Bi-Hamiltonian representation of Stäckel systems

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It is shown that linear separation relations are fundamental objects for integration by quadratures of Stäckel-separable Liouville-integrable systems (the so-called Stäckel systems). These relations are further employed for the classification of Stäckel systems. Moreover, we prove that *any* Stäckel-separable Liouville-integrable system can be lifted to a bi-Hamiltonian system of Gel'fand-Zakharevich type. In conjunction with other known result this implies that the existence of bi-Hamiltonian representation of Liouville-integrable systems is a necessary condition for Stäckel separability.

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I. INTRODUCTION

The Hamilton-Jacobi (HJ) theory seems to be one of the most powerful methods of integration by quadratures for a wide class of systems described by nonlinear ordinary differential equations, with a long history as a part of analytical mechanics. The theory in question is closely related to the Liouville-integrable Hamiltonian systems. The milestones of this theory include the works of Stäckel, Levi-Civitá, Eisenhart, Woodhouse, Kalnins, Miller, and Benenti. The majority of results were obtained for a very special class of integrable systems, important from the physical point of view, namely, for the systems with quadratic-in-momenta first integrals.

The first efficient construction of the separation variables for dynamical systems was discovered by Sklyanin [1]. He adapted the methods of soliton theory, i.e., the Lax representation and r-matrix theory for systematic derivation of separation coordinates. In this approach the integrals of motion in involution appear as coefficients of characteristic equation (*spectral curve*) of the Lax matrix. This method was successfully applied for separating variables in many integrable systems [1–8].

Recently, a modern geometric theory of separability on bi-Poissonian manifolds was developed [9–15]. This theory is closely related to the so-called Gel'fand-Zakharevich (GZ) bi-Hamiltonian systems [16,17]. The theory in question includes Liouville-integrable systems with integrals of motion being functions quadratic in momenta as a very special case. In this approach the constants of motion are closely related to the so-called *separation curve* which is intimately related to the Stäckel separation relations. The separation curve arising in the geometric approach is closely related to its counterpart in the r-matrix approach. In fact, these curves are identical for linear r matrix and related by exponentiation of momenta in the spectral curve for dynamical (quadratic) r matrix [6,14].

In the present paper we develop in a systematic fashion a separability theory of the Liouville-integrable systems which are of the GZ type, including as a special case the class of systems with quadratic-in-momenta first integrals. First of all, we treat Stäckel-separable systems according to the form of separation relations and make some observations related to their classification. Then we construct a quasi-bi-Hamiltonian representation of Stäckel systems on $2n$-dimensional phase space and lift them to the related GZ bi-Hamiltonian systems on the extended $(2n+k)$-dimensional phase space. This result proves that bi-Hamiltonian property is common to all classes of the Stäckel systems considered. In other words, we prove that the existence of bi-Hamiltonian representation for Liouville-integrable systems is a necessary condition for their Stäckel separability, i.e., for these systems for which separation relations are linear in all constants of motion which are in involution. Finally let us mention that up to now such a proof was available only for a distinguished class of the so-called Benenti systems [18], where $k=1$ and all constants of motion are quadratic in momenta.

II. SEPARABLE STÄCKEL SYSTEMS

Consider a Liouville-integrable system on a $2n$-dimensional phase space M. Thus, we have $M \ni u = (q_1, \ldots, q_n, p_1, \ldots, p_n)^T$ and there are n functions $H_i(q,p)$ in involution with respect to the canonical Poisson tensor π,

$$\{H_i, H_j\}_\pi = \pi(dH_i, dH_j) = \langle dH_i, \pi dH_j \rangle = 0, \quad i,j = 1, \ldots, n,$$

where $\langle \cdot, \cdot \rangle$ is the standard pairing of TM and T^*M. Canonicity of π means that the only nonzero Poisson brackets among the coordinates are $\{q_i, p_j\}_\pi = \delta_{ij}$. The functions H_i generate n Hamiltonian dynamic systems

$$u_{t_i} = \pi dH_i = X_{H_i}, \quad i = 1, \ldots, n, \tag{1}$$

where X_{H_i} are called the *Hamiltonian vector fields*.

The HJ method for solving Eq. (1) essentially amounts to the linearization of the latter via a canonical transformation,

$$(q,p) \to (b,a), \quad a_i = H_i, \quad i = 1, \ldots, n. \tag{2}$$

In order to find the conjugate coordinates b_i it is necessary to construct a generating function $W(q,a)$ of transformation (2) such that

$$b_j = \frac{\partial W}{\partial a_j}, \quad p_j = \frac{\partial W}{\partial q_j}.$$

The function $W(q,a)$ is a complete integral of the associated *Hamilton-Jacobi equations*

$$H_i\left(q_1,\ldots,q_n,\frac{\partial W}{\partial q_1},\ldots,\frac{\partial W}{\partial q_n}\right)=a_i,\quad i=1,\ldots,n. \tag{3}$$

In the (b,a) representation the t_i dynamics is trivial:

$$(a_j)_{t_i}=0,\quad (b_j)_{t_i}=\delta_{ij},$$

whence

$$b_j(q,a)=\frac{\partial W}{\partial a_j}=t_j+c_j,\quad j=1,\ldots,n, \tag{4}$$

where c_j are arbitrary constants.

Equation (4) provides implicit solutions for Eq. (1). Solving it for q_j is known as the *inverse Jacobi problem*. The reconstruction in explicit form of trajectories $q_j=q_j(t_i)$ is in itself a highly nontrivial problem from algebraic geometry, which is beyond the scope of this paper.

