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Painlevé-integrability of a (2+1)-dimensional reaction-diffusion equation: Exact solutions and their interactions

Kuetche Kamgang Victor,[1,2,*] Bouetou Bouetou Thomas,[1,2,3,†] and Timoleon Crepin Kofane[2,3,‡]

[1] *Ecole Nationale Supérieure Polytechnique, University of Yaounde I, P.O. Box. 8390, Cameroon*

[2] *Department of Physics, Faculty of Science, University of Yaounde I, P.O. Box. 812, Cameroon*

[3] *The Abdus Salam International Centre for Theoretical Physics, P.O. Box 586, Strada Costiera, II-34014 Trieste, Italy*

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We investigate the singularity structure analysis of a (2+1)-dimensional coupled nonlinear extension of the reaction-diffusion (NLERD) equation by means of the Painlevé (P) test. Following the Weiss *et al.*'s formalism [J. Math. Phys. **24**, 522 (1983)], we prove the arbitrariness of the expansion coefficients of the observables. Thus, without the use of the Kruskal's simplification, we obtain a Bäcklund transformation of the coupled NLERD equation via a consistent truncation procedure stemming from the Weiss *et al.*'s methodology [J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. **25**, 13 (1984)]. In the wake of such results, we unveil a typical spectrum of localized and periodic coherent patterns. We also investigate the scattering properties of such structures and we unearth two peculiar soliton phenomena, namely, the fusion and the fission.

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I. INTRODUCTION

Nonlinear evolution equations undoubtedly stand for an important tool in characterizing many complicated physical phenomena. Thus, in order to better understand the mechanism of these problems, it would be advantageous to study in detail the solutions to the associated equations. Physics and mathematics constitute the most important fields where many applications of integrable nonlinear equations are found. The search for exact analytical solutions to nonlinear physical models has long been a major concern for both mathematicians and physicists given that they can provide much physical information and more insight into the physical aspects of the problem and this results in further applications. There is a rich theory of such equations which is mostly devoted to the problem of integration and to the study of the underlying algebraic and analytic structures. Among the different approaches used to tackle this problem, are: the P conjecture [1,2], perturbative analyses [3], and approaches based on the existence of higher symmetries and conservation laws, just to name a few.

Over a half century ago, following the numerical experiments of the seminal work of Fermi *et al.* [4], Zabusky and Kruskal [5] discovered that an initial sine wave breaks into a train of stable solitary waves under the flow of the Korteweg-de Vries (KdV) equation. They found that this nonlinear wave collides with other solitary waves without the modification of its initial shape after the interaction process, although it suffers a shift in its phase. They thus coined the term soliton to designate these particlelike waves. Since then, the soliton structures have been widely extended to many nonlinear systems, and they appear as solutions to nonlinear partial differential evolution (NLPDE) equations. These robust objects result from the balance between the nonlinearity and the dispersion (or diffraction). The theory of solitons plays a very important role and has been applied in many natural sciences, especially in condensed-matter physics, field theory, fluid dynamics, etc [6–8]. It has been shown that some simplest NLPDE equations with soliton solutions possess an infinite set of conservation laws [9–11]. However, the full details showing the relationship of infinitely conserved quantities with soliton solutions have been shown to be unclear [12–14] until Wahlquist and Estabrook [15] developed a technique known as "prolongation structure," which has been generalized to nonlinear evolution equations. As applied to the KdV equation, this powerful method which is a set of interrelated potentials and pseudopotentials for NLPDE equations in two independent variables, generates infinite conservation laws leading directly to the soliton solutions, Bäcklund transformation (BT) between solutions, and inverse scattering transform (IST) to the initial value problem [14].

Following such results, Zhai *et al.* [16] recently investigated the integrable (2+1)-dimensional (modified) Heisenberg ferromagnet (HF) model [17] using the prolongation structure theory. The corresponding geometrical equivalent counterparts, such as the (2+1)-dimensional nonlinear Schrödinger equation and the coupled (2+1)-dimensional integrable equations, presented through the motion of Minkowski space curves endowed with an additional spatial variable, have been constructed. These last coupled (2+1)-dimensional integrable equations are given by [16,18]

$$\psi_t + \psi_{xy} + \gamma\psi = 0, \quad \phi_t - \phi_{xy} - \gamma\phi = 0, \quad \gamma_x + (\phi\psi)_y = 0, \tag{1}$$

where ψ, ϕ, and γ are physical observables and subscripts denote partial differentiation. Owing to the miscellaneous geometrical and physical applications of Eq. (1), it is worth investigating such a system both from the viewpoint of its integrability properties and from the viewpoint of the existence of stable excitations.

*vkuetche@yahoo.fr

†tbouetou@yahoo.fr

‡tckofane@yahoo.com

There exist many approaches to tackle the problem of integrability of a given system. In fact, the subject of integrability has considerably grown and progressed since the "early" integrability times of the 70's ([19–23] and references therein). Among such approaches, the Weiss-Tabor-Carnevale (WTC) formalism [1,2] is a powerful technique both for investigating the integrability properties of a given system and for constructing the miscellaneous solutions to such a system. In this way, Tang and Lou [24] used the WTC formalism [1,2] to formulate a "multilinear variable separation approach" (MLVSA) ([25] and references therein), which has been extended and developed for many (2+1)-dimensional integrable systems such as the Davey-Stewartson equation [26,27], the Nizhnik-Novikov-Veselov (NNV) equation [28], and the asymmetric NNV equation [29], just to name a few [30,31].

The fact that Eq. (1) is obtained by the prolongation structure has the merit that it not only satisfies the covariant geometry theory, but also possesses many other properties such as BT, Lax pair, IST. Thus, what we wish to show in this paper is that Eq. (1) exhibits an interesting variety of underlying solutions with different patterns [32–34]. First of all, after presenting the physical motivation of the present investigation, we aim at demonstrating the integrability properties of system (1) by means of the WTC formalism [1,2]. Next, based upon the results of this analysis, we aim at constructing a panel of underlying excitations followed by a study of their scattering properties.

Thus, the paper is organized as follows. In Sec. II, we present the physical motivation of our investigation. Next, in Sec. III, we study the singularity structure analysis of Eq. (1) using the WTC formalism [1,2]. Then, in Sec. IV, we derive the BT and Hirota's bilinearization of Eq. (1). Following these results, in Sec. V, we construct some interesting localized and periodic solutions followed by a study of their scattering behavior in Sec. VI. Finally, in Sec. VII, we end with a brief summary of the work.

