) + \delta|Q(X,T)|^2 Q(X,T) = 0, \tag{35}$$

where ε and δ are real constants. It is well known that standard NLS equation (35) is completely integrable and it has been thoroughly discussed in the literature. If $\delta\varepsilon > 0$, Eq. (35) is called self-focusing and it has bright soliton solutions. If $\delta\varepsilon < 0$, Eq. (35) is called self-defocusing and it has dark soliton solutions.

We look for a transformation in the form [35]

$$u(x,t) = Q(p(x,t),q(t))e^{ia(x,t)+c(t)}, \tag{36}$$

where $p(x,t)$, $q(t)$, $a(x,t)$, and $c(t)$ are real functions to be determined and $u(x,t)$ and $Q(X,T)$ are the solutions of Eqs. (25) and (35), respectively.

Substituting Eq. (36) into Eq. (25) and letting $p(x,t) = X$ and $q(t) = T$, we obtain

$$\begin{aligned} iQ_T q_t + fp_x^2 Q_{X,X} + ge^{2c}|Q|^2 Q + i[Q(c_t + fa_{x,x}) \\ + Q_X(p_t + 2fp_x a_x)] + fp_{x,x}Q_X - (a_t + fa_x^2 \\ - V_0 - V_1 x - V_2 x^2)Q = 0. \end{aligned} \tag{37}$$

Comparing this with Eq. (35) one has

$$\begin{aligned} c_t + fa_{x,x} = 0, \\ p_t + 2fp_x a_x = 0, \\ a_t + fa_x^2 - V_0 - V_1 x - V_2 x^2 = 0, \\ p_{x,x} = 0. \end{aligned} \tag{38}$$

From the first two and the fourth equations above we get

$$a(x,t) = -\frac{c_t(t)}{2f(t)}x^2 + h_1(t)x + h_2(t),$$

$$p(x,t) = xe^{2c(t)} - 2\int f(t)h_1(t)e^{2c(t)}dt, \tag{39}$$

where $h_1(t)$ and $h_2(t)$ are functions to be determined. Inserting $a(x,t)$ and $p(x,t)$ into the third equation in Eq. (38), collecting the coefficients of all powers of x, and further setting them as zero, we get

$$\begin{aligned} -c_{t,t}f + c_t f_t + 2fc_t^2 - 2V_2 f^2 = 0, \\ -2c_t h_1 + h_{1,t} - V_1 = 0, \\ fh_1^2 + h_{2,t} - V_0 = 0. \end{aligned} \tag{40}$$

From the last two equations in this system $h_1(t)$ and $h_2(t)$ can be solved:

$$h_1(t) = h(t)e^{2c(t)}, \tag{41}$$

$$h_2(t) = \int [V_0(t) - f(t)h(t)^2 e^{4c(t)}]dt + C_2, \tag{42}$$

where $h(t) = \int V_1(t)e^{-2c(t)}dt + C_1$. Here C_1 and C_2 are constants.

Finally in comparison to the standard NLS equation one has

$$\frac{ge^{2c}}{q_t} = \delta, \quad \frac{fe^{4c}}{q_t} = \varepsilon. \tag{43}$$

The above equations give

$$c(t) = \frac{1}{2}\ln\frac{\varepsilon g(t)}{\delta f(t)}, \quad q(t) = \frac{\varepsilon}{\delta^2}\int\frac{g^2(t)}{f(t)}dt + C_3, \tag{44}$$

where C_3 is a constant. Usually we choose C_3 such that $q(0) = 0$.

