PHYSICAL REVIEW E **79**, 041130 (2009)

Simplest nonequilibrium phase transition into an absorbing state

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(Received 4 January 2009; published 21 April 2009)

We study in further detail particle models displaying a boundary-induced absorbing state phase transition [Deloubrière and van Wijland Phys. Rev. E **65**, 046104 (2002) and Barato and Hinrichsen, Phys. Rev. Lett. **100**, 165701 (2008)]. These are one-dimensional systems consisting of a single site (the boundary) where creation and annihilation of particles occur, and a bulk where particles move diffusively. We study different versions of these models and confirm that, except for one exactly solvable bosonic variant exhibiting a discontinuous transition and trivial exponents, all the others display nontrivial behavior, with critical exponents differing from their mean-field values, representing a universality class. Finally, the relation of these systems with a (0+1)-dimensional non-Markovian process is discussed.

DOI: 10.1103/PhysRevE.79.041130 PACS number(s): 64.60.Ht, 68.35.Rh, 64.70.−p

I. INTRODUCTION

Phase transitions occurring in the bulk, but driven by specific conditions at its boundaries, are called *boundary-induced phase transitions* [1]. Examples include diffusive transport [2,3] and traffic flow [4] models. A simple example for this is provided by the one-dimensional totally asymmetric simple exclusion process [5], where particles enter the system at the left boundary, jump to the right in the bulk, and exit at the right boundary. Depending on the entering and exiting rate values, the system exhibits qualitatively different phenomenologies (maximal current, large current and low density, or small current and high density), with straightforward applications to traffic flow problems.

In the present work, we are interested in boundary-induced phase transitions in systems with absorbing states. An absorbing state is a dynamical trap which can be accessed but cannot be left [6–8]. Systems with absorbing phase transitions are controlled by a parameter, depending on which the system either enters the absorbing state with certainty or survives in a stationary fluctuating or active state. The most prominent family of phase transitions into an absorbing state is the very robust direct percolation (DP) universality class. A recent breakthrough has been the experimental observation of DP critical behavior for the first time [9].

A paradigmatic model in the DP universality class is the contact process (CP) [10]. It can be viewed as a simple model for the propagation of a disease where sick individuals can infect healthy neighbors or become healthy spontaneously. More precisely, in the CP in d spatial dimensions, a particle (infected individual) can be created at an "empty" site with a rate $\lambda n/2d$, where n is the number of nearest neighbors occupied by a particle, and an occupied site can become empty at rate 1. The empty configuration is an absorbing state. For λ larger than a certain critical threshold, λ_c, the process is able to sustain (in an infinite lattice) a nonvanishing density of particles, while for $\lambda<\lambda_c$ the dynamics ends up, ineluctably, in the absorbing state.

As continuous phase transitions involve long-range correlations, boundary effects may play an important role. In the context of absorbing phase transitions, previous studies focused primarily on DP confined to parabolas [11,12], active walls [13], as well as absorbing walls and edges [14,15]. Although such boundaries influence the dynamics deep into the bulk, the universality class of the bulk transition is not inherently changed; rather it is extended by an additional independent exponent describing the order parameter near the boundary. Therefore, the question arises whether it is possible to find boundary-induced absorbing phase transitions, absent in the corresponding systems without boundaries, constituting independent universality classes.

In this paper we present a detailed discussion of two slightly different models introduced in Refs. [16,17], respectively. Both of them exhibit a boundary-induced nonequilibrium phase transition into an absorbing state. These are one-dimensional particle systems consisting of a single site (and, at most, its nearest neighbor), where creation and annihilation of particles occur, and a bulk, where particles move diffusively. While in the first reference [16], the dynamics at the boundary is a contact process, in the second one [17] particles at the origin annihilate only pairwise ($A+A\to 0$). We study different versions of these models to compare them and scrutinize the relevance of relaxing the fermionic constraint (i.e., occupation number not restricted to 0 or 1) both at the bulk and at the boundary. Our study includes mean-field approximations, numerical analysis, and some field theoretical arguments, as well as the relation with a (0+1)-dimensional non-Markovian model [18].

The paper is organized as follows: In Sec. II we define the first model and present numerical results. In Sec. III we discuss various types of mean-field approximations and show that this model has indeed a nontrivial behavior. Section IV is concerned with two bosonic versions of the first model and the study of models with pair annihilation at the boundary. One of the bosonic versions is solved exactly and it is shown to have trivial critical behavior. Instead, the other bosonic version and models with pair annihilation are shown to share the same critical behavior as the first model. In Sec. V we discuss the relation with a (0+1)-dimensional non-

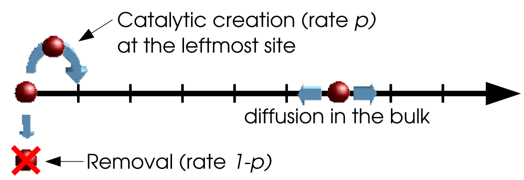

FIG. 1. (Color online) Model on a semi-infinite lattice with symmetric diffusion in the bulk and special dynamical rules at the left boundary (see main text).

Markovian model [18] and, finally, we present our main conclusions.

II. BASIC MODEL DEFINITION AND SIMULATIONS

A. Definition of the model

The model presented in [16] is defined on a one-dimensional semi-infinite discrete lattice where each site is either occupied by a particle ($s_i=1$) or empty ($s_i=0$) (see Fig. 1). All lattice sites have two neighbors, except for the boundary ($i=0$) with a single one. The dynamics is a combination of an unbiased random walk in the bulk and a contact processlike dynamics at the left boundary. It is implemented as follows:

(a) A particle is randomly selected.

(b) If it is located at the leftmost site, it generates another particle at site 1 with probability p, provided that it is empty ($s_1=0$), or it dies ($s_0=0$) with probability $1-p$.

