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Thermodynamics of feedback controlled systems

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We compute the entropy reduction in feedback controlled systems due to the repeated operation of the controller. This was the lacking ingredient to establish the thermodynamics of these systems, and in particular of Maxwell's demons. We illustrate some of the consequences of our general results by deriving the maximum work that can be extracted from isothermal feedback controlled systems. As a case example, we finally study a simple system that performs an isothermal information-fueled particle pumping.

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I. INTRODUCTION

Controllers are ubiquitous in science and technology with a number of purposes such as stabilizing unstable dynamics or increasing the performance [1]. Furthermore, many real systems in nature can be modeled as a system plus a controller. A controller is an external agent whose action is to modify the evolution of the system with a purpose. *Feedback or closed-loop* controllers use information about the state of the system. The feedback is the process performed by the controller of measuring the system, deciding on the action given the measurement output, and acting on the system. On the contrary, an open-loop controller operates on the system blindly, i.e., without information of its state. Although it is intuitively clear that the information about the state of the system can be used to improve the performance, there are still open questions on the connections between feedback control theory and information theory (see Ref. [1]). In particular, the understanding of the thermodynamics of feedback control is still incomplete. Much of the progress in the solution of this problem has come from the study of Maxwell's demon [2]. This is a being that gathers information about a system and is able to decrease the entropy of the system without performing work on it. The seminal work of Szilard [3] contains the basic ingredients of the trade off between information theory and thermodynamics, which is precisely stated in Landauer's principle: the erasure of 1 bit of information at temperature T implies an energetic cost of at least $k_B T \ln 2$ [4]. Bennett [5] pointed out that Landauer's principle is the key to preserving the second law of thermodynamics in feedback systems, as the controller must erase its memory after each cycle to allow the whole system to truly operate cyclically. How to achieve the shorter description for the memory record of the controller in order to minimize the energetic erasure cost was established by Zurek [6] by using an algorithmic complexity approach. On the other hand, Lloyd used in [7] a different point of view—that of the feedback controlled system. From this approach the effect of the interaction of the controller with the system is to reduce the entropy of the system due to the additional determination of the macrostate of the system through the information obtained from it. More recently, Touchette and Lloyd [8] computed the maximum additional reduction in entropy attainable in one control action when a feedback control is used instead of an open-loop control.

In this paper we also consider the point of view of the feedback controlled system. The thermodynamics of the interactions of the system with the controller and the environment are well studied for the heat and work exchanges. However, a complete understanding of the entropy reduction in the system due to its interaction with the feedback controller is still lacking. We solve here this problem and show how to compute this entropy reduction after one or several control steps. This result allows us to establish the thermodynamics of feedback controlled systems without assuming Landauer's principle. Several concepts and quantities defined in information theory [9] emerge naturally as one computes this entropy reduction. For the definition of the entropy we will use $k_B=1$ and natural logarithms. This implies that the information quantities that naturally appear will be in nats ($\ln 2$ nats$=1$ bit).

In Sec. II we compute the entropy reduction in a general feedback controlled system due to the repeated operation of the controller. The result allows us to establish the thermodynamics of feedback controlled systems. In Sec. III, we illustrate some of the consequences of our general result by deriving the maximum work that can be extracted from isothermal feedback controlled systems. In Sec. IV, we show the applicability and usability of the results in a simple dynamical system, a Markovian particle pump that is able to extract useful work from the entropy reduction due to the information used by an external feedback controller. Finally, we summarize the results of the paper in Sec. V.

II. ENTROPY REDUCTION IN FEEDBACK CONTROLLED SYSTEMS

Let us call $X_k := X(t_k)$ the macrostate of a general dynamical system at the kth control step of the controller (at time t_k). In a feedback controlled system the control step involves several operations by the controller: measuring the system, deciding the control action to take given the measurement

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output, and acting on the system following the selected control action. Therefore, the control action is the modification of the evolution of the system made by the external agent that we shall call the controller. The controller can perform several control actions on the system. By $C_1=c$ we denote that at the first control step, the controller has chosen to perform the action labeled by c. (It is *not* a specification of the state of the controller.) As the control actions are decided at their respective control steps, C_k represents only the decision taken at the kth control step.