II. PHYSICAL MOTIVATION

First of all, it is worth noting that with the transformation $\partial_x=\partial_y$, Eq. (1) straightforwardly reduces to a (1+1)-dimensional coupled NLPDE equation of diffusion type investigated by Nakayama [35], while surveying from the viewpoint of a geometrical approach the motion of curves in hyperboloid in the Minkowski space [35].

The development of highly complex organisms is an intriguing and fascinating problem. The genetic material is the same in each cell of an organism. One question that is worth investigating, therefore, is formulated as follows: How do cells produce spatial patterns under the influence of their common genes? To provide an answer based upon nonlinear interactions of at least two chemicals and on their diffusion, regarding autocatalysis and long-range inhibition as fundamental phenomena, Koch and Meinhardt [36] investigated some simple models of isotropic systems that describe the generation of patterns out of an initially nearly homogeneously state. Owing to the complexity of such biological systems, we aim at shedding light on the genuine anisotropic nature of these systems, which would be very helpful in the understanding of the dynamical behavior of these patterns.

Following the pioneer work of Turing [37] on the interactions of two substances with different rates, Gierer and Meinhardt [38] and independently Seger and Jackson [39] depicted two important features playing a central role in pattern formation: the local self-enhancement and long-range inhibition. Such features give rise to some important typical classification of biological apparatus, namely, the activator-inhibitor systems describing at least stripelike patterns in monkey and zebra and faceted eye of drosophila flies, just to name a few [40], the activator-substrate systems describing at least reticulated dragonflies, animal coat patterns and also Brusselator systems [41–43], and finally the biochemical switches [44–47]. Alongside such classifications, other kinds of interactions are possible, mediated for instance by mechanical forces [48,49], by electric potentials [50,51], or surface contact between cell membrane [52]. Cellular automata are also often used to explain the emergence of inhomogeneous patterns [42]. Nevertheless, chemical interactions coupled with the exchange of molecules are the main motor of primary pattern genesis in biological systems.

Following the seminal work of Koch and Meinhardt [36], taking into account the anisotropic properties of such systems, the models for complex biological pattern formation are described by the following reaction-diffusion equation:

$$U_{kt}=\nabla\cdot\mathbf{J}_k+R_k(U), \tag{2}$$

where the flow $\mathbf{J}_k$ is given by

$$\mathbf{J}_k=\bar{\bar{\mathbf{D}}}_k\,\nabla\,U_k. \tag{3}$$

The quantity $\bar{\bar{\mathbf{D}}}_k(k\in\mathbb{N})$ stands for the diffusion stress which components represent the diffusion coefficients with respect to a specified direction. The quantities $R_k(k\in\mathbb{N})$ are functions of the dynamical fields $U=\{U_k\}_{k\in\mathbb{N}}$ characterizing the nonlinear reactions among them. The gradient operator ∇ is expressed as $\nabla=(\partial_1,\partial_2,\partial_3,\cdots)$ with $\partial_i=\partial/\partial x_i$, $x_i(i\in\mathbb{N})$ being the spacelike coordinates. It is noted that in some cases where all the diffusion coefficients are different, the system can straightforwardly evolve toward an instability known as Turing instability [37] and complicated patterns related to morphogenesis, reaction front dynamics, and self-organization henceforward emerge [53–55]. It is also worth noting that negative diffusion coefficients deserve to be studied both mathematically [56,57] and physically [58]. Indeed, by considering a simple case consisting of two (1+1)-dimensional interacting chemicals with the diffusion stresses given by $\bar{\bar{\mathbf{D}}}_1=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $\bar{\bar{\mathbf{D}}}_2=\left(\begin{smallmatrix}-1&0\\0&0\end{smallmatrix}\right)$, and the nonlinear functions $R_1=2aU_1-2U_1^2U_2$ and $R_2=2U_1U_2^2-2aU_2$, the constant a being a fixed parameter, this system is shown to be gauge equivalent to a (1+1)-dimensional gravity [57], but also emerges as a good model for complex biological organisms such as zebra stripes or butterfly stripes where the nonlinear interacting cells are behaving as damped oscillators from the viewpoint of a thermofield approach [58]. Extending such a study to a (2+1)-dimensional system where the diffusion stresses $\bar{\bar{\mathbf{D}}}_1$ and $\bar{\bar{\mathbf{D}}}_2$ are given by

$$\bar{\bar{\mathbf{D}}}_1=\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix},\quad \bar{\bar{\mathbf{D}}}_2=\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}, \tag{4}$$

and considering the nonlinear functions R_1 and R_2 as

$$R_1=[\partial_2\chi-\partial_1\chi-a(t)]U_1,\quad R_2=[a(t)+\partial_1\chi-\partial_2\chi]U_2, \tag{5}$$

where the spaceless quantity $a\equiv a(t)$ stands for a time-dependent function while the pseudopotential quantity χ is introduced in order to characterize a typical nonlinear interaction between the two observables U_1 and U_2 as $U_1U_2=\partial_1\chi+\partial_2\chi-b(t)$ [the spaceless quantity $b\equiv b(t)$ depending on time], we derive a model valuable for a better understanding of the properties of some natural biological organisms such as zebra stripes, reticulated dragonflies, butterfly stripes, and faceted eye of drosophila flies, and constituting typical nonlinear systems of different kinds such as activator-substrate and activator-inhibitor, just to name a few. Following a variable transformation of the form $x=(x_1+x_2)/2$ and $y=(x_2-x_1)/2$, the system consisting of Eqs. (2)–(5) reduces to Eq. (1), provided $U_1=\psi$, $U_2=\phi$, and $\gamma=a(t)+\partial_1\chi-\partial_2\chi$. Furthermore, the extension toward a (2+1)-dimensional manifold is interesting both from the viewpoint of the investigation of its integrability properties and from the viewpoint of the existence of stable pattern formations.

III. SINGULARITY STRUCTURE ANALYSIS OF THE (2+1)-DIMENSIONAL COUPLED NLERD EQUATION

According to the standard WTC approach [1,2], if Eq. (1) is Painlevé (P) integrable, then all the available solutions to the model is written in the full Laurent series as follows:

$$\psi=\sum_{k=0}^{\infty}\psi_k g^{k+\alpha},\quad \phi=\sum_{k=0}^{\infty}\phi_k g^{k+\beta},\quad \gamma=\sum_{k=0}^{\infty}\gamma_k g^{k+\varsigma}, \tag{6}$$

with sufficient arbitrary functions among ψ_k, ϕ_k, γ_k, and g. The constants α, β, and ς should be negative integers. This means that the previous solutions are written as single-valued expressions among the arbitrary singularity manifold.