The main difficulty in applying the above method to a given Liouville-integrable system in given canonical coordinates (q,p) consists of solving system (3) for W. In general this is a hopeless task, as Eq. (3) is a system of nonlinear coupled partial differential equations. In essence, the only hitherto known way of overcoming this difficulty is to find distinguished canonical coordinates, denoted here by (λ,μ), for which there exist n relations

$$\varphi_i(\lambda_i,\mu_i;a_1,\ldots,a_n)=0,\quad i=1,\ldots,n,$$

$$a_i\in\mathbb{R},\quad \det\left[\frac{\partial\varphi_i}{\partial a_j}\right]\neq 0, \tag{5}$$

such that each of these relations involves only a single pair of canonical coordinates [1]. The determinant condition in Eq. (5) means that we can solve Eq. (5) for a_i and express a_i in the form $a_i=H_i(\lambda,\mu)$, with $i=1,\ldots,n$.

If the functions $W_i(\lambda_i,a)$ are solutions of a system of n *decoupled* ordinary differential equations obtained from Eq. (5) by substituting $\mu_i=\frac{dW_i(\lambda_i,a)}{d\lambda_i}$,

$$\varphi_i\left(\lambda_i,\mu_i=\frac{dW_i(\lambda_i,a)}{d\lambda_i},a_1,\ldots,a_n\right)=0,\quad i=1,\ldots,n, \tag{6}$$

then the function

$$W(\lambda,a)=\sum_{i=1}^{n}W_i(\lambda_i,a)$$

is an additively separable solution of *all* the Eq. (6), and *simultaneously* it is a solution of all Hamilton-Jacobi equations (3) because solving Eq. (5) to the form $a_i=H_i(\lambda,\mu)$ is a purely algebraic operation. The Hamiltonian functions H_i Poisson commute since the constructed function $W(\lambda,a)$ is a generating function for the canonical transformation $(\lambda,\mu)\to(b,a)$. The distinguished coordinates (λ,μ) for which the original Hamilton-Jacobi equations (3) are equivalent to a set of separation relations (6) are called the *separation coordinates*.

Of course, the original Jacobi formulation of the method was a bit different from the one presented above, and was made for a particular class of Hamiltonians. Nevertheless it contained all important ideas of the method. Jacobi himself doubted whether there exists a systematic method for construction of separation coordinates. Indeed, for many decades of development of separability theory, the method did not exist. Only recently, at the end of the 20th century, after more then 100 years of efforts, two different constructive methods were suggested, the first related to the Lax representation and the second related to the bi-Hamiltonian representation for a given integrable system.

We would like to stress that all results of the present paper are derived directly from separation relations (5), thus confirming their fundamental role in the modern separability theory.

In what follows we restrict ourselves to considering a special case of Eq. (5) when all separation relations are affine in H_i:

$$\sum_{k=1}^{n}S_i^k(\lambda_i,\mu_i)H_k=\psi_i(\lambda_i,\mu_i),\quad i=1,\ldots,n, \tag{7}$$

where S_i^k and ψ_i are arbitrary smooth functions of their arguments. Relations (7) are called the generalized *Stäckel separation relations* and the related dynamical systems are called the *Stäckel-separable* ones. The matrix $S=(S_i^k)$ will be called a *generalized Stäckel matrix*. The reason behind this name is the fact that conditions (7) with S_i^k being μ independent and ψ_i being quadratic in momenta μ are equivalent to the original Stäckel conditions for separability of Hamiltonians H_i. To recover the explicit Stäckel form of the Hamiltonians, it suffices to solve linear system (7) with respect to H_i.

Although the restriction of linearity appears to be very strong, for all known separable systems (at least to the knowledge of the author), the general separation conditions can be reduced to form (7) upon suitable choice of integrals of motion H_i. The possible explanation of this fact is that we simply have no mathematical tools for effective construction of separation coordinates for non-Stäckel-separable systems, so that part of separability theory is yet *terra incognita*.

Let us come back to the Stäckel case. If in Eq. (7) we further have $S_i^k(\lambda_i,\mu_i)=S^k(\lambda_i,\mu_i)$ and $\psi_i(\lambda_i,\mu_i)=\psi(\lambda_i,\mu_i)$, then the separation conditions can be represented by n copies of the curve

$$\sum_{k=1}^{n}S^k(\lambda,\mu)H_k=\psi(\lambda,\mu) \tag{8}$$

in (λ,μ) plane, called a *separation curve*. The copies in question are obtained by setting $\lambda=\lambda_i$ and $\mu=\mu_i$ for $i=1,\ldots,n$.

Remark. There is an important special case when Eq. (8) is an arbitrary nonsingular compact Riemann surface Γ, i.e., when $S^k(\lambda,\mu)$ and $\psi(\lambda,\mu)$ are polynomials of λ and μ of certain specific form. Then one can find the genus of this curve and basic holomorphic differentials in a standard fashion and Jacobi inversion problem (4) can be equivalently expressed by the Abel map of the Riemann surface Γ into its Jacobi variety and solved in the language of Riemann theta functions (see [19] and references therein).

From now on we will consider Stäckel-separable systems with separation relations of the most general form [Eq. (7)].

For reasons to be explained in Sec. III, we collect the terms from the left-hand side of Eq. (7) as follows:

$$\sum_{k=1}^{m} \varphi_i^k(\lambda_i,\mu_i) H^{(k)}(\lambda_i) = \psi_i(\lambda_i,\mu_i), \quad i=1,\ldots,n, \tag{9}$$

where

$$H^{(k)}(\lambda) = \sum_{i=1}^{n_k} \lambda^{n_k-i} H_i^{(k)}, \quad n_1+\cdots+n_m = n,$$

and impose the normalization $\varphi_i^m(\lambda_i,\mu_i)=1$.

As separation relations (7) play the fundamental role in the Hamilton-Jacobi theory, it is natural to employ them for classification of Stäckel systems. The form of separation relations (9) allows us to classify the associated Stäckel systems. Actually, any given class of Stäckel-separable systems can be represented by a fixed Stäckel matrix S and the functions ψ. The matrix S is uniquely defined by m vectors $\varphi^k = (\varphi_1^k,\ldots,\varphi_n^k)^T$, with $k=1,\ldots,m$, and the partition $(n_1,\ldots,n_m)$ of n. Note that in our normalization we have $\varphi^m=(1,\ldots,1)^T$.