Initially the entropy of the system is S_0, which is fixed by the probabilities $p_{X_0}(x)$ of each possible microstate x at time $t=0$. Subsequently, the system evolves with an entropy change from S_0 to S_1^b, which is the entropy just *before* the first control step. It is given by the statistical entropy

$$S_1^b = -\sum_{x\in\mathcal{X}} p_{X_1}(x)\ln p_{X_1}(x) =: H(X_1), \tag{1}$$

with $\mathcal{X}$ as the set of possible microstates of the system. At time t_1 the controller measures the state of the system. The result of this measurement determines, at least partially, the action the controller will take. The additional information on the system provided by the measure further determines the system macrostate [7], i.e., it defines a submacrostate that contains only microstates compatible with the measured value. However, from the point of view of the system, each set of measurement outputs that leads to the same control action can be considered as defining a single submacrostate of the system because the controller in its action on the system ignores the differences inside these sets. Thus, if the measurement implies a control action $C_1=c$, the entropy of the system decreases to

$$H(X_1|C_1=c) := -\sum_{x\in\mathcal{X}} p_{X_1|C_1}(x|c)\ln p_{X_1|C_1}(x|c). \tag{2}$$

Therefore, the average entropy *after* the first control step can be obtained by averaging over the set $\mathcal{C}$ of all possible control actions,

$$S_1^a = \sum_{c\in\mathcal{C}} p_{C_1}(c)H(X_1|C_1=c) =: H(X_1|C_1). \tag{3}$$

Hence the average variation in the entropy at the first step is

$$\Delta S_1 = S_1^a - S_1^b = H(X_1|C_1) - H(X_1) =: -I(X_1;C_1), \tag{4}$$

i.e., it is the (minus) mutual information [9] between X_1 and C_1.

Let us describe one more step. Each of the previous $|\mathcal{C}|$ submacrostates of the system with entropy $H(X_1|C_1=c)$ evolves to give an entropy $H(X_2|C_1=c)$ just before the second control step. Following the second control step, each one of these submacrostates of the system give $|\mathcal{C}|$ more submacrostates. The entropy of the system given that $C_1=c$ and $C_2=c'$ is $H(X_2|C_2=c',C_1=c)$. Therefore, the average entropy of the system after the second step is

$$S_2^a = \sum_{c,c'\in\mathcal{C}} p_{C_2C_1}(c',c)H(X_2|C_2=c',C_1=c) = H(X_2|C_2,C_1), \tag{5}$$

and the average variation in the entropy at this second control step is $\Delta S_2=S_2^a-S_2^b=H(X_2|C_2,C_1)-H(X_2|C_1)=-I(X_2;C_2|C_1)$. This conditioning of the mutual information shows that the entropy of the system is only reduced by the new information.

Analogously we get for the average entropy reduction in the kth step $\Delta S_k=-I(X_k;C_k|\mathbf{C}^{k-1})$, where $\mathbf{C}^{k-1}$ stands for $C_{k-1},C_{k-2},\ldots,C_1$. Using the properties of mutual information [9], this average entropy reduction can be written as

$$\begin{aligned}\Delta S_k &= -I(X_k;C_k|\mathbf{C}^{k-1})\\ &= -I(C_k;X_k|\mathbf{C}^{k-1})\\ &= -H(C_k|\mathbf{C}^{k-1}) + H(C_k|\mathbf{C}^{k-1},X_k).\end{aligned} \tag{6}$$

Finally, we find that the *total average entropy reduction due to the information used* in M control steps is $\Delta S_{\text{info}}=\sum_{k=1}^M \Delta S_k$, i.e.,

$$\Delta S_{\text{info}} = -\sum_{k=1}^{M} I(C_k;X_k|\mathbf{C}^{k-1}). \tag{7}$$

This general result indicates that this entropy reduction can be computed in terms of the joint probabilities for the state of the system and the control actions history. Using Eq. (6) and the chain rule for H (see Ref. [9]), we rewrite the last equation as

$$\Delta S_{\text{info}} = -H(\mathbf{C}^M) + \sum_{k=1}^{M} H(C_k|\mathbf{C}^{k-1},X_k). \tag{8}$$

Equation (7), or equivalently Eq. (8), is a central result of this paper. As a consistency check, note that for open-loop controlled systems the controller acts independently of the state of the system and it gets no information of it. Thus, $H(C_k|\mathbf{C}^{k-1},X_k)=H(C_k|\mathbf{C}^{k-1})$, which gives $\Delta S_{\text{info}}=0$ after applying the chain rule in Eq. (8), as expected. Note also that the mutual information in Eq. (7) between the system and the control actions is conditioned by the past control actions. This reflects that the correlations between measurements limit the attainable entropy reduction. Therefore, the entropy reduction in M consecutive measurements is equal or lower than in M independent measurements.