The formal way to find the constant α, β, and ς is known as the standard leading-order analysis. Thus, truncating the previous series given by Eq. (6) to the zeroth order, and then replacing them into Eq. (1) such as to compare the leading-order terms for $g\sim 0$, we find only one possible branch

$$\alpha=\beta=-1,\quad \varsigma=-2, \tag{7}$$

and

$$\psi_0\phi_0=2g_x^2,\quad \gamma_0=-2g_xg_y. \tag{8}$$

This implies that one of the three functions ψ_0, ϕ_0, and γ_0 is arbitrary. In general, there is no restriction on the valuedness of these functions. In fact, they can be real- or complex-valued expressions. Nonetheless, throughout the paper, the arbitrary functions shall be regarded as real-valued expressions.

In order to obtain the recursion relations to determine the functions ψ_k, ϕ_k, and γ_k, we substitute Eqs. (6)–(8) into Eq. (1). This leads us to the following algebraic system:

$$\mathcal{M}_k\mathcal{V}_k=\mathcal{T}_k, \tag{9}$$

where $\mathcal{M}_k$ is a square matrix, $\mathcal{V}_k=(\psi_k,\phi_k,\gamma_k)^T$ and $\mathcal{T}_k=(P_k,Q_k,U_k)^T$ with,

$$P_k=-\sum_{j=1}^{k-1}\gamma_{k-j}\psi_j-\psi_{k-2,xy}-\psi_{k-2,t}-(k-2)(\psi_{k-1}g_t+\psi_{k-1,x}g_y+\psi_{k-1,y}g_x+\psi_{k-1}g_{xy}), \tag{10}$$

$$Q_k=-\sum_{j=1}^{k-1}\gamma_{k-j}\phi_j-\phi_{k-2,xy}+\phi_{k-2,t}-(k-2)(-\phi_{k-1}g_t+\phi_{k-1,x}g_y+\phi_{k-1,y}g_x+\phi_{k-1}g_{xy}), \tag{11}$$

and

$$U_k=-\left[\sum_{j=0}^{k-1}(\psi_{k-j-1}\phi_j)_y+(k-2)g_y\sum_{j=1}^{k-1}\psi_j\phi_{k-j}+\gamma_{k-1,x}\right]. \tag{12}$$

The matrix $\mathcal{M}_k$ is given by

$$\mathcal{M}_k=\begin{bmatrix}A_{1k} & A_{2k} & A_{3k}\\ B_{1k} & B_{2k} & B_{3k}\\ C_{1k} & C_{2k} & C_{3k}\end{bmatrix}, \tag{13}$$

with

$$A_{1k}=k(k-3)g_xg_y,\quad A_{2k}=0,\quad A_{3k}=\psi_0, \tag{14}$$

$$B_{1k}=0,\quad B_{2k}=A_{1k},\quad B_{3k}=\phi_0, \tag{15}$$

$$C_{1k}=(k-2)\phi_0g_y,\quad C_{2k}=(k-2)\psi_0g_y,\quad C_{3k}=(k-2)g_x. \tag{16}$$

Thus, the determinant Δ_k of the matrix $\mathcal{M}_k$ is given by

$$\Delta_k=k(k-2)(k-3)(k-4)(k+1)g_y^2g_x^3. \tag{17}$$

The resonances are found at

$$k=-1,0,2,3,4. \tag{18}$$

The resonance at $k=-1$ corresponds to that of the singularity manifold g being arbitrary. If the model is P integrable, we require four resonance conditions at $k=0,2,3,4$, which are satisfied identically such that the other four arbitrary functions among ψ_k, ϕ_k, and γ_k can be introduced into the general series expansion given by Eq. (6). From the leading-order analysis, we know that the resonance at $k=0$ is satisfied identically and one of ψ_0, ϕ_0, and γ_0, is arbitrary. For $k=1$, ψ_1, ϕ_1, and γ_1 are explicitly found as follows:

$$\psi_1=-\frac{2g_xg_t\psi_0+2g_xg_y\psi_{0,x}-\psi_0^2\phi_{0,y}/2+\psi_{0,y}g_x^2}{4g_yg_x^2}, \tag{19a}$$

$$\phi_1=-\frac{2g_xg_y\phi_{0,x}-\phi_0^2\psi_{0,y}/2+\phi_{0,y}g_x^2-2g_xg_t\phi_0}{4g_yg_x^2}, \tag{19b}$$

$$\gamma_1 = 2g_{xy}. \tag{19c}$$

For $k=2$, one of ψ_2, ϕ_2, and γ_2 is arbitrary. For $k=3$, we also find that one of ψ_3, ϕ_3, and γ_3 is arbitrary. Finally, for the resonance at $k=4$, one of ψ_4, ϕ_4, and γ_4 is arbitrary. Consequently, the (2+1)-dimensional coupled NLERD Eq. (1) possesses a sufficient number of arbitrary functions. We conclude that this system is P integrable. Its complete integrability will be established if some essential properties such as the BT and the Hirota's bilinearization [59,60] are derived.

IV. BT AND HIROTA'S BILINEARIZATION OF THE (2+1)-DIMENSIONAL COUPLED NLERD EQUATION

It is well known that the P analysis is also useful in searching for other interesting properties [1,2] of a given system. In this section, we use the truncated P expansion to derive the BT and Hirota's bilinearization [59,60] of the (2+1)-dimensional coupled NLERD Eq. (1). Thus, setting

$$\psi_k = \phi_k = \gamma_{k+1} = 0, \quad k \geq 2, \tag{20}$$

Eq. (6) is transformed as follows:

$$\psi = \psi_0/g + \psi_1, \quad \phi = \phi_0/g + \phi_1, \quad \gamma = \gamma_0/g^2 + \gamma_1/g + \gamma_2. \tag{21}$$

Substituting Eq. (21) into Eq. (1) yields

$$\psi_{0,xy} + \psi_{0,t} + \psi_1\gamma_1 = -\psi_0\gamma_2, \quad \phi_{0,xy} - \phi_{0,t} + \phi_1\gamma_1 = -\phi_0\gamma_2 \tag{22}$$

and

$$\psi_{1,t} + \psi_{1,xy} + \gamma_2\phi_1 = 0, \quad \phi_{1,t} - \phi_{1,xy} - \gamma_2\phi_1 = 0,$$

$$\gamma_{2,x} + (\phi_1\psi_1)_y = 0. \tag{23}$$

From Eq. (23), it follows that $\{\psi_1,\phi_1,\gamma_2\}$ is a solution of the (2+1)-dimensional coupled NLERD Eq. (1). In order words, truncated expansion (21) actually stands for a BT. A seed solution is written as follows:

$$\psi_1 = \phi_1 = 0, \quad \gamma_2 \equiv \gamma_2(y,t). \tag{24}$$

This seed solution is a simple one and is actually useful for constructing many other solutions. For other existing seed solutions, many other classes of solutions are derived. It is that property of the P method for constructing various kinds of solutions by means of arbitrary functions that makes it potentially and powerfully underlying. The solutions are given by the Eq. (21) expressed in a truncated form. Due to the arbitrariness of these functions, many solutions are constructed in a straightforward way, provided to solve analytically or numerically some NLPD constraint equations. Many examples will be given in Sec. V while studying the interactions between such structures.