(c) Particles in the bulk perform a symmetric exclusion process, moving to any of their two neighbors with equal probability, provided that the destination site is empty (otherwise nothing happens).

Starting with a single particle at the leftmost site in an otherwise absorbing (i.e., empty) configuration, the process evolves as follows: the initial particle at site 0 either dies or generates another particle at the neighboring site 1. This last performs a random walk in the bulk until, eventually, it returns to the origin to create another offspring or disappear. A critical point located at $p_c=0.74435(15)$ has been reported to separate the absorbing phase, in which the total number of particles vanishes, from another with indefinitely sustained activity [16].

B. Order parameters

A possible order parameter for this model is the average density of particles at the leftmost site,

$$\rho_0 = \langle s_0 \rangle, \tag{1}$$

where $\langle\ \rangle$ stands for ensemble averages. This quantity is plotted in Fig. 2 as a function of $\Delta := p - p_c$. At the critical point, $\rho_0(t)$ decays algebraically in time as

$$\rho_0(t) \sim t^{-\alpha} \tag{2}$$

with an exponent $\alpha=0.50(1)$, compatible with a rational value $\alpha=1/2$.

Another possibility is to choose as an order parameter the average total number of particles, $\langle N(t)\rangle$, which, as shown in Fig. 2, goes to zero for $p<p_c$ and increases steadily for $p>p_c$ (actually, it is limited only by the system size). At criticality, $\langle N(t)\rangle$ is found to be constant in the large time limit.

In the usual scaling picture of absorbing phase transitions, the critical exponent β is related to the probability that a given site belongs to an infinite cluster generated from a fully occupied lattice at $t=-\infty$. This quantity tends to zero as the control parameter approaches the critical value from above. Similarly, the exponent β' is related to the probability that a localized seed generates an infinite cluster extending to $t=+\infty$. Therefore, in the supercritical phase ($\Delta>0$), the averaged activity of the site at the origin for $t\to\infty$ *measured in seed simulations* averaging over all runs, scales as $\rho^s \sim \Delta^{\beta+\beta'}$, where the superscript 's' stands for 'stationary'. At criticality, this function is expected to decay as $\rho(t) \sim t^{-(\beta+\beta')/\nu_\parallel}$, where $\nu_\parallel$ is the correlation time exponent. Moreover, in the DP class a special *time-reversal symmetry* implies that $\beta=\beta'$ [6].

As shown in [17], time-reversal symmetry also holds in the present type of models. This implies that, in supercritical seed simulations, the density of active sites at the boundary is expected to saturate as

$$\rho_0^s \sim \Delta^{2\beta}, \tag{3}$$

while, at criticality,

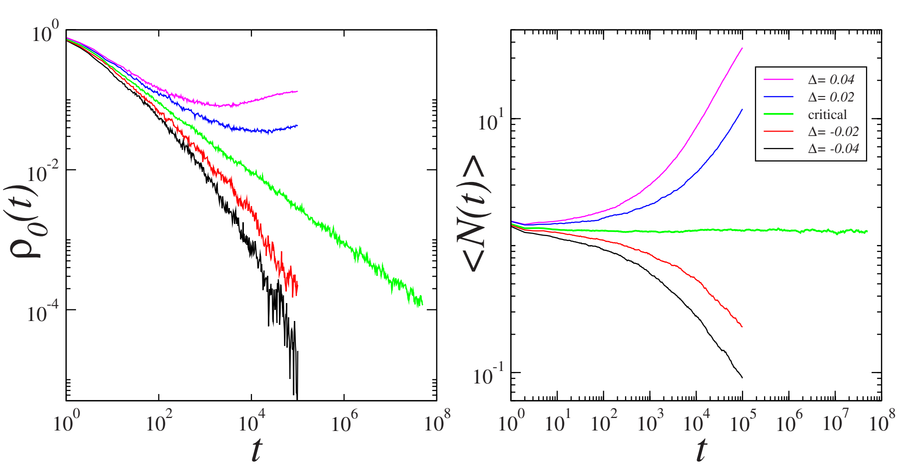

FIG. 2. (Color online) Density of particles at the leftmost site ρ_0 (left) and the average total number of particles N (right) as functions of time for different values of $\triangle$, below, above, and at criticality. At the critical point, $\rho_0(t)$ decays as $t^{-1/2}$ while and $\langle N(t)\rangle$ is essentially constant.

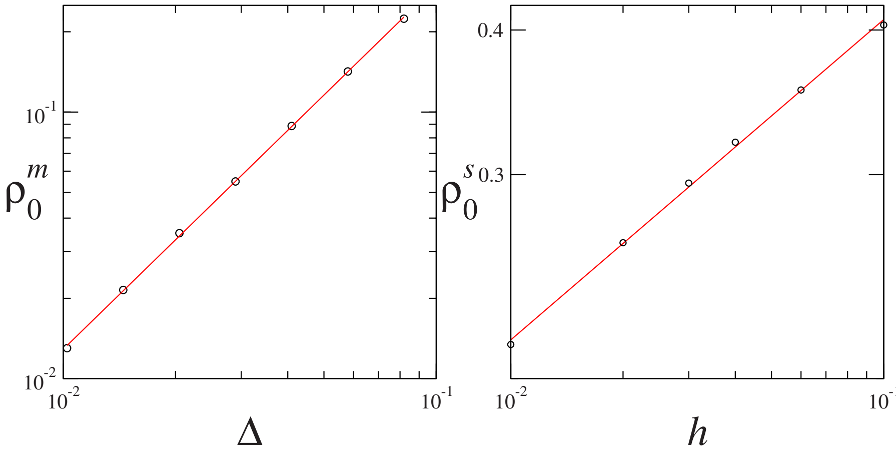

FIG. 3. (Color online) Left panel: Density of particles at the leftmost site, at the time when it reaches its minimum value, as a function of the distance from criticality Δ. This gives the exponent β=0.68(5). Right panel: The same quantity, at criticality and in the stationary state, as a function of the external field h, giving δ_h^{-1}=0.29(5), compatible with the conjectured value 1/3.

$$\rho_0(t) \sim t^{-2\beta/\nu_\parallel}, \tag{4}$$

implying that α in Eq. (2) is

$$\alpha = 2\beta/\nu_\parallel. \tag{5}$$

Assuming that α=1/2, then $\beta/\nu_\parallel$=1/4.