A. Deterministic feedback controllers

A relevant class of closed-loop controllers is *deterministic feedback controllers*. For them the control action is determined without uncertainty by the state of the system and the control actions history. Therefore

$$H(C_k|\mathbf{C}^{k-1},X_k) = 0, \tag{9}$$

and the entropy reduction in Eq. (8) simplifies to $\Delta S_{\text{info}}=-H(\mathbf{C}^M)$, which can be computed by just using the joint probability $p_{C_1,\ldots,C_M}(c_1,\ldots,c_M)$. Consequently, the average entropy reduction after a large number of control actions is given by the entropy rate $\bar{H}(\mathcal{C})$ of the stochastic process describing the control actions,

$$\lim_{M\to\infty}\frac{\Delta S_{\text{info}}}{M}=\lim_{M\to\infty}\frac{-H(\mathbf{C}^M)}{M}=:-\bar{H}(\mathcal{C}). \tag{10}$$

For a system and control dynamics without explicit dependencies in time, this average entropy reduction coincides with the asymptotic entropy reduction in one step [9], that is, $\lim_{M\to\infty}\Delta S_{\text{info}}/M=\lim_{M\to\infty}\Delta S_M$.

B. Nondeterministic feedback controllers

Feedback controllers satisfying Eq. (9) are error free. On the other hand, controllers affected by some *source of error* are common in real systems. In this case the decorrelation between the control actions and the state of the system reduces the attainable entropy reduction; see Eq. (8). For instance, consider a feedback controller with two possible actions, say "on" and "off," for which the system state and the previous control actions history determine which one of the actions is taken with probability $1-\epsilon$. For this system, $H(C_k|\mathbf{C}^{k-1},X_k)=H_b(\epsilon)$, with $H_b(\epsilon)$ as the binary entropy function $H_b(\epsilon):=-\epsilon\ln\epsilon-(1-\epsilon)\ln(1-\epsilon)$, and Eq. (8) gives

$$\lim_{M\to\infty}\frac{\Delta S_{\text{info}}}{M}=-\bar{H}(\mathcal{C})+H_b(\epsilon). \tag{11}$$

This shows that errors in the control operation limit the attainable entropy reduction.

C. Discussion

The new relation (7) sets the entropy reduction in the controlled system due to the information used by the external agent that operates on it. The reformulation of this relation as Eq. (8) allows us to understand the average entropy reduction per control step as two competing contributions: a negative term accounting for the entropy rate of the control actions and a positive term accounting for the decorrelation between the controller actions and the state of the system. This decorrelation can arise, for instance, from errors in the operation of the controller [see Eq. (11)]. These new relations, Eqs. (7) and (8), also show how the past control action history must be taken into account to avoid redundancy in the computation of the entropy reduction. They are consistent with the Zurek's computational interpretation of the controller as a memory record whose blocks occupied by past measurements must be compressed before the erasure process [6,10]. On the other hand, when only one control step is considered, Eq. (7) reduces to Eq. (4), which gives the well-known Landauer's energetic cost due to information [2], $k_BTI(X_1;C_1)$ (recovering units), also found for quantum systems [11].

The statement of the entropy reduction in terms of the control actions is an important point of this paper. It allows one to give a reachable bound for the efficiency. (If the controller performs the same action for two different measured values, the bound found for the efficiency considering the entropy reduction in terms of the measure could be non-reachable.) Note also that the overall reduction in the entropy of the system due to feedback control is expressed in terms of physical quantities and it can be computed without knowledge of internal details of the controller. In addition, this approach also allows one to compute the maximum entropy reduction attainable with a nondeterministic feedback control, Eq. (11), giving a reachable bound.

The entropy reduction in the system due to the information used by the controller is a fundamental ingredient in the thermodynamics of feedback controlled systems. It is the key to improving the performance in these systems compared with their open-loop counterparts. Once this entropy reduction is understood and we know how to compute it [Eq. (7) or (8)], the thermodynamics of feedback controlled systems is complete. In particular, we show in Sec. III how to compute thermodynamic relations for an *isothermal feedback controlled system*.