For example, the most intensively studied systems in the 20th century, those related to one-particle separable dynamics on Riemannian manifolds with flat or constant-curvature metrics, belong to the simplest class with $m=1$ and the functions ψ_i being quadratic in the momenta μ_i:

$$\sum_{j=1}^{n} H_j \lambda_i^{n-j} = \tfrac{1}{2} f_i(\lambda_i)\mu_i^2 + \gamma_i(\lambda_i), \quad i=1,\ldots,n. \tag{10}$$

This class, which will hereinafter be referred to as the Benenti class, includes systems generated by conformal Killing tensors [18,20–22], as well as bicofactor systems, generated by a pair of conformal Killing tensors [23–28]. Here the functions f_i define the Stäckel metric, while the functions γ_i define a separable potential. When $f_i=f(\lambda_i)$ and f is a polynomial of order not higher than $n+1$, then the associated Stäckel metric is of constant curvature.

Another class of separable systems also has $m=1$ but the functions ψ_i are now exponential in the momenta,

$$\sum_{j=1}^{n} H_j \lambda_i^{n-j} = \exp(a\mu_i) + \exp(-b\mu_i) + \gamma_i(\lambda_i), \quad i=1,\ldots,n,$$

where γ_i defines a separable potential. This class includes such systems as the periodic Toda lattice [13], the Korteweg–de Vries (KdV) dressing chain [14], and the Ruijsenaars-Schneider system [11].

We also know some particular examples from the classes with $m>1$. For instance, stationary flows of the Boussinesq hierarchy belong to the class with $m=2$, $n_1=1$, $n_2=n-1$, and $\varphi_i^1=\mu_i$ [11,12]. Dynamical system on loop algebra $\mathfrak{sl}(3)$ belongs to the class with $m=2$, $n_1=2$, $n_2=4$, and $\varphi_i^1=\mu_i$ [15]. In both cases the functions ψ_i are cubic in the momenta, so these separation relations belong to the following class:

$$\mu_i \sum_{j=1}^{n_1} H_j^{(1)} \lambda_i^{n_1-j} + \sum_{j=1}^{n_2} H_j^{(2)} \lambda_i^{n_1-j} = \tfrac{1}{3} f(\lambda_i)\mu_i^3 + \mu_i\gamma_1(\lambda_i) + \gamma_2(\lambda_i), \quad i=1,\ldots,n, \tag{11}$$

where $\mu\gamma_1$ and γ_2 give rise to the separable potentials.

Finally, systems from the classes with $1<m\le n$, $\varphi_i^k=\lambda_i^{\alpha_k}$, and $\alpha_k\in\mathbb{N}$ and with ψ_i quadratic in the momenta, i.e.,

$$\sum_{k=1}^{m} \lambda_i^{\alpha_k} H^{(k)}(\lambda_i) = \tfrac{1}{2} f_i(\lambda_i)\mu_i^2 + \gamma_i(\lambda_i), \quad i=1,\ldots,n, \tag{12}$$

were constructed in [29].

III. BI-HAMILTONIAN PROPERTY OF STÄCKEL SYSTEMS

We start this section with a few definitions important for further considerations. As the Hamiltonian formalism is of tensorial type, there is no need to restrict ourselves to nondegenerate canonical representation of Hamiltonian vector fields. Given a manifold $\mathcal{M}$, a *Poisson operator* π on $\mathcal{M}$ is a bivector (second-order contravariant tensor field) with vanishing Schouten bracket

$$[\pi,\pi]_S = 0.$$

Then the bracket

$$\{f_1,f_2\}_\pi := \langle df_1, \pi df_2\rangle, \quad f_1,f_2 \in C^\infty(\mathcal{M}),$$

is the Lie bracket; i.e., it is skew symmetric and satisfies the Jacobi identity. A function $c:\mathcal{M}\to\mathbb{R}$ is called the *Casimir function* of the Poisson operator π if for an arbitrary function $f:\mathcal{M}\to\mathbb{R}$ we have $\{f,c\}_\pi=0$ (or, equivalently, if $\pi dc=0$). A linear combination $\pi_\lambda=\pi_1-\lambda\pi_0$ ($\lambda\in\mathbb{R}$) of two Poisson operators π_0 and π_1 is called a *Poisson pencil* if the operator π_λ is Poissonian for any value of the parameter λ, i.e., when $[\pi_0,\pi_1]_S=0$. In this case we say that π_0 and π_1 are *compatible*. Given a Poisson pencil $\pi_\lambda=\pi_1-\lambda\pi_0$ we can often construct a sequence of vector fields X_i on $\mathcal{M}$ that have two Hamiltonian representations (the so-called *bi-Hamiltonian chain*),

$$X_i = \pi_1 dh_i = \pi_0 dh_{i+1}, \tag{13}$$

where $h_i\in C^\infty(\mathcal{M})$ are called the Hamiltonians of chain (13) and where i is a discrete index. This sequence of vector fields may or may not terminate in zero depending on the existence of the Casimir functions for the pencil.