Then it is easy to determine the remaining transformation parameters $a(x,t)$ and $p(x,t)$ as

$$\begin{aligned} a(x,t) = \frac{1}{4f(t)}\frac{d}{dt}\left(\ln\frac{f(t)}{g(t)}\right)x^2 + \frac{g(t)}{f(t)}z(t)x \\ - \int\left(\frac{g(t)^2}{f(t)}z(t)^2 - V_0(t)\right)dt + C_2, \end{aligned} \tag{45}$$

$$p(x,t) = \frac{\varepsilon g(t)}{\delta f(t)}x - \frac{2\varepsilon}{\delta}\int\frac{g(t)^2}{f(t)}z(t)dt. \tag{46}$$

Here we define $z(t)$ by

$$z(t) = \int\frac{f(t)}{g(t)}V_1(t)dt + C_1.$$

Now we have obtained the explicit expressions of $p(x,t)$, $q(t)$, $a(x,t)$, and $c(t)$ and then transformation (36) has also been explicitly and analytically determined. This result is independent of both the solutions of the standard NLS equation and of the parameters and the external potentials, which is in sharp contrast to the similarity transformation method. The general expression of the transformation is difficult to obtain explicitly in a general case [24].

IV. APPLICATIONS AND DISCUSSIONS

The existence of transformation (36) for conversion of nonautonomous NLS equation (25) to the standard NLS one [Eq. (35)] indicates that the analytical solitonlike solutions of Eq. (25) have a close relation with the conventional canonical solitons when integrability conditions (23) and (26) are satisfied. This is a fairly general conclusion and it can be applied to all solutions of the standard NLS equation. In order to show this generality and its wide applications in engineering integrable nonautonomous NLS equation, we discuss some explicit examples and make some comparison with the previous results in the literature.

A. Solitons in nonuniform media

As the first nontrivial generalization of the standard NLS equation, Chen and Liu [14] studied the propagation of a soliton in an inhomogeneous media governed by the following equation:

$$i\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2 u(x,t)}{\partial x^2} + 2|u(x,t)|^2 u(x,t) - 2\alpha x u(x,t) = 0, \quad (47)$$

where α is defined as in [14]. Apparently this equation is a special case of Eq. (25) under $f(t)=1$, $g(t)=2$, $V_0(t)=V_2(t)=0$, and $V_1(t)=-2\alpha$. In this case integrability conditions (23) and (26) are autonomically satisfied irrespective of the explicit form of $V_1(t)$, since it does not influence the integrability of the system studied. This is the reason that the model of Chen and Liu [14] can be studied by Lax pair method. The one-soliton solution of standard NLS equation (36) is

$$Q(X,T) = \mathrm{sech}(X) e^{(i/2)T}. \quad (48)$$

The corresponding one-soliton-like solution of Eq. (47) can be straightforwardly written as

$$u(x,t) = \mathrm{sech}(x + 2\alpha t^2 - 4t_0 t - x_0) \times e^{-i[2\alpha(t-t_0)x+(4/3)\alpha^2(t-t_0)^3]} e^{i(t+t_0)/2}, \quad (49)$$

where t_0 and x_0 are the initial time and position of the one-soliton-like solution. One notes that soliton (49) is the same as that in [14] except for some different notations.

Below we go beyond the one-soliton (49) to explore its dynamical properties for arbitrary $V_0(t)$ and $V_1(t)$. According to Eqs. (45) and (46) the dynamical behavior of the corresponding one-soliton-like solution can be described by the time-dependent wave number (let $\varepsilon=1$, $\delta=2$)

$$k(t) = \frac{\partial}{\partial x} a(x,t) = \int V_1(t) dt + 2C_1 \quad (50)$$

and the position- and time-dependent frequency shift

$$\Omega(x,t) = -\frac{\partial}{\partial t} a(x,t) = -V_1(t)x + k(t)^2 + V_0(t). \quad (51)$$