III. APPLICATION: ISOTHERMAL FEEDBACK CONTROLLED SYSTEMS

We study in this section the implications of the previous results for the case of an isothermal feedback controlled system. A general isothermal feedback controlled system is a system that is coupled to a feedback controller, to a thermal bath of temperature T, and to another external system on which it does work. When the system is operated cyclically, the initial state is recovered after a cycle, and the variations in internal energy and entropy of the system in the cycle are zero. During such a cycle the system releases a quantity of heat Q to the thermal bath and does work W on the external system. The transfer of the internal energy of the controller ΔU_{cont} to the system is given by the first law of thermodynamics,

$$\Delta U_{\text{cont}}+Q+W=0. \tag{12}$$

On the other hand, the second law of thermodynamics gives

$$T\Delta S_{\text{cont}}+Q\geq 0, \tag{13}$$

with ΔS_{cont} as the entropy increase in the controller. Combining both relations we get the inequality

$$W\leq-\Delta U_{\text{cont}}+T\Delta S_{\text{cont}}=-\Delta F_{\text{cont}}, \tag{14}$$

where ΔF_{cont} is the variation in the Helmholtz free energy of the controller in the cycle. From this relation it is natural to define the efficiency of a feedback controlled system as

$$\eta=\frac{W}{-\Delta F_{\text{cont}}}. \tag{15}$$

In addition, if the controller only interacts with the system and without heat transfer, we have $\Delta S_{\text{cont}}\geq-\Delta S_{\text{info}}$, i.e., the increase in entropy of the controller should be greater than or equal to the reduction in the entropy of the system due to the actions of the controller. This implies that the maximum efficiency that can be attained with an isothermal feedback controlled system is

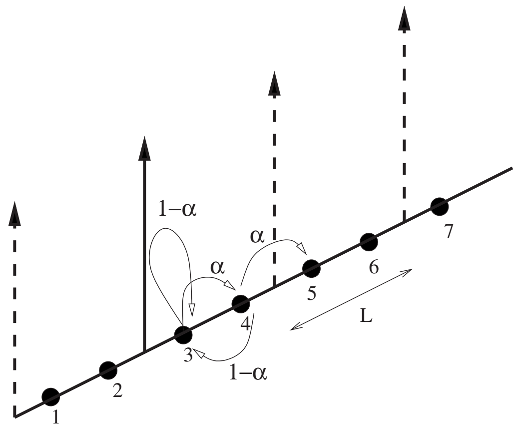

FIG. 1. Illustration of the Markovian particle pump with $n=2$ lattice sites between barriers. This is a simple feedback controlled system that extracts useful work from the entropy reduction due to the information about the system used by the external feedback controller.

$$\eta = \frac{W}{-\Delta U_{\text{cont}} - T\Delta S_{\text{info}}}, \qquad (16)$$

where W is the work extracted from the system, $-\Delta U_{\text{cont}}$ is the work done by the controller on the system, and ΔS_{info} is the entropy reduction in the system due to the information-dependent operation of the controller, which can be computed with Eq. (7).

IV. EXAMPLE: MARKOVIAN PARTICLE PUMP

We shall illustrate how to apply our results in a simple dynamical system, a *Markovian particle pump*, which is able to extract useful work from the entropy reduction due to the information about the system used by an external feedback controller. Consider a particle in a one-dimensional lattice that is in contact with a thermal bath at temperature T. An external controller can activate reflecting barriers separated by a distance L with n lattice sites between two consecutive barriers; see Fig. 1. For the discussion of this example we will consider units of $k_BT=1$ and $L=1$. In the absence of external forces, the particle jumps to the left or to the right site with the same probability, 1/2, at each time step. Now let us have a force f pointing in the negative direction. The probability of jumping to the right decreases and becomes $\alpha := 1/(1+e^{f/n})$, as follows from detailed balance. We aim to move the particle to the right (against the force). For this purpose the controller measures the particle location and consecutively raises from left to right the reflecting barriers to trap the particle further and further to the right. The next barrier to the right is raised when the measurement indicates that the particle has crossed to the right-hand side. This implies that when the particle moves to the left until the raised barrier location, it finds a reflecting boundary condition, while the particle has no bounds to its displacements to the right.