Consider a bi-Poissonian manifold (M,π_0,π_1) of dim $M=2n+m$, where π_0, π_1 is a pair of compatible Poisson tensors of rank $2n$. We further assume that the Poisson pencil π_λ admits m Casimir functions which are polynomial in the pencil parameter λ and have the form

$$h^{(j)}(\lambda) = \sum_{i=0}^{n_j} \lambda^{n_j-i} h_i^{(j)}, \quad j=1,\ldots,m, \tag{14}$$

so that $n_1+\cdots+n_m=n$ and $h_i^{(j)}$ are functionally independent. The collection of n bi-Hamiltonian vector fields

$$\pi_\lambda dh^{(j)}(\lambda) = 0 \Leftrightarrow X_i^{(j)} = \pi_1 dh_i^{(j)} = \pi_0 dh_{i+1}^{(j)},$$

$$i = 1, \dots, n_j, \quad j = 1, \dots, m, \tag{15}$$

is called the GZ system of the bi-Poissonian manifold $\mathcal{M}$. Notice that each chain starts from a Casimir of π_0 and terminates with a Casimir of π_1. Moreover, all $h_i^{(j)}$ pairwise commute with respect to both Poisson structures,

$$X_i^{(j)}(h_l^{(k)}) = \langle dh_l^{(k)}, \pi_0 dh_{i+1}^{(j)} \rangle = \langle dh_l^{(k)}, \pi_1 dh_i^{(j)} \rangle = \{h_l^{(k)}, h_{i+1}^{(j)}\}_{\pi_0}$$
$$= \{h_l^{(k)}, h_i^{(j)}\}_{\pi_1} = 0.$$

In Sec. IV we prove that an arbitrary Stäckel system on the phase space M with separation conditions given by Eq. (9) can be lifted to a GZ bi-Hamiltonian system in the extended phase space $\mathcal{M}$.

As recently proved in [15], the Stäckel Hamiltonians from separation relations (7) admit the following quasi-bi-Hamiltonian representation:

$$\Pi_1 dH_i = \sum_{j=1}^{n} F_{ij} \Pi_0 dH_j, \quad i = 1, \dots, n, \tag{16}$$

where Π_0 is a canonical Poisson tensor

$$\Pi_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

I_n is an $n \times n$ unit matrix, Π_1 is a noncanonical Poisson tensor of the form

$$\Pi_1 = \begin{pmatrix} 0 & \Lambda_n \\ -\Lambda_n & 0 \end{pmatrix}, \quad \Lambda_n = \mathrm{diag}(\lambda_1, \dots, \lambda_n),$$

compatible with Π_0, and the *control matrix* F has the form

$$F = (S^{-1} \Lambda_n S), \tag{17}$$

where S is the associated Stäckel matrix.

To have a better insight into the functions F_{ij}, we will find another representation for the entries $F_{ij} = (S^{-1} \Lambda_n S)_{ij}$ of F. To this end consider a system of n linear equations for V_k, with $k = 1, \dots, n$;

$$\sum_{k=1}^{n} S_i^k(\lambda_i, \mu_i) V_k = \sum_{j=1}^{n} \lambda_i S_i^j(\lambda_i, \mu_i) a_j, \quad i = 1, \dots, n, \tag{18}$$

where a_i, with $i = 1, \dots, n$, are some parameters. The solution of this system has the form

$$V_r = \sum_{p=1}^{n} \alpha_{rp} a_p, \quad \alpha_{rp} = \frac{\det(S^{(rp)})}{\det S}, \tag{19}$$

where $S^{(rp)}$ is the matrix S with the rth column replaced by $(\lambda_1 S_1^p(\lambda_1 \mu_1), \dots, \lambda_n S_n^p(\lambda_n \mu_n))^T$, the string of coefficients at the parameter a_p. On the other hand, as $V = (V_1, \dots, V_n)^T$ and $a = (a_1, \dots, a_n)^T$, system (18) can be written in the matrix form as

$$SV = \Lambda_n S a \Rightarrow V = S^{-1} \Lambda_n S a = \alpha a,$$

where $\alpha_{ij} = (S^{-1} \Lambda_n S)_{ij}$. Comparing this result with Eqs. (17) and (19) we find

$$F_{ij} = (S^{-1} \Lambda_n S)_{ij} = \frac{\det(S^{(ij)})}{\det S}. \tag{20}$$

Now, the important question is which entries F_{ij} are nonzero when the separation relations take form (7). In other words, we want to know for which i, j $\det(S^{(ij)}) \neq 0$, i.e., the matrix $S^{(ij)}$ has no linearly dependent columns.

To answer this question, we first rewrite quasi-bi-Hamiltonian chain (16) in the equivalent form

$$\Pi_1 dH_i^{(k)} = \sum_{l=1}^{m} \sum_{j=1}^{n_l} F_{i,j}^{k,l} \Pi_0 dH_j^{(l)}, \quad k = 1, \dots, m, \quad i = 1, \dots, n_k, \tag{21}$$

adapted to the separation relations written in form (9). Then a simple inspection shows that

$$F_{i,i+1}^{k,k} = 1, \quad F_{i,1}^{k,l} \equiv F_i^{k,l} \neq 0.$$

Hence, quasi-bi-Hamiltonian representation (21) takes the form

$$\Pi_1 dH_i^{(k)} = \Pi_0 dH_{i+1}^{(k)} + \sum_{l=1}^{m} F_i^{k,l} \Pi_0 dH_1^{(l)}, \quad H_{n_k+1}^{(k)} = 0, \tag{22}$$

where

$$F_i^{k,l} = \frac{\det S_i^{(k,l)}}{\det S},$$

and $S_i^{(k,l)}$ is the Stäckel matrix S with $(n_1 + \cdots + n_{k-1} + i)$th column replaced by $(\varphi_1^l \lambda_1^{n_l}, \dots, \varphi_n^l \lambda_n^{n_l})^T$.