Using the seed solution given by Eq. (24), Eqs. (19a) and (19b) are written in the following compact form:

$$\mathcal{A}\mathcal{V}_{0,x} + \mathcal{B}\mathcal{V}_{0,y} + \mathcal{C}\mathcal{V}_0 = 0, \tag{25}$$

where $\mathcal{V}_0=(\psi_0,\phi_0)^T$ and

$$\mathcal{A} = \begin{bmatrix} 2g_xg_y & 0 \\ 0 & 2g_xg_y \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} g_x^2 & -\frac{\psi_0^2}{2} \\ -\frac{\phi_0^2}{2} & g_x^2 \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} 2g_xg_t & 0 \\ 0 & -2g_xg_t \end{bmatrix}. \tag{26}$$

Thus, solving Eq. (25) by means of the characteristics method, it yields

$$\mathcal{V}_0 = \mathcal{G}_0\left(x - \int \frac{dy}{\mathcal{A}^{-1}\mathcal{B}}\right)\exp\left(\int \frac{dx}{\mathcal{C}^{-1}\mathcal{A}}\right), \tag{27}$$

where $\mathcal{G}_0$ stands for an arbitrary array function of $(x - \int \frac{dy}{\mathcal{A}^{-1}\mathcal{B}})$ to be determined.

Now, substituting Eqs. (21) and (24) into Eq. (1), the following bilinear system is derived as follows:

$$(D_t + D_xD_y - \nu)\psi_0 \cdot g + \psi_0\gamma_1 = 0, \tag{28a}$$

$$(-D_t + D_xD_y - \nu)\phi_0 \cdot g + \phi_0\gamma_1 = 0, \tag{28b}$$

$$(D_xD_y - \mu)g \cdot g - \gamma_0 = 0, \tag{28c}$$

$$D_x\gamma_0 \cdot g + D_y(\psi_0\phi_0) \cdot g + gD_x\gamma_1 \cdot g = \gamma_0 g_x + \psi_0\phi_0 g_y, \tag{28d}$$

where $\nu-\gamma_2-\mu=0$ and μ, ν, δ, and ϱ stand for arbitrary quantities to be determined. We note that the symbols D_x, D_y, and D_t represent the Hirota's operators [59,60] with respect to the variables x, y, and t. By expanding the functions g, ψ_0, ϕ_0, γ_0, γ_1, and γ_2 as formal power series, and using them in system (28), the one-, two- and N-soliton solutions (N being an integer) to system (1) can straightforwardly be constructed. However, such solutions will be studied in detail in a separate paper. Knowing that the BT of system (1) has been found and its related Hirota's bilinearization derived, we conclude that the (2+1)-dimensional coupled NLERD Eq. (1) is completely integrable. This confirms the power of the "prolongation structure" coined by Wahlquist and Estabrook [15], establishing the integrability properties of a given NLPDE equation. Nonetheless, the usefulness of the WTC formalism also stems from its allowance to construct interesting solutions based upon the previous results of the P analysis.

Now, owing to the arbitrariness of some functions derived from the P analysis, there are many interesting solutions to investigate. Thus, in Secs. V and VI, we aim at focusing our interest to solutions for which the quantities $\psi\phi$ and γ are expressed as follows:

$$\psi\phi = 2(\partial_x \ln|g|)^2, \quad \gamma = \gamma_2 + \frac{D_xD_yg \cdot g}{g^2}, \tag{29}$$

and which stem from Eqs. (8) and (21).

From a seed solution to system (23), as given by Eq. (24), it is seen that this solution does not depend on whether γ_1 is different from zero or not. Thus, by setting $\gamma_1=0$ and using Eq. (19), one gets $g_{xy}=0$, which shows that g is the sum of two arbitrary functions $g_1 \equiv g_1(x,t)$ and $g_2 \equiv g_2(y,t)$. Now,

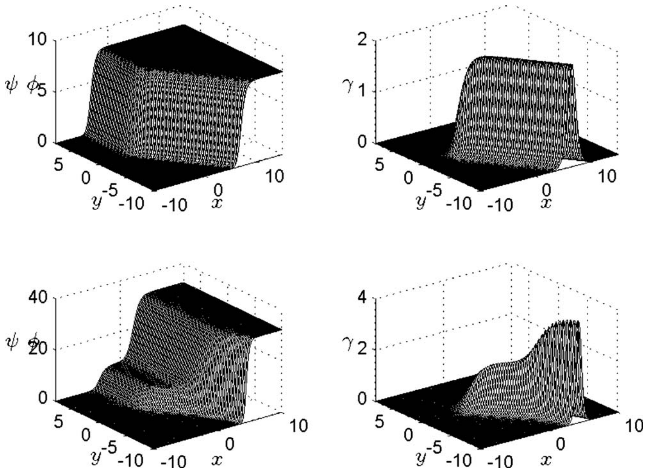

FIG. 1. Typical half-straight-line soliton and kinks depicted through γ and $\psi\phi$, respectively, under the following selections: $p=\exp(2x+7)$ and $q=\exp(-2y)$ for the upper panels; $p=\exp(2x+7)+\exp(4x+3)$ and $q=\exp(-2y)+\exp(-y)$ for lower panels. $a_0=1$, $a_1=a_2=1$, and $a_3=0$.