This defines a deterministic feedback control that pumps the particle by using information about the location of the jumping particle. We stress that a blind open-loop control strategy for the lifting of the barriers cannot achieve direct flux against the load. In addition, our closed-loop controller does not introduce any extra energy in the system. Thus, the entropy reduction in the system thanks to the information-gathering operation is the only responsible for the pumping. In particular, we highlight that a naive definition of efficiency as $\eta=W/(-\Delta U_{\text{cont}})$ is meaningless for engines that work due to an information-dependent operation. Our general results allow us to compute the maximum possible efficiency of this pump as a case example, not only in the quasistatic regime (large time intervals between two operations of the controller) but also when it is operated nonquasistatically (for instance every time step).

Let us first compute the maximum efficiency attainable when the controller operates every time step. We consider the particle initially at the origin with the reflecting barrier to the left raised. At time t_k the controller takes the value $C_k=1$ when the next right barrier is raised or $C_k=0$ if the barrier remains off. As the feedback control in this example satisfies the deterministic condition (9), the average entropy reduction per step is given by Eq. (10). Furthermore, in order to simplify the computation of the entropy rate, it is useful to change to a description in terms of a new stochastic process $\tilde{C}$, with $\tilde{C}_s$ defined as the number of steps between the raise of the barrier $s-1$ and that of the barrier s (first passage time). For example, the event $(C_1,\ldots,C_7)=(0,0,0,1,0,0,1)$ corresponds to the event $(\tilde{C}_1,\tilde{C}_2)=(4,3)$. It is clear that we can establish a one-to-one correspondence between C and $\tilde{C}$, as both represent univocally the control actions history. Calling $\langle\tau\rangle$ as the average first passage time through the next barrier position, we have that Eq. (10) reads

$$\lim_{t\to\infty}\frac{T\Delta S_{\text{info}}}{t} = \lim_{t\to\infty}\frac{-H(\mathbf{C}^t)}{t} = \lim_{s\to\infty}\frac{-H(\tilde{\mathbf{C}}^s)}{s\langle\tau\rangle}. \qquad (17)$$

[That is, $\bar{H}(\mathcal{C})=\bar{H}(\tilde{\mathcal{C}})/\langle\tau\rangle$.] As the new tilde variables are independent and identically distributed we have $H(\tilde{\mathbf{C}}^s)=sH(\tilde{C}_1)$. Thus,

$$\lim_{t\to\infty}\frac{T\Delta S_{\text{info}}}{t} = \frac{-H(\tilde{C}_1)}{\langle\tau\rangle} = \frac{\sum_{k=1}^{\infty} p_\tau(k)\ln p_\tau(k)}{\sum_{k=1}^{\infty} k p_\tau(k)}, \qquad (18)$$

where $p_\tau(k)$ is the probability mass function of the first passage time being $\tau=k$. This asymptotic value, Eq. (18), is reached in a characteristic time $\langle\tau\rangle$. The probability $p_\tau(k)$ can be obtained from the transition probabilities between the states of the jumping particle.

On the other hand, the average potential increase is $W=f/\langle\tau\rangle$. Therefore, the maximum efficiency attainable at this nonquasistatic regime is obtained from Eq. (16) that reads

$$\eta_{\text{nq}} = \frac{f}{H(\tilde{C}_1)}. \qquad (19)$$

A. One lattice site between consecutive barriers

For instance, for the case with a single lattice site between two barriers $p_\tau(k)=\alpha(1-\alpha)^{k-1}$, implying $H(\widetilde{C}_1)=H_b(\alpha)/\alpha$ and $\langle\tau\rangle=1/\alpha$. Thus, the average entropy reduction per step is $H_b(\alpha)$, and the average potential increase is $W=f/\langle\tau\rangle=\alpha f$. Finally, the maximum efficiency attainable at this nonquasistatic regime is $\eta_{\text{nq}}=\alpha f/H_b(\alpha)$. This result for the model with a single site between two consecutive barriers can also be obtained without using Eq. (18). For this simple case operation steps at different times are independent and $T\Delta S_k=-H(C_k)$ with $p_{C_k}(1)=\alpha$. This gives an entropy reduction per step $H_b(\alpha)$. On the other hand, the average potential-energy gain per step is αf because the particle gains an energy f with probability α. In view of these considerations we recover $\eta_{\text{nq}}=\alpha f/H_b(\alpha)$.