In the rest of this section we show that representation (22) can be lifted to a GZ bi-Hamiltonian form. First, we extend the $2n$-dimensional phase space M to $\mathcal{M} = M \times \mathbb{R}^m$ with additional coordinates c_i, where $i = 1, \dots, m$, on $\mathbb{R}^m$. Then, we extend the Hamiltonians as follows:

$$H_i^{(k)}(q,p) \to h_i^{(k)}(q,p,c) = H_i^{(k)}(q,p) - \sum_{l=1}^{m} F_i^{k,l}(q,p) c_l. \tag{23}$$

From Eqs. (18)–(20)we infer that the separation relations for $h_i^{(k)}$ read

$$\sum_{k=1}^{m} \varphi_i^k(\lambda_i, \mu_i) h^{(k)}(\lambda_i) = \psi_i(\lambda_i, \mu_i), \quad i = 1, \dots, n, \tag{24}$$

where

$$h^{(k)}(\lambda) = \sum_{i=0}^{n_k} \lambda^{n_k - i} h_i^{(k)}, \quad h_0^{(k)} = c_k, \quad n_1 + \cdots + n_m = n.$$

Moreover, for the functions $F_i^{k,l}$ we have the same quasi-bi-Hamiltonian representation as for $H_i^{(k)}$:

$$\Pi_1 dF_i^{k,l} = \Pi_0 dF_{i+1}^{k,l} + \sum_{r=1}^{m} F_i^{k,r} \Pi_0 dF_1^{r,l}. \tag{25}$$

This was proved for arbitrary Stäckel systems in [15].

Denote the push-forwards of the Poisson tensors Π_0 and Π_1 to $\mathcal{M}$ by π_0 and π_{1D}. Both π_0 and π_{1D} are degenerate and possess common Casimirs c_i, with $i=1,\ldots,m$. We have

$$\pi_0=\left(\begin{array}{c|c}\Pi_0 & 0\\ \hline 0 & 0\end{array}\right),\quad \pi_{1D}=\left(\begin{array}{c|c}\Pi_1 & 0\\ \hline 0 & 0\end{array}\right). \tag{26}$$

Relations (22)–(26) imply that on $\mathcal{M}$ we have a quasi-bi-Hamiltonian representation with respect to the Poisson tensors π_0 and π_{1D} of the form

$$\pi_{1D}dh_i^{(k)}=\pi_0 dh_{i+1}^{(k)}+\sum_{l=1}^{m}F_i^{k,l}\pi_0 dh_1^{(l)},\quad F_0^{k,l}=-\delta_{kl},\quad h_{n_k+1}^{(k)}=0. \tag{27}$$

Now introduce the bivector

$$\pi_1 \coloneqq \pi_{1D}+\sum_{k=1}^{m}X_1^{(k)}\wedge Z_k,$$

where

$$X_1^{(k)}=\pi_0 dh_1^{(k)},\quad Z_k=\frac{\partial}{\partial c_k}.$$

First we show that the bivector π_1 is Poissonian. Using the properties of the Schouten bracket we have

$$[\pi_1,\pi_1]_S=2\sum_i Z_i\wedge L_{X_1^{(i)}}\pi_{1D}+2\sum_{i,j}[X_1^{(i)},Z_j]\wedge Z_i\wedge X_1^{(j)}, \tag{28}$$

where L_X means the Lie derivative in the direction of X, and $[\cdot,\cdot]$ is the commutator of vector fields. Now, let us prove that

$$L_{X_1^{(r)}}\pi_{1D}=\sum_l \pi_0 dF_1^{r,l}\wedge X_1^{(l)}. \tag{29}$$

From Eq. (27) we have

$$Y_k \coloneqq \pi_{1D}dh_{n_k}^{(k)}=\sum_l F_{n_k}^{k,l}X_1^{(l)},$$

$$\pi_{1D}dF_{n_k}^{k,l}=\sum_r F_{n_k}^{k,r}\pi_0 dF_1^{r,l}.$$

From the Poissonian property of π_{1D} it follows that

$$\begin{aligned}0&=L_{Y_k}\pi_{1D}\\&=\sum_l \left(F_{n_k}^{k,l}L_{X_1^{(l)}}\pi_{1D}-\pi_{1D}dF_{n_k}^{k,l}\wedge X_1^{(l)}\right)\\&=\sum_l\left[F_{n_k}^{k,l}L_{X_1^{(l)}}\pi_{1D}-\left(\sum_r F_{n_k}^{k,r}\pi_0 dF_1^{r,l}\right)\wedge X_1^{(l)}\right]\\&=\sum_r F_{n_k}^{k,r}L_{X_1^{(r)}}\pi_{1D}-\sum_r F_{n_k}^{k,r}\sum_l \pi_0 dF_1^{r,l}\wedge X_1^{(l)}\\&=\sum_r F_{n_k}^{k,r}\left[L_{X_1^{(r)}}\pi_{1D}-\sum_l \pi_0 dF_1^{r,l}\wedge X_1^{(l)}\right].\end{aligned}$$

Hence Eq. (29) is satisfied. On the other hand,

$$[X_1^{(i)},Z_j]=\pi_0 dF_1^{i,j},$$

so Eq. (28) becomes

$$\begin{aligned}[\pi_1,\pi_1]_S&=2\sum_{i,j}Z_i\wedge\pi_0 dF_1^{i,j}\wedge X_1^{(j)}+2\sum_{i,j}\pi_0 dF_1^{i,j}\wedge Z_i\wedge X_1^{(j)}\\&=0,\end{aligned}$$

and thus π_1 is Poissonian.

Moreover, the Poisson bivectors π_0 and π_1 are compatible as

$$[\pi_0,\pi_1]_S=\sum_i\left(Z_i\wedge L_{X_1^{(i)}}\pi_0-X_1^{(i)}\wedge L_{Z_i}\pi_0\right)=0.$$

Finally, the vector fields $X_i^{(k)}$ form bi-Hamiltonian chains with respect to π_0, π_1. Indeed, we have

$$\begin{aligned}\pi_1 dh_i^{(k)}&=\pi_0 dh_{i+1}^{(k)}+\sum_{l=1}^{m}F_i^{k,l}\pi_0 dh_1^{(l)}+\sum_{l=1}^{m}(X_1^{(l)}\wedge Z_l)dh_i^{(k)}\\&=\pi_0 dh_{i+1}^{(k)}=X_{i+1}^{(k)},\end{aligned}$$

as

$$\sum_{l=1}^{m}(X_1^{(l)}\wedge Z_l)dh_i^{(k)}=-\sum_{l=1}^{m}F_i^{k,l}X_1^{(l)}=-\sum_{l=1}^{m}F_i^{k,l}\pi_0 dh_1^{(l)}.$$

Quite obviously,

$$\pi_1 dh_{n_k}^{(k)}=0,\quad X_1^{(l)}=\pi_1 dc_l=\pi_1 dh_0^{(l)},$$

so $h^{(k)}(\lambda)$ are polynomial in λ Casimir functions of the Poisson pencil $\pi_\lambda=\pi_1-\lambda\pi_0$.