In addition the central position of the one-soliton moves with the velocity

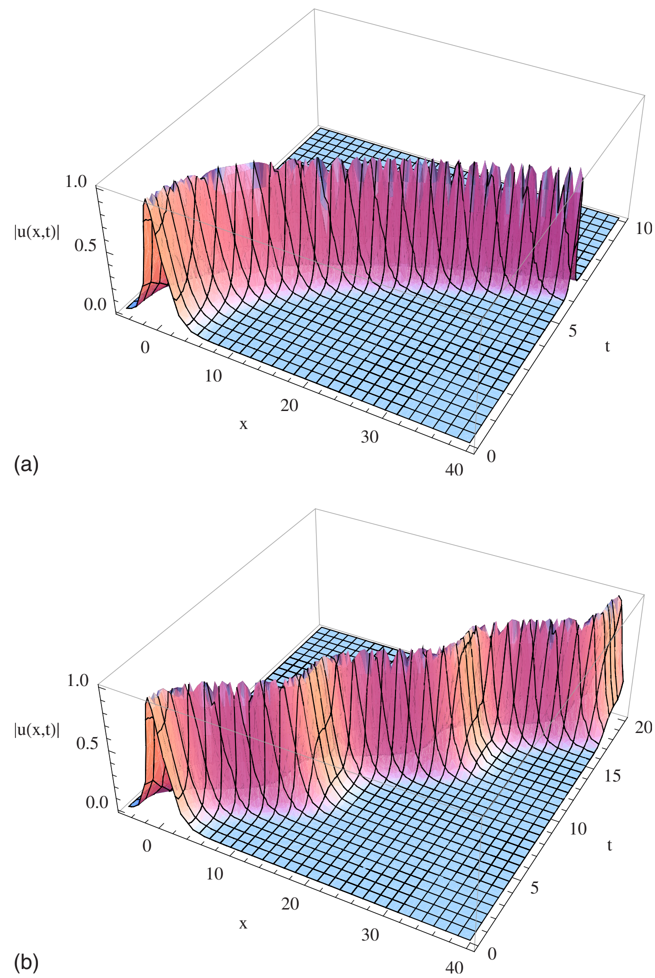

FIG. 1. (Color online) The one-soliton dynamical behavior. Upper plot: $V_0(t)=0$, $V_1(t)=1$; lower plot: $V_0(t)=V_1(t)=\sin(\omega t)$ with $\omega=1$. The other parameters used are $f(t)=g(t)=1$, $C_1=C_2=C_3=0$, and $V_2(t)=0$.

$$v_g(t) = -\frac{\partial}{\partial t} p(x,t) = 2k(t). \quad (52)$$

In Fig. 1 we show the one-soliton evolution with time. The upper one is the model studied by Chen and Liu [14] with $V_0(t)=0$ and $V_1(t)=1$. The lower one shows that the one-soliton oscillates apparently in space since the linear external potential is modulated periodically as $V_1(t)=\sin(\omega t)$. Obviously, when $\omega\to\infty$, the one-soliton behaves like standard stationary one-soliton (48) since in this case $\int \sin(\omega t) dt \to 0$. Of course, one can take the form of $V_0(t)$ and $V_1(t)$ as desired since the presence of these two external potentials does not break down the integrability of the system. Thus the soliton moves as a whole under the potentials $V_0(t)$ and $V_1(t)$. This can be seen from Fig. 1, in which the amplitude and width of the one-soliton remain unchanged. However, this is not the case in the presence of the quadratic external potential, as discussed below.

B. Presence of quadratic external potential

It is well known that the presence of the quadratic external potential is important to generate BEC in experiments. In

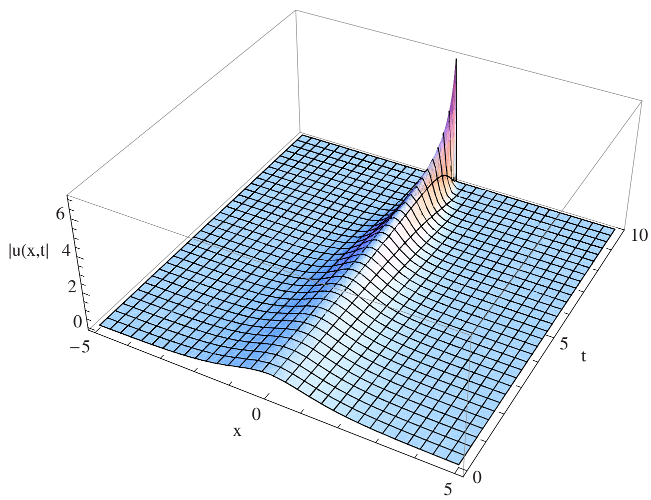

FIG. 2. (Color online) The one-soliton compression as the magnetic field tuned by Eq. (55) in the absence of the quadratic external potential. Here $V_0(t)=V_1(t)=0$ and $f(t)=1$, $C_1=C_2=C_3=0$, and $\alpha=-0.1$.