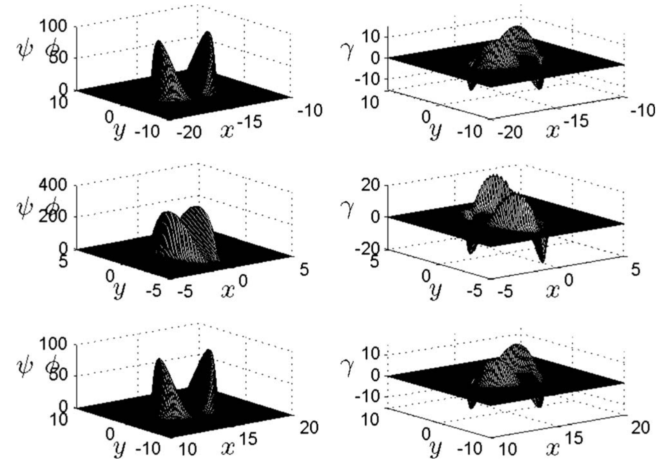

FIG. 2. Typical breatherlike ring soliton and another type of breatherlike pattern depicted by γ and $\psi\phi$, respectively. These plots correspond to the following selection: $p=\exp\{-[(\cos(t)+4/3)x-20\sin(t)]^2+5\}$ and $q=\exp(y^2-5)$. $a_0=0$, $a_1=1$, $a_2=1$, and $a_3=0$. For upper panels, $t=-\pi/2$, for middle panels, $t=0$ and for lower panels, $t=\pi/2$.

considering the case where $\gamma_1\neq 0$, and searching for a class of solutions generalizing the previous ones such that $\gamma_1=f_1(x,t)f_2(y,t)$ with $f_1(x,t)$ and $f_2(y,t)$ being arbitrary functions, from Eq. (19), the function g is expressed as follows:

$$g=a_0+a_1p+a_2q+a_3pq, \tag{30}$$

where $p=p(x,t)$ and $q=q(y,t)$ stand for arbitrary functions and the parameters $a_i(i=0,1,2,3)$ are arbitrary constants. With the above form of g given by Eq. (30), we combine the two Eqs. (22) and (27) in order to get a nonlinear system expressed in terms of $\mathcal{G}_0$ and γ_2, which can be solved analytically or numerically. For simplicity, it is interesting to take $\gamma_2=0$ as considered in Sec. V while investigating the scattering behavior of some localized excitations.

V. DISCUSSION OF THE LOCALIZED AND PERIODIC EXCITATIONS TO THE (2+1)-DIMENSIONAL COUPLED NLERD EQUATION

First of all, let us concentrate on time-independent functions $p\equiv p(x)$ and $q\equiv q(y)$. The first interesting choice we make is given by

$$p=\sum_{k=1}^{n} p_k\exp(\alpha_k x+x_{0k}),\quad q=\sum_{k=1}^{m} q_k\exp(\beta_k y+y_{0k}), \tag{31}$$

where $m,n\in\mathbb{N}$, and p_k, α_k, x_{0k} $(k=1,\cdots,n)$, q_k, β_k, and y_{0k} $(k=1,\cdots,m)$ are arbitrary constants. Under the selections $(n=1,m=1)$ and $(n=2,m=2)$ the detail of which is presented in the caption of Fig. 1, we depict a typical structure of two static half-straight-line solitons for the quantity γ. Owing to the resonance effect, the two half-straight-line solitons can overlap and become only one half-straight-line soliton well known in the literature [61–63]. Correspondingly, under the previous selections, $\psi\phi$ follows a typical kink presented in Fig. 1 as a concatenation of two or three kinklike waves. Many other structures can also be found by choosing other values for integers n and m.

Another type of important nonlinear excitation is the breather solution [25,29]. Because of the arbitrariness of the functions p and q, we can include some periodic-time functions in the different selections. There are many different ways to construct breather solutions. We make a simple selection as presented in Fig. 2, where the details are provided in the caption. From this figure, the amplitude of the γ-breather soliton varies from ~ 13 to ~ 24, the radius in x direction from ~ 5 to ~ 3, and the center from ~ -15 to ~ 15. Besides, amplitude of the $\psi\phi$-breather soliton varies from ~ 90 to ~ 275 and the center from ~ -15 to ~ 15.

Among the family class of periodic excitations, many can be constructed by different techniques. For example, by choosing Jacobi elliptic functions, namely, sn and cn functions. It is worth noting that the Jacobi transformation implies that any solution found by one Jacobi elliptic function can be transformed into equivalent one that is obtained by another. First of all, we express the multiperiodic solutions as follows:

$$p=\sum_{k=1}^{k=N} A_k\,\mathrm{sn}(\xi_k|m_k),\quad q=\sum_{k=1}^{k=M} B_k\,\mathrm{sn}(\eta_k|n_k), \tag{32}$$

where $\xi_k=\alpha_k x+\beta_k t$ and $\eta_k=\gamma_k y+\delta_k t$, and A_k, B_k, α_k, β_k, γ_k, and δ_k being arbitrary constants. Taking $N=1$ and $M=1$, we construct the following doubly-periodic waves:

$$p=\mathrm{sn}(\xi|m),\quad q=\mathrm{sn}(\eta|n), \tag{33}$$

with $\xi=\xi_1=2x+t$, $\eta=\eta_1=y+4t$, $m=m_1=0.95$, and $n=n_1=0.97$. This type of doubly periodic wave is depicted in Fig. 3 at $t=0$ where we have taken $a_0=4$.

Another interesting periodic wave is the y-periodic wave shaped as a worm and constructed as follows:

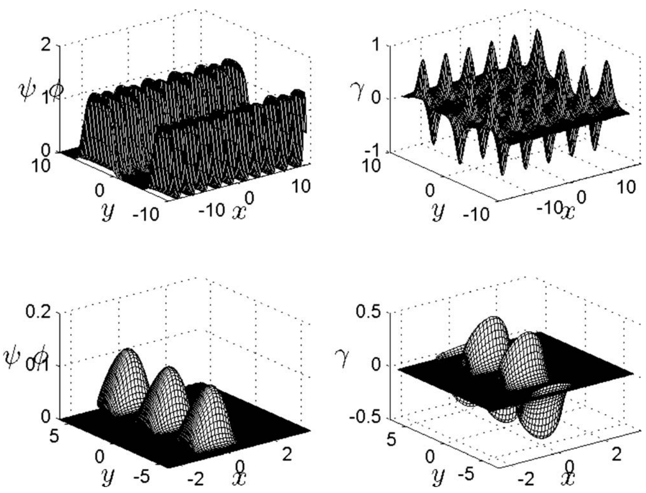

FIG. 3. Doubly periodic waves (upper panels) and typical y-periodic pattern (lower panels) depicted by γ and $\psi\phi$ at initial time.

$$p_x = -\operatorname{sech}^2(\xi - t), \quad q = \operatorname{sn}(\eta|n), \tag{34}$$

with

$$x = \xi + k_4 \tanh(\xi - t), \quad \eta = 2(y + 2t). \tag{35}$$

This periodic wave is depicted in Fig. 3 at $t=0$ with $k_4 = -2.5$, $n=0.8$, and $a_0=8$. It should be noted that all periodic waves do not straightforwardly have elastic properties. However, many of them retain their properties such as amplitudes, velocities, and shapes, just to name a few, and remain unchanged after interactions. In Sec. VI, we shall present the conditions under which the elastic interactions occur. It is actually worth noting that many other underlying solutions can be constructed in a somewhat endless series of figures. This is due to the arbitrariness of the functions p and q making the WTC method an underlying and powerful technique.