C. Stationary properties

In numerical simulations in the active phase, it takes a very long time, specially for small values of Δ, to reach the steady state. Moreover, we observed the unusual fact that, for $\Delta > 0$, the density ρ_0 goes through a minimum before reaching the stationary state (see Fig. 2 and also [19], where similar nonmonotonous curves were reported). However, it turns out that the value ρ_0^m at the minimum and the saturation value ρ_0^s differ by a constant factor, entailing that both quantities scale in the same way, i.e.,

$$\rho_0^m \sim \Delta^{2\beta}. \tag{6}$$

Note that this can be true only if the density $\rho_0(t)$ in seed simulations obeys the scaling relation

$$\rho_0(t) = \Delta^{2\beta} R(t\Delta^{\nu_\parallel}), \tag{7}$$

i.e., if it is possible to collapse the data by plotting $\rho_0\Delta^{-2\beta}$ versus $t\Delta^{4\beta}$. Indeed, this will be shown to be the case in Sec. V for a 0-dimensional non-Markovian process argued to be in the same universality class.

Relying on this observation, one can determine the value of the exponent β by measuring the density ρ_0^m at the minimum, which is reached much earlier than the stationary state. In Fig. 3 we plot ρ_m as a function of Δ, inferring β =0.68(5).

D. External field

In ordinary directed percolation, an external field, conjugate to the order parameter, can be implemented by creating active sites at some constant rate h, thereby destroying the absorbing nature of the empty configuration. At criticality, the external field is known to drive a $d+1$-dimensional DP process toward a stationary state with $\rho^s \sim h^{1/\delta_h}$ where $\delta_h^{-1} = \beta/(\nu_\parallel + d\nu_\perp - \beta')$, and $\nu_\perp$ is the correlation length critical exponent.

In the present model, the external field, conjugate to the order parameter ρ_0, corresponds to spontaneous creation of activity at the leftmost site at rate h. The above hyperscaling relation for δ_h is thus expected to be fulfilled by taking d=0,

$$\rho_0^s \sim h^{1/\delta_h} \tag{8}$$

with

$$\delta_h^{-1} = \beta/(\nu_\parallel - \beta'). \tag{9}$$

From this expression, exploiting the fact that $\beta=\beta'$ and using Eq. (5) as well as the conjectured rational value α=1/2, a prediction δ_h^{-1}=1/3 is obtained. Our numerical estimate, δ_h^{-1}=0.29(5) (see Fig. 3) is compatible with this result.

E. Survival probability

The survival probability $P_s(t)$ is defined as the fraction of runs that, starting with a single seed at the boundary, survive *at least* until time t. At criticality, this quantity is expected to decay algebraically,

$$P_s(t) \sim t^{-\delta}, \tag{10}$$

with the so-called survival exponent δ, while in the supercritical regime it saturates in the long time limit. Since $P_s(\infty)$ coincides with the probability for a seed to generate an infinite cluster, the saturation value of the survival probability as a function of the distance from criticality gives the exponent β'. As in DP, one expects $P_s(t)$ to decay in time with an exponent $\delta=\beta'/\nu_\parallel$=1/4. However, as shown in Fig. 4, one finds a much smaller exponent δ=0.15(2). Therefore, the usual relation $\delta=\beta'/\nu_\parallel$ does not hold. We also observed that it is not possible to collapse different curves of $P_s(t)$ for different values of Δ, i.e., the survival probability seems to exhibit an anomalous type of scaling behavior. We expect that off-critical simulations of the survival probability give the exponent β' but the simulation times needed to reach steady state are prohibitively long. An explanation for the value δ=0.15(2), differing from $\beta/\nu_\parallel$, is given in the following subsection.

F. Time-reversal symmetry

In ordinary bond DP, the statistical weight of a configuration of percolating paths does not depend on the direction

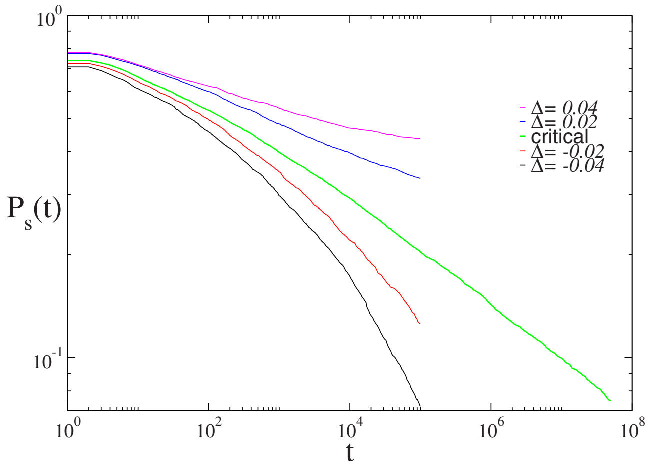

FIG. 4. (Color online) Survival probability $P_s(t)$ as function of time below, above, and at criticality. At the critical point, it decays with the exponent $\delta=0.15(2)$, different from $\beta/\nu_\parallel$, and in agreement with the conjectured value 1/6.

of time. More specifically, the probability to find an open path from at least one site at time $t=0$ to a particular site at time t coincides with the probability to find an open path from a particular site at time $t=0$ to at least one site at time t. This implies that, in bond DP, (i) the density $\rho(t)$ in simulations with fully occupied initial state and (ii) the survival probability $P_s(t)$ in seed simulations coincide; hence $\beta=\beta'$. In other realizations of DP (e.g., site DP), this time-reversal symmetry is not exact but only asymptotically realized.

