Comment on "Ising model on the scale-free network with a Cayley-tree-like structure"

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A clarification is due about the paper by Hasegawa and Nemoto [Phys. Rev. E **75**, 026105 (2007)], where a clear distinction between the Zeta and Zipf power-law distributions offers an alternative interpretation of the behavior of susceptibility of the model at hand. More precisely, their conclusion that susceptibility diverges for this scale-free network model with power-law distribution $P(k) \sim k^{-\gamma}$ for the coordination number k for all temperatures, for values of exponent $\gamma \leq 4$ (as observed in real networks), stems from the (infinite domain) Zeta distribution power-law assumption for the coordination number distribution. On the other hand, by assuming the Zipf power-law distribution (with an arbitrary finite upper bound on the coordination number), the susceptibility is well behaved, diverges in the interval $0 \leq T < T_S$, and is finite for $T \geq T_S$, where T_S depends on $P(k)$.

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In a recent work Hasegawa and Nemoto [1] have derived exact closed-form expressions for magnetization and zero-field susceptibility of the Ising model on Cayley-tree-like structures with arbitrary distribution of the tree coordination number by generalizing the approach presented in [2] for the regular Cayley tree. Their formulas are quite general, expressed in terms of the first three moments of the coordination number k. They further apply these expressions to analyze behavior of susceptibility on scale-free (SF) networks with power-law distribution of the coordination number $P(k) \sim k^{-\gamma}$, and arrive at conclusion that, for $\gamma > 4$, susceptibility diverges below temperature T_S given by $\tanh^2(J/k_B T_S) = \langle k \rangle / \langle k(k-1) \rangle$ (where J is the nearest-neighbor ferromagnetic interaction parameter, and k_B is the Boltzmann constant), and remains finite above T_S while for $\gamma \leq 4$ susceptibility diverges at all temperatures. In what follows we offer an alternative interpretation pertinent to scale-free networks by assuming a finite domain of the power-law distribution.

The general expression for zero-field susceptibility of a Cayley-tree-like structure of radius n, derived by Hasegawa and Nemoto [1], is given by

$$\chi_n = \frac{c(1-t^2)}{k_B T}\left(\frac{\alpha^2 t^2 + (\alpha - \eta)t^2 - 1}{(\alpha t - 1)^2(\alpha t^2 - 1)} \frac{\alpha^n - 1}{\alpha - 1} + \frac{2\alpha^{n-1} t(\eta t - \alpha^2 t - \alpha t + \alpha)}{(\alpha t - 1)^2(\alpha t^2 - 1)} \frac{t^n - 1}{t - 1} + \frac{(\alpha^2 - \eta)(t+1)\alpha^{n-1} t^2}{(\alpha - 1)(t-1)(\alpha t^2 - 1)} \frac{t^{2n} - 1}{t^2 - 1} + \frac{(\eta - \alpha)\alpha^n t^2}{(\alpha t - 1)^2(\alpha - 1)} \frac{(\alpha t^2)^n - 1}{\alpha t^2 - 1} \right) \Bigg/ \left(1 + c\frac{\alpha^n - 1}{\alpha - 1}\right), \tag{1}$$

where

$$c = \langle k \rangle, \quad \alpha = \frac{\langle k^2 \rangle}{\langle k \rangle} - 1, \quad \eta = \frac{\langle k^3 \rangle - \langle k^2 \rangle}{\langle k \rangle}, \tag{2}$$

and $\langle k \rangle$, $\langle k^2 \rangle$, and $\langle k^3 \rangle$ are the first three moments of the coordination number, following (an arbitrary) distribution $P(k)$. Setting $P(k) = \delta(k-3)$, where δ is the Kronecker delta function, recovers results [2] for the regular Cayley tree with (constant) coordination number $k=3$ while setting $P(k) \sim k^{-\gamma}$ corresponds to scale-free networks.

The crux of the matter in the SF network case is the choice of the domain of the probability distribution. If one opts for the infinite domain (as done in [1] for the analysis of the zero-field susceptibility behavior), the coordination number is drawn from the Zeta distribution with the density mass function $f(k;\gamma) = k^{-\gamma}/\zeta(\gamma)$ and infinite domain $k \in \{1, 2, \ldots, \infty\}$, where $\zeta(\gamma)$ is the Riemann zeta function. In this case however, the first moment is defined only for $\gamma > 2$, the second moment for $\gamma > 3$, and the third moment for $\gamma > 4$, and the conclusion of Hasegawa and Nemoto [1] about the zero-field susceptibility stems precisely from this assumption.

On the other hand, for description of real networks it appears more reasonable to choose a finite domain by setting an (arbitrary) upper bound K on the coordination number k, corresponding to the choice of the Zipf distribution with the probability mass function $f(k;\gamma,K) = k^{-\gamma}/H_{K,\gamma}$ and the finite domain $k \in \{1, 2, \ldots, K\}$, where $H_{K,\gamma} = \sum_{k=1}^{K} k^{-\gamma}$ is the Kth generalized harmonic number of order γ. In this case all the moments are finite, and Hasegawa and Nemoto [1] in fact use this option to produce their Fig. 2 by setting maximum degree to $K=40$.

Furthermore, by taking the limit $n \to \infty$, it is simple to show that susceptibility diverges due to diverging radius only (there are no intrinsic temperature induced critical points), and the limiting behavior is described by the formula

$$\lim_{n\to\infty} \frac{\ln(\chi_n)}{n} = \begin{cases} \ln(\alpha t^2), & T < T_S \\ 0, & T \geq T_S \end{cases} \tag{3}$$

where T_S is determined by $\alpha \tanh^2(J/k_B T_S) = 1$. We demonstrate this scaling behavior on Fig. 1, which should be com-

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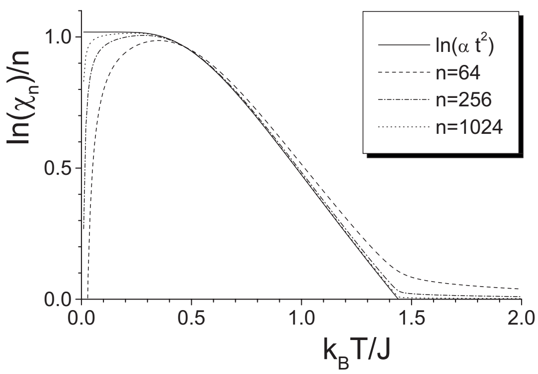

FIG. 1. Scaled zero-field susceptibility for $\gamma=2.7$, calculated using formula (1), for several network sizes $n=64,256,1024$. The full line represents the limiting curve $\ln(\alpha t^2)$.

pared with Fig. 2 of Hasegawa and Nemoto [1] (we use linear temperature scale as opposed to logarithmic, and we divide the logarithm of susceptibility by radius n). It should be noted that the observed maxima represent a finite-size effect that fades away as the thermodynamic limit is approached but may be relevant for large finite-size networks.

In summary, this Comment is meant to clarify some conclusions drawn by Hasegawa and Nemoto [1] on the zero-field susceptibility behavior of the Ising model on scale-free networks. There is nothing special about $\gamma \leq 4$ for scale-free networks except the fact that infinite domain power-law assumption lacks the definition of the third moment. Assuming a finite (arbitrarily large) domain removes this problem, and the susceptibility is shown to be well behaved, with finite-size scaling governed by the network radius.

[1] T. Hasegawa and K. Nemoto, Phys. Rev. E **75**, 026105 (2007).

[2] T. Stošić, B. Stošić, and I. P. Fittipaldi, J. Magn. Magn. Mater. **177-181**, 185 (1998).