VI. SCATTERING BEHAVIOR: ELASTIC AND NONELASTIC INTERACTION

First of all, it is important to present the asymptotic behaviors of the localized excitations derived from Eq. (29). Recently, Tang *et al.* [30] studied the elastic and nonelastic interaction between some localized waves such as special types ring solitons and peakons. Also, Lou [64] proposed a method for constructing solitary waves with compact support on the basis of the universal formula from the MLVSA method, and this method has been extended to generate (2+1)-dimensional solitary waves and solitons, namely, plateau-type, basin-type, and bowl-type ring solitons for the (2+1)-dimensional sine-Gordon equation [65]. Besides, Tang and Lou [66] provided an interesting way for the construction of foldons (folded solitary waves with soliton structure). In this section, we extend such an investigation to the (2+1)-dimensional coupled NLERD Eq. (1).

We assume that $F_i(\omega_i) \equiv F_i(\xi - v_i t) \equiv \int P_i dx|_{\omega_i \to \pm\infty} \to F_i^{\pm}$ and ω_i are invariant as $t \to \infty$. We consider that q and $a_i (i=0,\cdots,3)$ are time independent. At the ith excitation, the interaction properties among the localized excitations are described by the following equations:

$$\gamma_{t\to\mp\infty} \to \sum_{i=1}^{N} 2 \frac{A_i}{(a_0 + a_2 q(y) + [a_1 + a_3 q(y)][F_i(\omega_i) + \Omega_i^{\mp}])^2}, \tag{36}$$

$$\psi\phi|_{t\to\mp\infty} \to \sum_{i=1}^{N} 2 \left\{ \frac{P_i(\omega_i)[a_1 + a_3 q(y)]}{a_0 + a_2 q(y) + [a_1 + a_3 q(y)][F_i(\omega_i) + \Omega_i^{\mp}]} \right\}^2, \tag{37}$$

$$x|_{t\to\mp\infty} \to \xi + \Gamma_i^{\mp} + X_i(\xi - v_i t), \tag{38}$$

where

$$A_i = -[(a_1 + a_3 q)(a_2 + a_3[F_i(\omega_i) + \Omega_i^{\mp}]) \times (a_0 + a_2 q + [a_1 + a_3 q(y)][F_i(\omega_i) + \Omega_i^{\mp}]) - a_3] q_y P_i(\omega_i), \tag{39}$$

$$\Omega_i^{\mp} = \sum_{j<i} F_j^{\mp} + \sum_{j>i} F_j^{\mp}, \quad \Gamma_i^{\mp} = \sum_{j<i} G_j^{\mp} + \sum_{j>i} G_j^{\mp}. \tag{40}$$

Thus, the ith excitation preserves its shape if $\Omega_i^{+} = \Omega_i^{-}$, and its total phase shift is $\Gamma_i^{+} - \Gamma_i^{-}$. Therefore, in order to construct completely elastic interaction properties, it is suggested to select suitable localized functions F_i (and then P_i) such that $\Omega_i^{+} = \Omega_i^{-} (i=1,\ldots,N)$. Multiple (2+1)-dimensional localized solitonic excitations with completely elastic interaction properties are then built-up from the (1+1)-dimensional multiple localized excitations, provided the above properties are satisfied. For example, one can derive multiple folded solitary waves ($\Omega_i^{+} \neq \Omega_i^{-}$, at least for one i) or multiple foldons ($\Omega_i^{+} = \Omega_i^{-}$ for all i) from the (1+1)-dimensional localized multivalued functions generating loop solitons [67,68].

As an illustration, we first consider interactions among special exponentially decaying ring solitons and half-straight-line solitons, whose expressions are derived from

$$p = \tfrac{1}{3}\sin^{-1}[\tanh(x - 4t)] + \tfrac{1}{3}\sin^{-1}[\tanh(x)],$$

$$q = \tfrac{1}{3}\sin^{-1}[\tanh(y)]. \tag{41}$$

These features are depicted in Fig. 4 and it is seen that these structures are stationary along the y axis and moving along the x axis. Thus, initially along the x axis, the solitary waves are located at $x \sim -11.6$ and $x \sim 0$, respectively, at time $t = -3$. Then, during the interaction, they merge at time $t=0$ at $x \sim 0$ to form a single entity. At time $t=3$, after they exchange their amplitudes, the two initial structures recover their initial properties such as velocities and shapes, just to name a few, with corresponding location at $x \sim 0$ and $x \sim 11.6$, respectively.

Alongside the elastic interactions, there are scattering among some typical y-periodic waves which are written as follows:

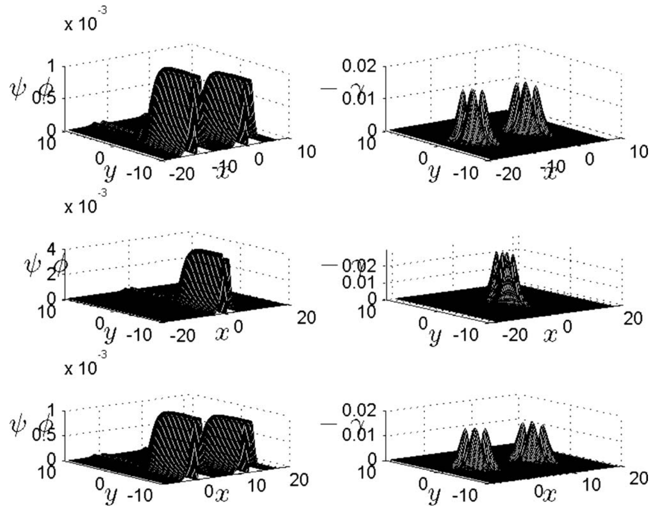

FIG. 4. Elastic interaction between special exponentially decaying ring solitons and typical quasi-half-straight-line solitons depicted by $-\gamma$ and $\psi\phi$, respectively. For upper panels, $t=-3$, for middle panels, $t=0$ and for lower panels, $t=3$.

$$p = 1.2\,\mathrm{sech}^2(\xi) + 0.8\,\mathrm{sech}^2(\xi - 0.25t), \quad q = \mathrm{sn}(\eta|0.8), \tag{42}$$

where

$$x = \xi - 1.5\tanh(\xi) - 1.5\tanh(\xi - 0.25t), \quad \eta = y + 4t. \tag{43}$$

As depicted in Fig. 5, the initial waves located at $x\sim-3$ and $x\sim-1.5$, respectively, at time $t=-18$ interact in a peculiar process where the amplitude of the higher structure decreases whereas the small one increases its amplitude. After the scattering process, at time $t=18$, they recover their velocities and shapes, but with a small modification of their initial amplitudes. At that time, their locations merged to $x\sim3$ and $x\sim1.5$, respectively. This kind of attractive interaction is regarded as approximately elastic even though there may be some exchange of energy.