IV. EXAMPLES

Here we illustrate the above results with three examples of separable systems with three degrees of freedom. Two of them are classical Stäckel systems with separation relations quadratic in the momenta, while the third example has separation relations cubic in the momenta.

A. Example 1

Consider the separation relations on a six-dimensional phase space given by the bare (potential-free) separation curve

$$H_1\lambda^2+H_2\lambda+H_3=\tfrac{1}{8}\mu^2$$

from class (10). This curve corresponds to geodesic motion for a classical Stäckel system (of Benenti type). The transformation $(\lambda,\mu)\to(q,p)$ to the flat coordinates of associated metric follows from the point transformation

$$\sigma_1(q)=q_1=\lambda_1+\lambda_2+\lambda_3,$$

$$\sigma_2(q)=\tfrac{1}{4}q_1^2+\tfrac{1}{2}q_2=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3,$$

$$\sigma_3(q)=\tfrac{1}{4}q_1q_2+\tfrac{1}{4}q_3=\lambda_1\lambda_2\lambda_3.$$

In the flat coordinates the Hamiltonians take the form

$$H_1 = p_1p_3 + \tfrac{1}{2}p_2^2,$$

$$H_2 = \tfrac{1}{2}q_3p_3^2 - \tfrac{1}{2}q_1p_2^2 + \tfrac{1}{2}q_2p_2p_3 - \tfrac{1}{2}p_1p_2 - \tfrac{1}{2}q_1p_1p_3,$$

$$H_3 = \tfrac{1}{8}q_2^2p_3^2 + \tfrac{1}{8}q_1^2p_2^2 + \tfrac{1}{8}p_1^2 + \tfrac{1}{4}q_1p_1p_2 + \tfrac{1}{4}q_2p_1p_3 - \tfrac{1}{4}q_1q_2p_2p_3 - \tfrac{1}{2}q_3p_2p_3,$$

and admit a quasi-bi-Hamiltonian representation (16) with the operators Π_0 and Π_1 of the form

$$\Pi_0 = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}, \tag{30}$$

$$\Pi_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & q_1 & -1 & 0 \\ 0 & 0 & 0 & q_2 & 0 & -1 \\ 0 & 0 & 0 & 2q_3 & q_2 & q_1 \\ -q_1 & -q_2 & -2q_3 & 0 & p_2 & p_3 \\ 1 & 0 & -q_2 & -p_2 & 0 & 0 \\ 0 & 1 & -q_1 & -p_3 & 0 & 0 \end{pmatrix}, \tag{31}$$

and the control matrix

$$F = \begin{pmatrix} q_1 & 1 & 0 \\ -\frac{1}{4}q_1^2 - \frac{1}{2}q_2^2 & 0 & 1 \\ \frac{1}{2}q_1q_2 + \frac{1}{4}q_3 & 0 & 0 \end{pmatrix}.$$

On the extended phase space of dimension seven, with an additional coordinate c, we have the extended Hamiltonians (23)

$$h_0 = c,$$

$$h_1 = p_1p_3 + \tfrac{1}{2}p_2^2 - cq_1,$$

$$h_2 = \tfrac{1}{2}q_3p_3^2 - \tfrac{1}{2}q^1p_2^2 + \tfrac{1}{2}q_2p_2p_3 - \tfrac{1}{2}p_1p_2 - \tfrac{1}{2}q_1p_1p_3 + \left(\tfrac{1}{4}q_1^2 + \tfrac{1}{2}q_2^2\right)c,$$

$$h_3 = \tfrac{1}{8}q_2^2p_3^2 + \tfrac{1}{8}q_1^2p_2^2 + \tfrac{1}{8}p_1^2 + \tfrac{1}{4}q_1p_1p_2 + \tfrac{1}{4}q_2p_1p_3 - \tfrac{1}{4}q_1q_2p_2p_3 - \tfrac{1}{2}q_3p_2p_3 - \left(\tfrac{1}{2}q_1q_2 + \tfrac{1}{4}q_3\right)c.$$

They form a bi-Hamiltonian chain

$$\pi_0 dh_0 = 0,$$

$$\pi_0 dh_1 = X_1 = \pi_1 dh_0,$$

$$\pi_0 dh_2 = X_2 = \pi_1 dh_1,$$

$$\pi_0 dh_3 = X_3 = \pi_1 dh_2,$$

$$0 = \pi_1 dh_3,$$

with the Poisson operators π_0 and π_1, where

$$\pi_0 = \left(\begin{array}{c|c} \Pi_0 & 0 \\ \hline 0 & 0 \end{array}\right) \quad , \quad \pi_1 = \left(\begin{array}{c|c} \Pi_1 & X_1 \\ \hline -X_1^T & 0 \end{array}\right).$$

The separation curve for the extended system takes the form

$$c\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = \tfrac{1}{8}\mu^2.$$

B. Example 2

Consider now separation relations on a six-dimensional phase space given by the bare separation curve

$$\bar{H}_1\lambda^3 + \bar{H}_2\lambda^2 + \bar{H}_3 = \tfrac{1}{8}\mu^2$$

from class (12). When written using notation (9), this curve takes the form

$$\lambda^2(H_1^{(1)}\lambda + H_2^{(1)}) + H_1^{(2)} = \tfrac{1}{8}\mu^2$$

and again represents geodesic motion for a classical Stäckel system (this time of non-Benenti type). Using the coordinates, the Hamiltonians, and the functions σ_i from the previous example, we find that

$$\bar{H}_1 = H_1^{(1)} = -\frac{1}{\sigma_2}H_2,$$

$$\bar{H}_2 = H_2^{(1)} = H_1 - \frac{\sigma_1}{\sigma_2}H_2,$$

$$\bar{H}_3 = H_1^{(2)} = H_3 - \frac{\sigma_3}{\sigma_2}H_2.$$

Thus we see that the Hamiltonians $\bar{H}_i$ are related to H_i through the so-called generalized Stäckel transform (see [30] for further details on the latter). One can show that the metric tensor associated to the Hamiltonian $\bar{H}_1$ is not flat anymore.