the mean-field level the BEC can be well described by the GP equation. Thus the analytic solution of the GP equation is most relevant to the experimental observation of BEC dynamics. For simplicity we fix $f(t)=1$ in the case of BEC. Thus Eq. (26) becomes

$$V_2(t) = \frac{1}{4} g(t) \frac{d^2}{dt^2} \frac{1}{g(t)}, \tag{53}$$

which indicates how to tune the quadratic external potential when the nonlinear interaction is tuned by the technique of Feshbach resonance in order to ensure the integrability of the matter-wave system. A similar condition was also given by Hernandez *et al.* [36] and Zhang *et al.* [37] in discussing how to control the soliton interactions in BEC. It is very interesting to note that if the quadratic external potential is absent, i.e., $V_2(t)=0$, condition (53) means that

$$g(t) = \frac{1}{1+\alpha t}, \tag{54}$$

where α is a constant. When $\alpha=0$, $g(t)=1$; there is no nonlinear modulation, which is the standard NLS equation. When $\alpha>0$, the nonlinearity decreases with time and as a result the soliton broadens with time. On the contrary, when $\alpha<0$ and $t\in(0,-1/\alpha)$, the nonlinearity increases rapidly with time and correspondingly the soliton is compressed rapidly, as shown in Fig. 2. This case is consistent with that in [38]. According to the theory of Feshbach resonance

$$a_s(t) = a_{bg}\left(1 + \frac{\Delta}{B_0 - B(t)}\right),$$

and so

$$g(t) = \frac{a_s(t)}{a_{bg}} = 1 + \frac{\Delta}{B_0 - B(t)},$$

which gives

$$B(t) = B_0 + \Delta\left(1 + \frac{1}{\alpha t}\right). \tag{55}$$

When the magnetic field varies like Eq. (55) [note that $t \in (0,-1/\alpha)$], the nonlinear interaction follows like Eq. (54). This effect provides a simple way to compress the soliton.

Below we focus on the effect of the quadratic external potential tuned by Eq. (53). We first discuss one-soliton case (48) and then provide a multisoliton solution later. To describe the one-soliton dynamics the time-dependent wave number becomes

$$k(t) = -\frac{x}{2}\frac{g_t(t)}{g(t)} + g(t)\left[\int \frac{V_1(t)}{g(t)}dt + C_1\right], \tag{56}$$

and the position- and time-dependent frequency shift is

$$\Omega(x,t) = \frac{x^2}{4}[\ln g(t)]_{tt} - x\left[g_t(t)\left(\int \frac{V_1(t)}{g(t)}dt + C_1\right) + V_1(t)\right] + g(t)^2\left(\int \frac{V_1(t)}{g(t)}dt + C_1\right)^2 - V_0(t). \tag{57}$$

The central position of the one-soliton moves with velocity

$$v_g(x,t) = -\frac{\varepsilon}{\delta}g_t x + \frac{2\varepsilon}{\delta}g(t)^2\left(\int \frac{V_1(t)}{g(t)}dt + C_1\right), \tag{58}$$

which is space and time dependent.

In Fig. 3 we present the one-soliton dynamics when the nonlinear interaction is tuned as $g(t)=1+d\sin(\omega t)$ [39] as $(d,\omega)=(0.5,2)$. To ensure the integrability of the system studied, the corresponding quadratic external potential $V_2(t)$ should be tuned as Eq. (53), as shown in the upper plot of Fig. 3. It is noted that not only the magnitude of the quadratic external potential changes periodically but also its sign does, which means changing between the repulsive and attractive quadratic external potentials. With these modulations the one-soliton oscillates periodically due to $V_1(t)$, as discussed above. However, different from the case discussed above, here the amplitude and width of the one-soliton change periodically. When the amplitude of the one-soliton increases, the width of the one-soliton becomes narrow correspondingly. This result is due to the periodic modulation of the nonlinear interaction $g(t)$, and the expressions of $c(t)$ and $p(x,t)$.