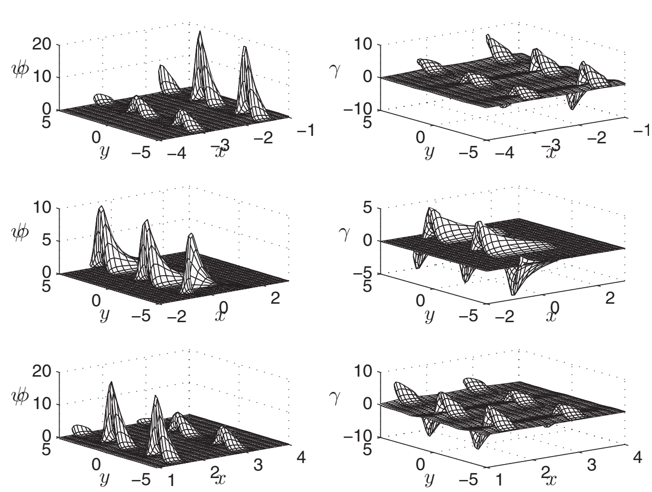

FIG. 5. Quasielastic interaction between typical periodic patterns depicted by γ and $\psi\phi$. For upper panels, $t=-18$, for middle panels, $t=3$ and for lower panels, $t=18$.

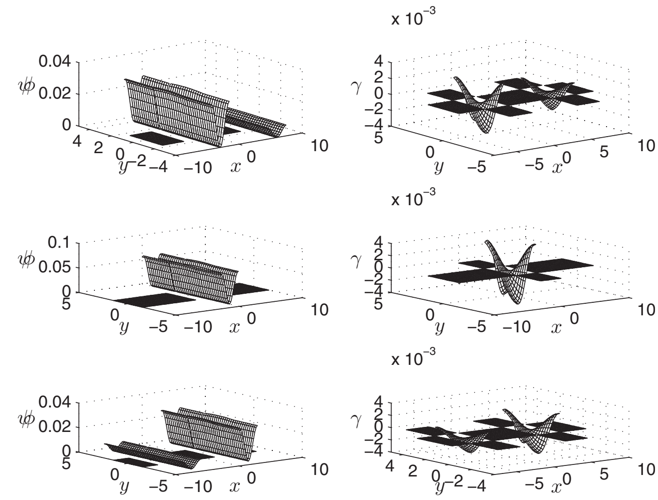

FIG. 6. Elastic interaction among two typical pattern formations depicted by $\psi\phi$ and γ under the following selection: $R_1^p(\varphi_1)=-2\cos(\varphi_1)-2$, $R_2^p(\varphi_2)=-\cos(\varphi_2)-1$, and $R^q(\phi)=\cos(\phi)+1$ with $\varphi_1=x-t$, $\varphi_2=x+2t$, $\phi=y$, $\varphi_{11}=\varphi_{12}=\phi_1=-\pi/2$, and $\varphi_{21}=\varphi_{22}=\phi_2=\pi/2$. $a_0=17$, $a_1=1$, $a_2=1$, and $a_3=1/14$. For upper panels, $t=-3$, for middle panels, $t=0$ and for lower panels, $t=3$.

Among other interesting excitations, there are solitons with compact support which are derived from the following equation [25]:

$$p = \sum_{i=1}^{M} \begin{cases} 0, & \varphi_i \le \varphi_{1i} \\ R_i^p(\varphi_i) - R_i^p(\varphi_{1i}), & \varphi_{1i} < \varphi_{2i} \\ R_i^p(\varphi_{2i}) - R_i^p(\varphi_{1i}), & \varphi_{2i} < \varphi_i, \end{cases}$$

$$q = \sum_{i=1}^{N} \begin{cases} 0, & \phi_i \le \phi_{1i} \\ R_i^q(\phi_i) - R_i^q(\phi_{1i}), & \phi_{1i} < \phi_{2i} \\ R_i^q(\phi_{2i}) - R_i^q(\phi_{1i}), & \phi_{2i} < \phi_i, \end{cases} \tag{44}$$

where $\varphi_i=x-c_it(i=1,\cdots,M)$ and $\phi_i=y-v_it(i=1,\cdots,M)$, c_i and v_i being arbitrary constants standing for velocities of waves. $R_i^p(\varphi_i)(i=1,\cdots,M)$ and $R_i^q(\phi_i)(i=1,\cdots,N)$ are differentiable functions yielding many kinds of (2+1)-dimensional solitons with compact support. We consider one kind of such waves provided in the caption of Fig. 6. The interaction is utterly elastic such that the two initial structures which are localized at $x\sim-3$ and $x\sim6$ at time $t=-3$, retain their shapes after the scattering process in such a way that at time $t=3$, their locations now becomes $x\sim3$ and $x\sim-6$, respectively. During this attractive interaction, the two structures merge at $t=0$ at location $x\sim0$ to give a single soliton with compact support with higher amplitude.

Moreover, there is still too much to deal with interactions in (2+1)-dimensional systems, thanks to the arbitrariness of the functions p and q. As usual, when two soliton structures interact mutually, in general, they attract or repel each other. It is rather rare to observe a coalescence phenomenon merging to a resonance process. Among the fauna of exotic localized excitation solutions to the (2+1)-dimensional coupled

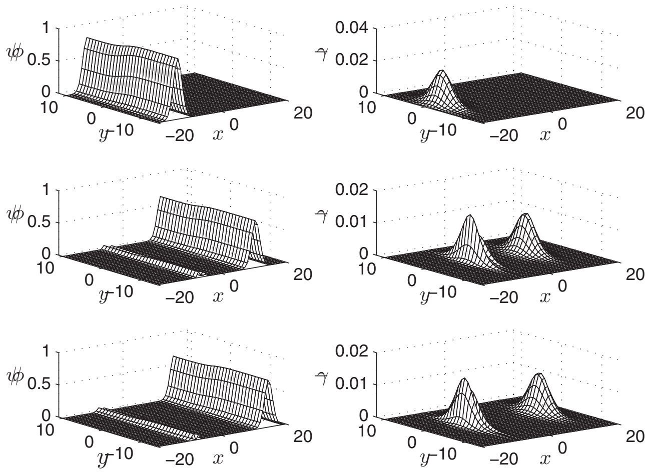

FIG. 7. Fission phenomenon depicted by $\psi\phi$ and $-\gamma$, respectively. For upper panels, $t=-3$, for middle panels, $t=3$ and for lower panels, $t=4$.