The Hamiltonians $\bar{H}_i$ form a quasi-bi-Hamiltonian chain (16) with Poisson tensors (30) and (31) and the control matrix

$$F = \begin{pmatrix} \sigma_1 - \dfrac{\sigma_3}{\sigma_2} & 1 & -\dfrac{1}{\sigma_2} \\ -\sigma_2 + \dfrac{\sigma_1\sigma_3}{\sigma_2} & 0 & \dfrac{\sigma_1}{\sigma_2} \\ \dfrac{\sigma_3^2}{\sigma_2} & 0 & \dfrac{\sigma_3}{\sigma_2} \end{pmatrix}.$$

On the extended phase space of dimension eight, with additional coordinates c_1, c_2, we have the extended Hamiltonians (23)

$$h_0^{(1)} = c_1,$$

$$h_1^{(1)} = H_1^{(1)} - \left(\sigma_1 - \frac{\sigma_3}{\sigma_2}\right)c_1 + \frac{1}{\sigma_2}c_2,$$

$$h_2^{(1)} = H_2^{(1)} + \left(\sigma_2 - \frac{\sigma_1\sigma_3}{\sigma_2}\right)c_1 - \frac{\sigma_1}{\sigma_2}c_2,$$

$$h_0^{(2)} = c_2,$$

$$h_1^{(2)} = H_1^{(2)} - \frac{\sigma_3^2}{\sigma_2}c_1 - \frac{\sigma_3}{\sigma_2}c_2.$$

They form two bi-Hamiltonian chains

$$\begin{aligned}
&\pi_0 dh_0^{(1)} = 0, \\
&\pi_0 dh_0^{(1)} = X_1^{(1)} = \pi_1 dh_0^{(1)}, \quad \pi_0 dh_0^{(2)} = 0, \\
&\pi_0 dh_2^{(1)} = X_2^{(1)} = \pi_1 dh_1^{(1)}, \quad \pi_0 dh_1^{(2)} = X_1^{(2)} = \pi_1 dh_0^{(2)}, \\
&0 = \pi_1 dh_2^{(1)}, \qquad\qquad\qquad\quad 0 = \pi_1 dh_1^{(2)}
\end{aligned}$$

with the Poisson operators π_0 and π_1 of the forms

$$\pi_0 = \left(\begin{array}{c|cc} \Pi_0 & 0 & 0 \\ \hline 0 & & \\ 0 & & 0 \end{array}\right), \qquad \pi_1 = \left(\begin{array}{c|cc} \Pi_1 & X_1^{(1)} & X_1^{(2)} \\ \hline -(X_1^{(1)})^T & & \\ -(X_1^{(2)})^T & & 0 \end{array}\right).$$

The separation curve for the extended system takes the form

$$\lambda^2(c_1\lambda^2 + h_1^{(1)}\lambda + h_2^{(1)}) + c_2\lambda + h_1^{(2)} = \tfrac{1}{8}\mu^2.$$

C. Example 3

Consider separation relations on a six-dimensional phase space given by the following bare separation curve cubic in the momenta

$$\mu H_1^{(1)} + H_1^{(2)}\lambda + H_2^{(2)} = \mu^3$$

from class (11). The transformation $(\lambda,\mu)\to(q,p)$ to new canonical coordinates in which all Hamiltonians are of a polynomial form is obtained from the following two transformations:

$$u_1 = 3q_2 - 3q_3,$$

$$u_2 = -q_1p_2 - q_1p_3 + 3q_3^2 + 5q_1^3 - 6q_2q_3,$$

$$u_3 = -q_3^3 - 9q_1^3q_3 + q_1q_3p_2 + q_1q_3p_3 - \tfrac{2}{27}q_1^3q_2 + q_1^2p_1 + 3q_2q_3^2,$$

$$v_1 = -\frac{1}{q_1},$$

$$v_2 = \frac{3q_2 - 2q_3}{q_1},$$

$$v_3 = p_3 + \frac{2}{3}p_2 - \frac{q_3^2}{q_1} + 3\frac{q_2q_3}{q_1} - 4q_1^2,$$

and

$$u_1 = \lambda_1 + \lambda_2 + \lambda_3,$$

$$u_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,$$

$$u_3 = \lambda_1\lambda_2\lambda_3,$$

$$\mu_i = v_1\lambda_i^2 + v_2\lambda_i + v_3, \quad i = 1,2,3.$$

In the (q,p) coordinates the Hamiltonians take the form

$$\begin{aligned}
H_1^{(1)} = {}& p_2p_3 + \tfrac{1}{3}p_2^2 + p_3^2 - 7q_1^2p_3 - 4q_1^2p_2 - 3q_2p_1 + 18q_1q_2^2 \\
&+ 13q_1^4 + 12q_3q_1q_2,
\end{aligned}$$

$$H_1^{(2)} = 12q_1^3q_2 + 8q_1^3q_3 - 2q_1^2p_1 + (-6q_1q_2 - 4q_1q_3)p_3 + p_1p_3,$$

$$\begin{aligned}
H_2^{(2)} = {}& \tfrac{1}{3}p_2p_3^2 + \tfrac{1}{3}p_2^2p_3 + \tfrac{2}{27}p_2^3 - q_1^2p_3^2 - \tfrac{4}{3}q_1^2p_2^2 - q_2p_1p_2 - q_1p_1^2 \\
&- \tfrac{10}{3}q_1^2p_3p_2 + (q_3 - 3q_2)p_1p_3 + (21q_1^2q_2 + 6q_3q_1^2)p_1 \\
&+ \left(4q_3q_1q_2 + 6q_1q_2^2 + \tfrac{22}{3}q_1^4\right)p_2 \\
&+ (7q_1^4 + 18q_1q_2^2 + 6q_3q_1q_2 - 4q_1q_3^2)p_3 - 8q_1^3q_3^2 \\
&- 72q_3q_1^3q_2 - 90q_1^3q_2^2 - 12q_1^6.
\end{aligned}$$