Applying the same arguments to the present model, the survival probability $P_s(t)$ in seed simulations should scale in the same way as the density of active sites at the boundary $\rho_0(t)$ in a process starting with a *fully occupied lattice* in the bulk. A numerical test, which approximates such a situation, confirms this conjecture, i.e., one has $\rho_0(t)\sim t^{-\delta}$ with $\delta\approx 0.15$ for a fully occupied initial state.

Following the arguments of [18] in a related model, this observation can be used to provide an heuristic explanation for the fact that $\delta\neq\beta/\nu_\parallel$. It is known that if the boundary acts as a sink or perfect trap (e.g., if $p=0$), then, in a process starting with a fully occupied lattice, one observes a growing depletion zone around the boundary whose linear size $l'(t)$ increases as $l'\sim t^{\alpha_l}$, with $\alpha_l=1/2$ (see [20] and Sec. II G). Thus, the density of active sites decays as $t^{-1/2}$. Hence, the influx of particles from the bulk to the leftmost site may be considered as an effective time-dependent external field $h(t)\sim t^{-1/2}$. Making the assumption that this field varies so slowly that the response of the process (i.e., the actual average activity at the boundary) behaves adiabatically, as if the field was constant, then in a critical process starting from an initially fully occupied state,

$$\rho_0(t)\sim t^{-1/2\delta_h}\sim t^{-1/6}. \tag{11}$$

Owing to the time-reversal property, this quantity should decay as the survival probability. This chain of heuristic arguments leads to the conjecture that the survival exponent is given by $\delta=1/6$, in agreement with the numerical estimate $\delta=0.15(2)$.

This unusual value of the exponent δ is clearly related to the fact that the present problem is inhomogeneous. The argumentation presented above does not work for the CP, for example, since there is no special site and, therefore, a fully occupied lattice cannot be interpreted as a time-dependent field acting on a special site.

G. Density profile

Now, we consider the density profile $\rho(x,t)$ in the bulk, where $x\in\mathbb{N}$ is the spatial coordinate (distance to the boundary), computed at the critical point. In the left panel of Fig. 5, we compare the data collapse of the curves $\rho(x,t)t^{1/2}$ as a function of $x/t^{1/2}$ with a Gaussian and observe an excellent agreement, indicating random-walk-like behavior with a dynamical exponent $z=2$. However, in contrast to a simple random walk, particles are mutually correlated. This is illustrated in the right panel of Fig. 5, where the connected correlation function between two nearest neighbors

$$\rho^{\text{pair}}(x,t)=\langle\rho(x+1,t)\rho(x,t)\rangle-\langle\rho(x+1,t)\rangle\langle\rho(x,t)\rangle \tag{12}$$

in a system at the critical point is plotted against time. One observes an algebraic decay, $x^{-1/2}$, with distance. According to the standard scaling theory this implies that $\beta/\nu_\perp=1/2$, confirming that $z=\nu_\parallel/\nu_\perp=2$. Moreover, these results are in full agreement with field theoretical calculations presented in Ref. [17] (see Sec. IV C), which predict $z=2$ and $\alpha=1/2$.

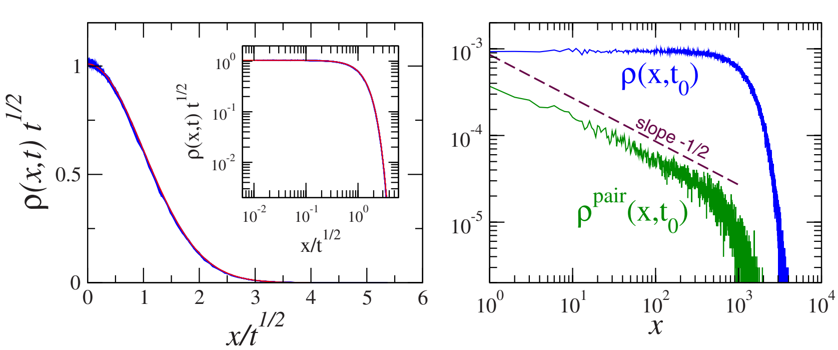

FIG. 5. (Color online) Left: Data collapse of the rescaled profiles of the particle density at criticality for $t_0=64,128,\ldots,8192$ (blue) compared to a Gaussian distribution (red). Inset: The same data collapse in a double-logarithmic representation. Right: Density of particles (blue) and pairs (green) at $t_0=1\times10^6$, showing the presence of correlations which decay in space as $x^{-1/2}$, indicating that $\beta/\nu_\perp=1/2$.

III. MEAN-FIELD APPROXIMATION

Here, we study mean-field approximations at different levels. Let us denote by η_i the probability to find a particle at site i; the temporal evolution within a simple (one-site) mean-field approximation is given by

$$\frac{d\eta_0}{dt} = -(1-p)\eta_0 + \frac{1}{2}\eta_1(1-\eta_0), \tag{13}$$

$$\frac{d\eta_1}{dt} = p\eta_0(1-\eta_1) + \frac{1}{2}(\eta_2 + \eta_0\eta_1 - 2\eta_1), \tag{14}$$

$$\frac{d\eta_i}{dt} = \frac{1}{2}(\eta_{i+1} + \eta_{i-1} - 2\eta_i), \quad \text{for } i = 2,3,\ldots . \tag{15}$$

Note that the equations for the boundary site and its neighbor, Eqs. (13) and (14), include quadratic terms due to the exclusion constraint, while the equation for sites at the bulk, Eq. (15), describes in this approximation a symmetric random walk, i.e., it is a diffusion equation. The critical point within simple mean-field theory (where the equation for η_1 also becomes a diffusion equation) is $p_c=1/2$.