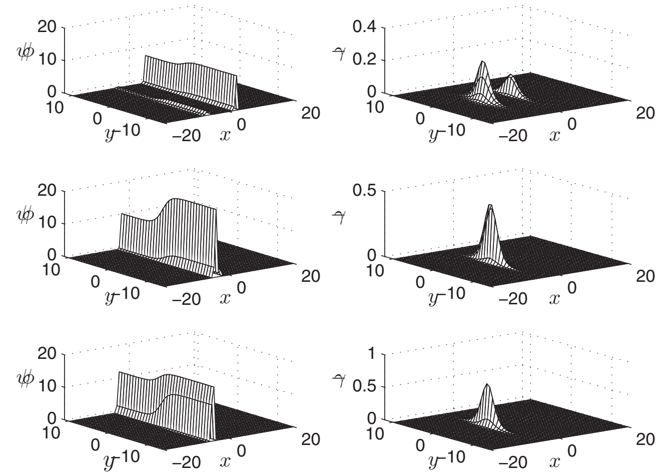

FIG. 8. Fusion phenomenon depicted by $\psi\phi$ and $-\gamma$, respectively. For upper panels, $t=-3$, for middle panels, $t=3$ and for lower panels, $t=4$.

NLERD Eq. (1), by choosing some appropriate parameters, two peculiar soliton phenomena are observed, which we may call "fission" and "fusion." Indeed, such phenomena have earlier been investigated in a number of systems such as the (2+1)-dimensional generalized Sasa-Satsuma equation [69] and the Burgers and Sharma-Tasso-Olver equations ([70] and references therein), just to name a few. Let us consider the following expressions of arbitrary functions p and q [69]:

$$p = \alpha \tanh(k_1 x + \omega_1 t) + \beta \tanh^3(k_2 x + \omega_2 t)/2,$$

$$q = \gamma \tanh(k_3 y), \tag{45}$$

such that with the following parameters: $\alpha=18$, $\beta=1/2$, $\gamma=1/3$, $k_1=1/2$, $k_2=1$, $k_3=1/3$, $\omega_1=-2$, and $\omega_2=2$ with $a_0=20$, $a_1=1$, $a_2=1$, and $a_3=1/50$, the fission phenomenon is depicted. In Fig. 7, it is seen that the initial solitary wave moves toward positive x direction and then splits into two other solitary waves with different amplitudes which repel each other. On the contrary, while trying the following parameters: $\alpha=1$, $\beta=18$, $\gamma=1$, $k_1=1$, $k_2=2$, $k_3=1/2$, $\omega_1=2$, and $\omega_2=-2$ with $a_0=20$, $a_1=1$, $a_2=1$, and $a_3=1/50$, a fusion phenomenon appears. As presented in Fig. 8, two initial x-moving solitary waves with different amplitudes coalesce together merging to a single solitary wave.

VII. SUMMARY

In this paper, we have investigated the P property of the (2+1)-dimensional coupled NLERD Eq. (1) and proven that it is completely integrable as revealed earlier also by Zhai *et al.* [16], following a the "prolongation structure" analysis originally due to Wahlquist and Estabrook [15]. We have then judiciously made use of the truncated P analysis to derive Hirota's bilinearization [59,60] of the system prior to a construction of some soliton solutions. Besides, based upon the arbitrariness of some basic functions, we have constructed a wide class of localized and periodic single-valued and multivalued excitations. The interaction between half-straight-line solitons presents different features. By selecting appropriate functions, we have constructed a such structure and a typical kink structure of two or three kinklike waves. Another type of localized breather has been constructed. We were interested in particular in some kind of breathing waves between times $t=-\pi/2$ and $t=\pi/2$. By using Jacobi elliptic functions, we have constructed some special periodic waves. Due to the arbitrariness of the functions p and q, there is still many more possible excitation solutions to the (2+1)-dimensional coupled NLERD Eq. (1) that can be produced.

More interesting perhaps, is the soliton structure of the above excitations. We presented a systematic way of assessing the soliton structure of a solitary wave solution to the (2+1)-dimensional coupled NLERD Eq. (1). While choosing appropriate arbitrary functions, we investigated the scattering properties of solitons with compact support and also typical half-straight-line structures. As a result, they retain their shapes and velocities after the interaction. We also investigated the interaction between periodic multivalued waves with "worm"-like shape generated by Jacobi elliptic functions. This interaction (even though suffering some peculiarities at the coalescence area) is considered as elastic since these periodic waves recover their initial velocities and shapes after the scattering process. It may also be attractive since the initial waves pass through each other during the interaction process. This kind of attractive scattering is also investigated in the scattering behavior of two other kinds of waves with compact support.

Moreover, when two waves interact, they do not only repel or pass through each other. They can also coalesce to form a single moving wave. This is a fusion phenomenon. Besides, when a single solitary wave moves, a peculiar phenomenon sometimes appears where the initial wave splits into two, three, or more other similar waves, but with different amplitudes. This kind of process is termed as fission. We have shown that the (2+1)-dimensional coupled NLERD Eq. (1) possesses these kinds of soliton phenomena. As illustration, we have shown that an initial exponentially decaying ring soliton can split into two baby solitons repelling each other further. Besides, two exponentially decaying ring solitons with different amplitudes can fuse together to form a single moving structure.

Although the standard WTC's P expansion method used in this paper is distinguished among other techniques for efficiently investigating integrability properties and finding some exact solutions to nonlinear systems, it is still not easy to find physically significant nonsingular localized solutions to some model systems. Thus, Conte [71] developed an alternative P-analysis approach, the invariant P analysis for such models. A modification of the truncated Conte's expansion has been presented by Pickering [72,73], and generalized further by Lou [74–76]. He showed that the standard and nonstandard truncations of the generalized P expansion lead to the construction of explicit exact solutions. As a future perspective, it would be worth applying this generalized method to the (2+1)-dimensional coupled NLERD Eq. (1) to find more significant solutions in order to further understand the dynamical behavior of system (1). It is also important to mention here that the integrability properties of a given system do not necessarily imply that all solutions scatter through an elastic process. In fact, when a system is shown to be integrable, that means it possesses an infinite set of conserved quantities. However, the conservation of some quantities such as the momentum and the amplitude are not straightforward. Such results have been revealed recently through many investigations. As an example, nonelastic interaction have been found recently, while investigating some (2+1)-dimensional integrable systems such as the Boiti-Leon-Pempinelli equation, the Nizhnik-Novikov-Veselov (NNV) system, dispersive long wave equation, or the modified NNV system [24,77,78].

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