They form a quasi-bi-Hamiltonian chain (16) with the noncanonical Poisson operator

$$\Pi_1 = \begin{pmatrix}
0 & 0 & 0 & -q_3 & 3q_1 & 2q_2 \\
0 & 0 & -\tfrac{1}{3}q_1 & A & 3q_2 - q_3 & -q_2 \\
0 & \tfrac{1}{3}q_1 & 0 & 2q_1^2 & 0 & -q_3 \\
-q_3 & -A & -2q_1^2 & 0 & B & C \\
1 - 3q_1 & -3q_2 + q_3 & 0 & -B & 0 & -24q_1^2 \\
2q_1 & q_2 & q_3 & -C & 24q_1^2 & 0
\end{pmatrix},$$

where $A=-\frac{1}{3}p_2+\frac{1}{3}p_3-3q_1^2$, $B=54q_1q_2+24q_1q_3-3p_1$, $C=-24q_1q_2-12q_1q_3+p_1$, and the control matrix

$$F = \begin{pmatrix}
-q_3 & -q_1 & 0 \\
-\tfrac{1}{3}p_2 + q_1^2 & -2q_3 + 3q_2 & 1 \\
5q_3q_1^2 + 6q_1^2q_2 - q_1p_1 - \tfrac{1}{3}q_3p_2 & -4q_1^3 - q_3^2 + 3q_2q_3 + \tfrac{2}{3}q_1p_2 + q_1p_3 & 0
\end{pmatrix}.$$

On the extended phase space of dimension eight, with additional coordinates c_1, c_2, the extended Hamiltonians (23) are

$$h_0^{(1)} = c_1,$$

$$h_1^{(1)} = H_1^{(1)} + q_3 c_1 + q_1 c_2,$$

$$h_0^{(2)} = c_2,$$

$$h_1^{(2)} = H_1^{(2)} + \left(\tfrac{1}{3}p_2 - q_1^2\right)c_1 + (2q_3 - 3q_2)c_2,$$

$$h_2^{(2)} = H_2^{(2)} - \left(5q_3q_1^2 + 6q_1^2q_2 - q_1p_1 - \tfrac{1}{3}q_3p_2\right)c_1 - \left(-4q_1^3 - q_3^2 + 3q_2q_3 + \tfrac{2}{3}q_1p_2 + q_1p_3\right)c_2.$$

They form two bi-Hamiltonian chains

$$\begin{array}{ll} \pi_0 dh_0^{(1)} = 0, & \pi_0 dh_0^{(2)} = 0, \\ \pi_0 dh_1^{(1)} = X_1^{(1)} = \pi_1 dh_0^{(1)}, & \pi_0 dh_1^{(2)} = X_1^{(2)} = \pi_1 dh_0^{(2)}, \\ 0 = \pi_1 dh_1^{(1)}, & \pi_0 dh_2^{(2)} = X_2^{(2)} = \pi_1 dh_1^{(2)}, \\ & 0 = \pi_1 dh_2^{(2)}, \end{array}$$

with the corresponding Poisson operators π_0 and π_1,

$$\pi_0 = \left(\begin{array}{c|cc} \Pi_0 & 0 & 0 \\ \hline 0 & & \\ 0 & & 0 \end{array}\right), \quad \pi_1 = \left(\begin{array}{c|cc} \Pi_1 & X_1^{(1)} & X_1^{(2)} \\ \hline -(X_1^{(1)})^T & & \\ -(X_1^{(2)})^T & & 0 \end{array}\right).$$

The separation curve for the extended system takes the form

$$\mu(c_1\lambda + h_1^{(1)}) + c_2\lambda^2 + h_1^{(2)}\lambda + h_2^{(2)} = \mu^3.$$

V. SUMMARY

We have considered the Stäckel systems classified using their separation relations. The most general form of the separation relations considered in the present paper is

$$\sum_{k=1}^{m} \varphi_i^k(\mu_i, \lambda_i) H^{(k)}(\lambda_i) = \psi_i(\lambda_i, \mu_i), \quad m \le n, \quad i = 1, \ldots, n,$$

where

$$H^{(k)}(\lambda_i) = \sum_{j=1}^{n_k} \lambda_i^{n_k - j} H_j^{(k)}, \quad n_1 + \cdots + n_m = n,$$

and φ_i^k, ψ_i are smooth functions of their arguments. Moreover, we have proved that all systems whose separation relations are of the above form admit (after the lift to an extended phase space) Gel'fand-Zakharevich bi-Hamiltonian representation. This confirms universality of the latter property for the Stäckel systems. As a consequence, a geometric separability theory, based on the existence of GZ bi-Hamiltonian representation of a given system, is applicable for all Liouville-integrable systems from the classes we considered.

Quite obviously, the knowledge of quasi-bi-Hamiltonian representation is sufficient for separability of a given Liouville-integrable system. Unfortunately, there is no systematic method available for the construction of such representation. On the other hand, there are some systematic methods for finding the bi-Hamiltonian representation. From this point of view the result presented in this paper is of interest, as it shows that the existence of bi-Hamiltonian representation is an inherent property of Stäckel systems.

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