Considering a localized initial condition at the boundary, $\eta_i=\delta_{i,0}$, after a transient time the densities at sites 0 and 1 should, approximately, coincide. Therefore, from Eq. (13) with $\eta_0 \approx \eta_1$, it follows that, at criticality, $\eta_0 \sim t^{-1/2}$.

In the stationary regime, Eq. (13) leads to $\eta_0 \sim (p-1/2)$ for $p \geq 1/2$. From these results we have

$$\alpha^{\rm MF} = 1/2, \quad \beta^{\rm MF} = 1. \tag{16}$$

To obtain the survival exponent, δ, we follow the arguments of the preceding section and study the decay of activity from a fully occupied lattice, $\eta_i=1$ for all i. Integrating Eqs. (13)–(15) numerically with this initial condition, we obtain an exponent in agreement with

$$\delta^{\rm MF} = 1/4. \tag{17}$$

A more accurate approximation can be obtained by keeping the correlation between the first two sites, which is expected to be more relevant than the correlation between other neighboring sites. Such a pair-approximation was used recently in a model where a boundary site also plays a special role [21]. In this approximation, the master equation reads

$$\frac{d\sigma_{00}}{dt} = (1-p)\sigma_{10} + \tfrac{1}{2}[\sigma_{01}(1-\eta_2) - \sigma_{00}\eta_2],$$

$$\frac{d\sigma_{01}}{dt} = (1-p)\sigma_{11} + \tfrac{1}{2}[\sigma_{00}\eta_2 - \sigma_{01}(2-\eta_2)],$$

$$\frac{d\sigma_{10}}{dt} = -\sigma_{10} + \tfrac{1}{2}[\sigma_{01} - \sigma_{10}\eta_2 + \sigma_{11}(1-\eta_2)],$$

$$\frac{d\sigma_{11}}{dt} = p\sigma_{10} - (1-p)\sigma_{11} + \tfrac{1}{2}[\sigma_{10}\eta_2 - \sigma_{11}(1-\eta_2)],$$

$$\frac{d\eta_2}{dt} = \tfrac{1}{2}(\eta_3 + \sigma_{11} + \sigma_{01} - 2\eta_2),$$

$$\frac{d\eta_i}{dt} = \tfrac{1}{2}(\eta_{i+1} + \eta_{i-1} - 2\eta_i) \quad \text{for } i = 3,4,\ldots , \tag{18}$$

where $\sigma_{s_0 s_1}$ is the probability that the occupation numbers of the first two sites are s_0 and s_1. Numerical integration of these equations leads to an improved critical-point estimation, $p_c \approx 0.634$, but to the same mean-field exponents as above.

IV. RELATED MODELS AND FIELD THEORETICAL APPROACHES

A. Bosonic variant

The model defined above is fermionic in the sense that each site can be occupied by, at most, one particle. We now consider a bosonic variant without such a constraint. This means that diffusion is independent of the configuration of particles and that particles can be created at the boundary site without restriction. More specifically, the update rules are the following:

(a) A particle is chosen randomly.

(b) If the particle is located at the leftmost site it can create another particle at the leftmost site ($s_0=s_0+1$) at rate λ, die ($s_0=s_0-1$) at rate σ, or diffuse to the next neighbor at rate D.

(c) If the particle is located in the bulk, it diffuses to the right or to the left at equal rates D.

The corresponding master equation is

$$\begin{aligned}\frac{dP(\{n\},t)}{dt} &= \lambda[(n_0-1)P(n_0-1,\ldots,t) - n_0P(\{n\},t)] \\ &\quad + \sigma[(n_0+1)P(n_0+1,\ldots,t) - n_0P(\{n\},t)] \\ &\quad + D\Bigg[\sum_{ij} P(\ldots,n_i-1,n_j+1,\ldots,t) \\ &\quad + P(\ldots,n_i+1,n_j-1,\ldots,t) - 2P(\{n\},t)\Bigg],\end{aligned} \tag{19}$$

where $P(\{n\},t)$ is the probability to find a given configuration $\{n\}=n_0,n_1,n_2\ldots$ and the sum runs over all nearest neighbors, j, of site i (recall that site 0 has only one neighbor). Defining the state vector

$$|\psi(t)\rangle = \sum_{\{n\}} P(\{n\},t)|\{n\}\rangle, \tag{20}$$

where $|\{n\}\rangle = \otimes_i |n_i\rangle$ denotes the usual configuration basis, the master equation can be expressed in the form

$$\frac{d}{dt}|\psi(t)\rangle = -\hat{H}|\psi(t)\rangle, \tag{21}$$

where $\hat{H}$ is the time evolution operator. Using bosonic creation and annihilation operators, defined by $\hat{a}_i|n_i\rangle = n_i|n_i-1\rangle$ and $\hat{a}_i^\dagger|n_i\rangle = |n_i+1\rangle$, the master equation Eq. (19) can be shown to correspond to the time evolution operator

$$\hat{H}=D\sum_{\langle ij\rangle}(\hat{a}_i^{\dagger}-a_j^{\dagger})(\hat{a}_i-\hat{a}_j)+\sigma(\hat{a}_0^{\dagger}-1)\hat{a}_0+\lambda\hat{a}_0^{\dagger}(1-\hat{a}_0^{\dagger})\hat{a}_0. \tag{22}$$