Now a brief summary is in order. In a generalized one-dimensional integrable NLS equation (25), $V_0(t)$ term contributes only to the frequency shift of the one-soliton and it can be removed by a time-dependent phase in the wave function as mentioned above. The linear external potential $V_1(t)$ term contributes both the frequency shift and the central position of the one-soliton. These two terms do not influence the integrability of the system. However, the quadratic external potential $V_2(t)$ term affects the integrability of the system and it should satisfy a constraint condition (26) to ensure the integrability of the system. Surprisingly the $V_2(t)$ term does not directly contribute to the soliton motion and possible deformations.

After the one-soliton dynamics has been clearly explored, below we discuss multisoliton dynamics. As an example we

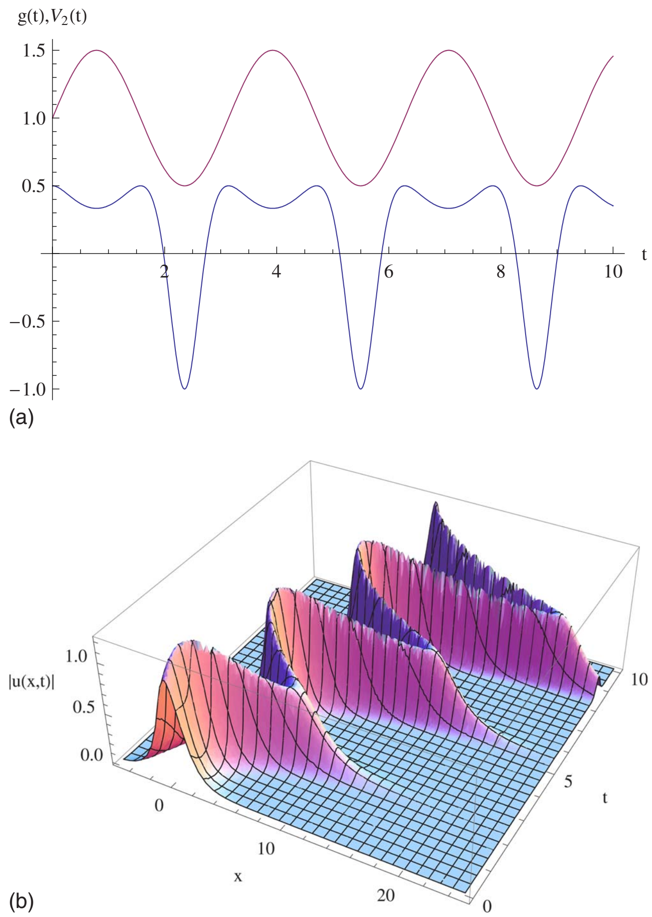

FIG. 3. (Color online) Upper plot: the nonlinear interaction tuned as $g(t)=1+d\sin(\omega t)$ with $(d,\omega)=(0.5,2)$ and the quadratic external potential given by Eq. (53) with $\varepsilon=1/2$. Lower plot: the corresponding one-soliton time evolution with $V_0(t)=\sin(\omega t)$ and $V_1(t)=g(t)\sin(\omega t)$. The other parameters used are $f(t)=1$ and $C_1=C_2=C_3=0$.

consider the following two-soliton solution of the standard NLS equation [40]:

$$Q(X,T)=4e^{(i/2)T}\frac{\cosh(3X)+3e^{4iT}\cosh X}{\cosh(4X)+4\cosh(2X)+3\cos(4T)}. \tag{59}$$

In Fig. 4 we show the time evolution of the two-soliton when the nonlinear interaction modulation and the external potential are applied, as in the one-soliton case. One notes that the overall features are completely similar to the one-soliton case; i.e., the two-soliton oscillates periodically in space and its amplitude and width change dramatically, as compared with the standard one (the upper plot of Fig. 4). Here we do not observe the soliton splitting phenomenon, as discussed in [4]. The reason is that the present case is completely integrable, but the system (see, e.g., Eq. (5.5) of [4]) does not satisfy integrability condition (26) since $V_2=0$ in that system.