In this formalism, the expectation value of an operator $\hat{B}$ is given by $\langle\hat{B}\rangle=\langle 1|\hat{B}|\psi(t)\rangle$, where $\langle 1|=\Sigma_{\{n\}}\langle\{n\}|$. As is the case for the bosonic contact process [22], the equations for the time evolution of the density of particles close. From the Heisenberg equation of motion, $\frac{d\hat{B}}{dt}=[\hat{H},\hat{B}]$ and Eq. (22), one obtains

$$\frac{d\rho_0}{dt}=D(\rho_1-\rho_0)+\Delta\rho_0$$

$$\frac{d\rho_i}{dt}=D(\rho_{i+1}+\rho_{i-1}-2\rho_i)\quad i=1,2,3\ldots, \tag{23}$$

where $\rho_i(t)=\langle a_i^{\dagger}(t)a_i(t)\rangle=\langle a_i(t)\rangle$ and $\Delta=\lambda-\sigma$. Alternatively, one could have written a Langevin equation equivalent to Eq. (22), and from it, averaging over the resulting noise, one readily arrives at the same set of equations [Eq. (23)].

From these equations, we can see that the critical point is $\Delta=0$, where Eq. (23) is a diffusion equation. In the continuum limit, Eq. (23) reads

$$\frac{\partial\rho(x,t)}{\partial t}=\frac{\partial^2\rho(x,t)}{\partial x^2}+\Delta\delta(x)\rho(x,t), \tag{24}$$

where x is the spatial coordinate and, without loss of generality, we have set $D=1$. We note that in order to take the continuum limit in Eq. (23), a site -1, with $\rho_{-1}=\rho_0$, has to be introduced so that appropriate boundary conditions are satisfied. The solution of this inhomogeneous diffusion equation is

$$\rho(x,t)=\int_0^{\infty}\delta(\zeta)G(x,\zeta,t)d\zeta+\int_0^t\int_0^{\infty}\Delta\delta(\zeta)\rho(\zeta,\tau)G(x,\zeta,t-\tau),d\zeta d\tau \tag{25}$$

where $G(x,\zeta,t)=(e^{-(x+\zeta)^2/(4t)}+e^{-(x-\zeta)^2/(4t)})/(\sqrt{\pi t})$ is the Green function and the first term in the right-hand side comes from the initial condition $\rho(x,0)=\delta(x)$. From Eq. (25) we have

$$\rho_0(t)=\frac{2}{\sqrt{\pi t}}+2\Delta\frac{d^{-1/2}}{dt^{-1/2}}\rho_0(t), \tag{26}$$

where $\rho_0(t)=\rho(0,t)$, and the operator $\frac{d^{-1/2}}{dt^{-1/2}}$, defined by

$$\frac{d^{-1/2}}{dt^{-1/2}}f(t)=\int_0^t\frac{f(\tau)}{\sqrt{\pi(t-\tau)}}d\tau, \tag{27}$$

is a half integral operator [23]. Equation (26) involves [owing to the delta function in the interaction term in Eq. (24)] only the density at the leftmost site. This justifies the mapping of this model onto an effective one-site non-Markovian process (see Sec. V). Using some rules for half integration [23] to solve Eq. (26), we find

$$\rho_0(t)=\frac{2}{\sqrt{\pi t}}+4\Delta\,\exp(4\Delta^2t)\mathrm{erf}(-2\Delta\sqrt{t}), \tag{28}$$

where erf(x) is the error function. This implies that, above the critical point, ρ_0 grows exponentially in the long time limit, and does not reach a stationary value, i.e., there is a first-order transition and, hence, $\beta=0$ in this bosonic model. From equation Eq. (28), we deduce $\beta'=1$ and $\nu_{\parallel}=2$. We have not been able to calculate the survival-probability exponent exactly, but numerical simulations suggest $\delta=1/4$, in agreement with the mean-field exponent.

B. Partially bosonic variant

Let us now introduce a *partially bosonic* variant of the previous model by retaining the exclusion constraint only at the boundary, but not in the bulk. The rules, in this case, are:

(a) A particle is randomly chosen.

(b) If it is at the leftmost site, it can generate a particle at site 1 (provided that $s_1=0$) with probability p or die ($s_0:=0$) with probability $1-p$.

(c) Particles in the bulk diffuse to the right or to the left with the same probability, 1/2.

Numerical simulations show that this variant exhibits the same critical behavior as the original model even if the critical point is shifted to $p_c=0.6973(1)$. This shows that the fermionic constraint is relevant only at the boundary, where it induces a saturation of the particle density and leads the transition to become continuous.

C. Models with pair annihilation at the boundary

In the models discussed so far, particles at the boundary either create an offspring or die spontaneously at some rate. Instead, a very similar model was introduced in Ref. [17], for which particles at the boundary annihilate only in *pairs*. In its fermionic variant, particles at sites 0 and 1 annihilate with each other (provided that both sites are occupied) at some rate, while isolated particles at the boundary cannot disappear,

$$\text{present models: } A\to 2A,\quad A\to\varnothing\,,$$

$$\text{models of Ref. [17]: } A\to 2A,\quad 2A\to\varnothing\,.$$

Analogously, one can define a bosonic version, in which two particles at the boundary can annihilate. In the following discussion we consider these two variants in d spatial dimensions where, as is the case $d=1$, only a single site has "special" dynamics.

A detailed field theoretical analysis of these pair-annihilating models was presented in [17]. In the bosonic case, proceeding as above [see Eq. (22)] one obtains the following time evolution operator:

$$\hat{H}=D\sum_{\langle ij\rangle}(\hat{a}_i^{\dagger}-\hat{a}_j^{\dagger})(\hat{a}_i-\hat{a}_j)+\sigma[(\hat{a}_0^{\dagger})^2-1]\hat{a}_0^2+\lambda\hat{a}_0^{\dagger}(1-\hat{a}_0^{\dagger})\hat{a}_0, \tag{29}$$

which, after eliminating higher-order terms and taking the continuum limit, is equivalent to a Langevin equation iden-

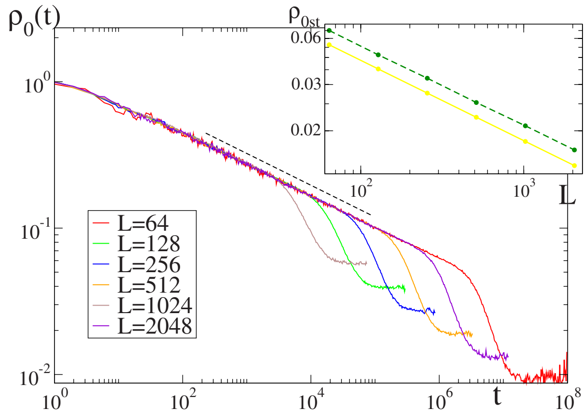

FIG. 6. (Color online) Temporal behavior of ρ_0 for the bosonic pair-annihilating model, starting from a homogeneous initial condition for different system sizes (from $L=64$ to $L=2048$). The exponent $\beta/\nu_\perp$ can be measured from the scaling of the different saturation values as a function of system size (see inset; yellow line). Also, in the inset (dashed green line), we show the scaling of saturation values for the fermionic version of the same model, showing the same type of scaling.

tical to the one for DP except for the fact that all terms, except for the Laplacian, are multiplied by a δ function at the boundary; i.e., the nondiffusive part of the dynamics operates only at the boundary. An ϵ-expansion analysis of Eq. (29) (see [17]) leads to $\alpha=1/2$ and $z=2$ as exact results in all orders of perturbation theory, and to $\beta=1-3(4-3d)/8$, up to first order in $\epsilon=4/3-d$ around the critical dimension $d_c=4/3$. Also, it was shown that the time-reversal symmetry is preserved.

We have verified all these predictions in computer simulations of the bosonic annihilation model. For instance, from the time decay of $\rho_0(t)$, as shown in Fig. 6, we determine $\delta=0.21(3)$, while from a finite-size scaling analysis of the saturation values of the order parameter at criticality we measure $\beta/\nu_\perp=0.51(2)$ (see Fig. 6), in reasonable agreement with the expected results, $\delta=1/6$ and $\beta/\nu_\perp=1/2$, respectively. Moreover, from spreading simulations (not shown) we estimate $\alpha\approx 1/2$ and $z\approx 2$. All the exponents are in agreement with the ones presented in the previous section for single-particle annihilation models.

Actually, a simple argument explains why the model of Sec. II and the pair-annihilation model share the same critical behavior. This is plausible because the chain reaction $A\to 2A\to\varnothing$ in the model with pair annihilation generates effectively the reaction $A\to\varnothing$ of the model considered with CP-like dynamics.

Hence, the field theoretical predictions discussed above [17,16] apply also to the CP-like model. In $d=1$, the one-loop prediction $\beta=5/8=0.625$ [17] is not far from the exponent measured in Sec. II, $\beta=0.68(5)$.

On the other hand, the fermionic version of the pair-annihilating model has been conjectured to yield in a different universality class, and a prediction for its critical exponents is made in [17] (for instance, $\beta=1$). Our numerical simulations disprove such a claim; all the measured critical exponents for the fermionic variant of the pair-annihilation model are numerically indistinguishable from their bosonic counterparts (see Fig. 6).

In summary, all the defined models, either with single-particle annihilation or with pair annihilation fermionic or bosonic, exhibit a boundary-induced phase transitions, and, except for one of them, they all are continuous and share the same critical behavior. The exception to this rule is the CP-like model without a fermionic constraint at the boundary, which lacks of a saturation mechanism in the active phase, leading to unbounded growth of particle density at the leftmost site above the critical point and to a discontinuous transition.

V. RELATION TO A (0+1)-DIMENSIONAL NON-MARKOVIAN PROCESS

In Ref. [17], by integrating out the fields related to diffusion in the bulk from the corresponding action, it was shown that the class of boundary-induced phase transitions into an absorbing state considered here can be related to a non-Markovian single-site process. The properties of such a spreading process on a time line have been studied in further detail in Ref. [18].

On a heuristic basis, the relation can be explained as follows: consider the CP-like model only from the perspective of the leftmost site. A particle at the origin may die or create a new particle that will go for a random walk coming back to the origin after a time τ. What happens during this random walk is irrelevant from the perspective of the leftmost site; the only relevant aspect is the time needed for a created particle to come back to the boundary. Once it returns it may die or create new offsprings which, on their turn, will undergo random walks in the bulk.

Our simulations above show that the fermionic constraint is irrelevant in the bulk. Therefore, we can consider without lost of generality the bulk-bosonic version in which there is no effective interaction among diffusing particles. In this case, the probability distribution of the returning time to the origin has the well-known asymptotic form [24]

$$P(\tau)\sim\tau^{-3/2}. \tag{30}$$

Taking all these elements into account we define the following non-Markovian model on a single site [16]:

(a) Set initially $s(t):=\delta_{t,0}$ for all times, t.

(b) Select the lowest t for which $s(t)=1$.

(c) With probability μ, generate a waiting time τ according to the distribution Eq. (30), truncate it to an integer, and set $s(t+\tau):=1$; otherwise (with probability $1-\mu$) set $s(t):=0$.

(d) Go back to (b).

The process runs until the system enters the absorbing state ($s(t')=0$ for all $t'>t$) or a predetermined maximum time is exceeded.