The above studies of the one- and two-soliton dynamics show that the transformation we obtained is quite powerful

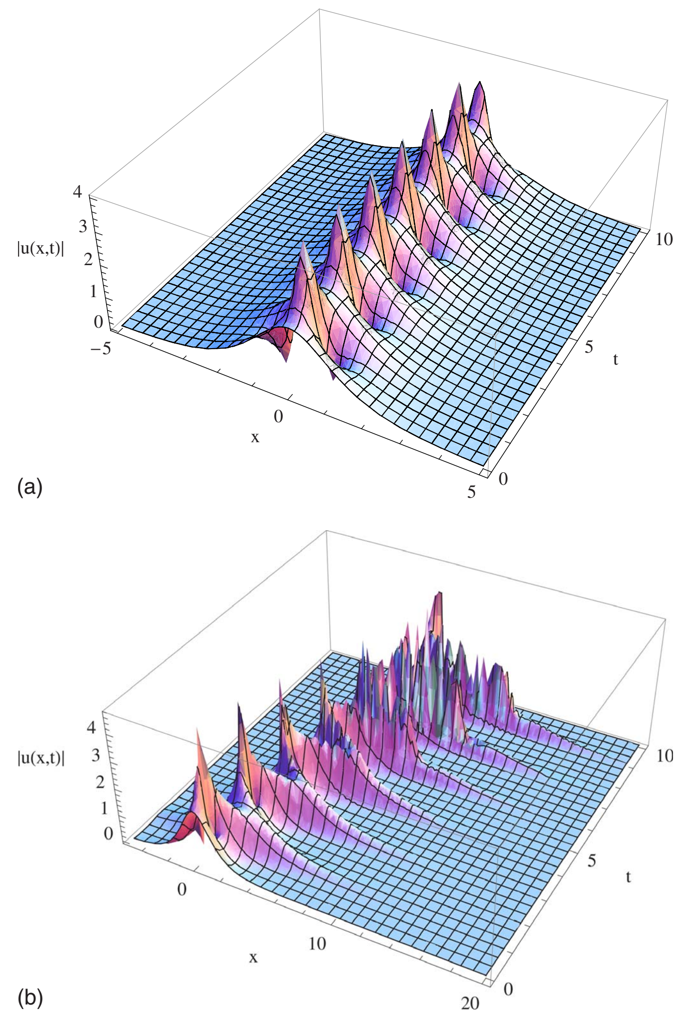

FIG. 4. (Color online) Upper plot: the two-soliton solution of the standard NLS equation. Lower plot: time evolution of the two-soliton tuned by the nonlinear interaction and the external potentials as used in Fig. 3. All parameters used are the same as those in Fig. 3 except for $\omega=4$.

and can be applied to all solutions of the standard NLS equation, which are vast in the literature [31,41–47].

V. SUMMARY

Based on the Painlevé test we have thoroughly analyzed the integrability conditions of the generalized one-dimensional nonautonomous NLS equation and found the general constraint conditions on the dispersion and nonlinearity managements and the external potentials. It provides many ways to engineer integrable nonautonomous NLS systems. Through a general transformation the solutions of these integrable nonautonomous NLS equations have been obtained analytically from the solutions of the standard NLS equation. The corresponding soliton dynamics can be controlled as desired by tuning the related dispersion, nonlinearity, and external potential parameters. Moreover, the characteristic contributions of different control parameters to the soliton dynamics have been clearly identified, which is of significance to guide experiment to control the soliton dynamics. The same idea can also be used to other nonlinear systems and provides a general way to engineer integrable nonautonomous nonlinear systems.

ACKNOWLEDGMENTS

This work was supported partly by the Program for NCET, NSF, and the National Program for Basic Research of China.

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