The density of particles at the leftmost site of the original model is related to $\langle s(t)\rangle$ in the single-site model, the survival probability at time t is given by the fraction of runs surviving at least up to t, and the initial condition $s(t):=\delta_{t,0}$ corresponds to start with a single particle at the boundary in the full model. Critical exponents can be defined as in the original model. However, the simulation results for the single-site non-Markovian model are more reliable because it is pos-

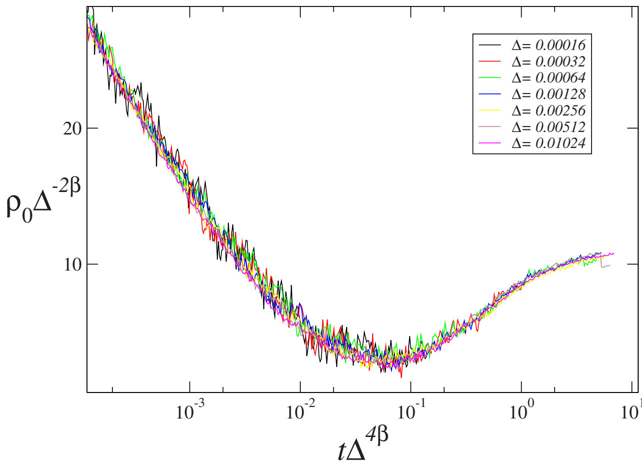

FIG. 7. (Color online) Off-critical data collapse with the one-site model: $\langle s(t)\rangle\Delta^{-2\beta}$ as a function of $t\Delta^{4\beta}$ for different values of $\triangle$, with β=0.71(2).

sible to perform much longer runs and, in the case of off-critical simulations, one can work with smaller values of Δ. With time-dependent simulations at the critical point μ_c =0.574 262(2), we obtained α=0.500(5) and δ=0.165(3), in good agreement with the conjectured values $\alpha=1/2$ and $\delta=1/6$. As an example, we show the results of supercritical simulations in Fig. 7, where we obtained a convincing data collapse by plotting $\langle s(t)\rangle\Delta^{-2\beta}$ as a function of $t\Delta^{4\beta}$ for different values of Δ with β=0.71(2). The latter estimate is in agreement with β=0.68(5), coming from the original model.

As shown in previous studies (see e.g., [25] and references therein), a non-Markovian time evolution with algebraically distributed waiting times $P(\tau)\sim\tau^{-1-\kappa}$ is generated by so-called fractional derivatives ∂_t^κ, which are defined by

$$\partial_t^\kappa\rho(t) = \frac{1}{\mathcal{N}_\parallel(\kappa)}\int_0^\infty dt' t'^{-1-\kappa}[\rho(t)-\rho(t-t')], \quad (31)$$

where $\kappa\in[0,1]$ and $\mathcal{N}_\parallel(\kappa)=-\Gamma(-\kappa)$ is a normalization constant. Hence, we expect this model to be described by a DP-like 0-dimensional Langevin equation with a half-time derivative, instead of the usual one, to account for the non-Markovian character of the model

$$\partial_t^{1/2}\rho(t) = a\rho(t) - \rho(t)^2 + \xi(t), \quad (32)$$

where a is proportional to the distance from criticality and ξ is a multiplicative noise with correlations $\langle\xi(t)\xi(t')\rangle=\rho(t)\delta(t-t')$. This equation can be obtained from the effective action that arises when the fields related to diffusion in the bulk are integrated out, and the relation of the order of the fractional derivative in a generalized one-site model with the dimension in the full model is $\kappa=(2-d)/2$ [17]. An analysis of this one-site model with general κ and a comparison with the results coming from field theory is presented in [18].

VI. CONCLUSION

We have studied boundary-induced phase transitions into an absorbing state in one-dimensional systems with creation and annihilation dynamics at the boundary and simple diffusive dynamics in the bulk. The nontrivial dynamics at the boundary induces a phase transition in the bulk. We have analyzed such a transition for different though similar models, including different ingredients: either single-particle annihilation or pairwise annihilation, fermionic constraint or lack of it, etc.

A particular bosonic version can be exactly solved; owing to the lack of any saturation mechanism, the density of particles grows unboundedly in the active phase, leading to a discontinuous transition with trivial critical exponents.

The rest of the analyzed models exhibit a continuous transition and define a unique universality class. At the bulk, the dynamics is governed by random walks, entailing the exponent values $z=2$ and $\alpha=1/2$. On the other hand, some critical exponents take nontrivial values: (i) the survival probability from a localized seed at the boundary exponent, which from an heuristic argument supported by simulations results, turns out to be $\delta=1/6$, as well as (ii) the order-parameter exponent, β=0.71(2). The remaining exponents can be obtained from these ones using scaling relations.

Finally, it has been shown that the class of boundary-induced phase transitions studied here can be related to a single-site non-Markovian process. This process is particularly suitable for numerical simulations and it is also of conceptual interest in the sense that it shows that nonequilibrium phase transitions can occur even in 0+1 dimensions by choosing an adequate non-Markovian dynamics. It is also convenient for the comparison of the results obtained form the ϵ expansion and simulations [18,17].

The models studied here possibly constitute the simplest universality class of nonequilibrium phase transition into an absorbing state, in the sense that the transition occurs because of the special dynamics of just one site and, in contrast to DP, some critical exponents can be obtained exactly from the field theory.

ACKNOWLEDGMENTS

We thank X. Durang and M. Henkel for helpful discussions. Financial support by the Deutsche Forschungsgemeinschaft (Grant No. HI 744/3-1), by the Spanish MEyC-FEDER (Project No. FIS2005-00791), and from Junta de Andalucía as group FQM-165 is gratefully acknowledged.

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