B. Several lattice sites between consecutive barriers

As α is the probability of jumping to the right, the probability of the first passage time being $\tau=k$ is obtained from the probability $p_{X_{k-1}}(n)$ of finding the particle at site n (just to the left to the first barrier) at instant time $k-1$ as $p_\tau(k)=\alpha p_{X_{k-1}}(n)$. To evaluate this probability we only need to know the transition probabilities of jumping between the different spatial positions (see Fig. 1). We shall call Π as the matrix such that its (i,j)th entry is the probability $p_{j\to i}$ of jumping from the j site to the i site. Then, for the particle pump with n sites between barriers, Π is the $n\times n$ tridiagonal matrix

$$\Pi=\begin{pmatrix} 1-\alpha & 1-\alpha & & & \\ \alpha & 0 & \ddots & & \\ & \alpha & \ddots & 1-\alpha & \\ & & \ddots & 0 & 1-\alpha \\ & & & \alpha & 0 \end{pmatrix}. \tag{20}$$

Assuming that the particle is initially situated at the origin, the probability $p_{X_{k-1}}(n)$ is given by the $(n,1)$th element of the $(k-1)$th power of Π. Hence,

$$p_\tau(k)=\alpha\Pi^{k-1}(n,1). \tag{21}$$

For instance, for $n=1$ we recover $p_\tau(k)=\alpha(1-\alpha)^{k-1}$, with $\alpha=1/(1+e^f)$. For $n=2$ we get, after some straightforward calculus, $p_\tau(k)=a(b_+^{k-1}-b_-^{k-1})$, where $a:=\alpha^2/\sqrt{1+2\alpha-3\alpha^2}$ and $b_\pm:=(1-\alpha\pm\sqrt{1+2\alpha-3\alpha^2})/2$, with $\alpha=1/(1+e^{f/2})$.

Once the probabilities $p_\tau(k)$ are obtained, the entropy reduction and the efficiency can be computed with Eqs. (18) and (19), respectively. We plot in Fig. 2 this entropy reduction $\lim_{t\to\infty}T\Delta S_{\text{info}}/t$ for the particle pump with $n=5$ lattice sites between barriers, together with the time dependence of the average entropy reduction per time step obtained by means of computer simulations of the dynamics in the maximum measurement regime. As expected, this time evolution tends to the theoretical asymptotic value in a characteristic time of order $\langle\tau\rangle=\sum_{k=1}^{\infty}kp_\tau(k)$.

The numerical results in Fig. 2 have been obtained evolving the particle distribution according to the known transition probabilities. The entropy reduction in each measurement is given by the entropy difference between the particle distributions before and after the measurement. After the measurement we keep one of the two possible particle distributions chosen randomly with the probability of the corresponding measurement output, and we evolve this particle distribution until the next measurement. Following this procedure we have performed several realizations of the control actions history, and thereafter we have performed an average over realizations to obtain the average entropy per time step as a function of time. For these simulations we have considered $n=5$ lattice sites and force $f=1$ (in units of $k_BT=1$ and $L=1$) or equivalently $\alpha=1/(1+e^{1/5})\approx 0.45$.

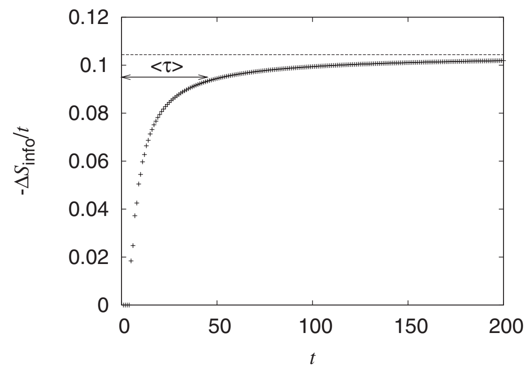

FIG. 2. Average entropy reduction per time step as a function of time for the particle pump with $n=5$ lattice sites between barriers: numerical simulations (+ signs) and asymptotic value (dashed line). The asymptotic value is approached in a characteristic time of the order of the mean first passage time $\langle\tau\rangle$. Force $f=1$. Units $k_BT=1$ and $L=1$.

C. Quasistatic regime

To conclude the analysis of the illustrating example, the Markovian particle pump, we shall compute its maximum efficiency in the quasistatic regime. Consider again the particle initially situated at the origin. As the time between measurements is large enough, the system has reached equilibrium when the controller measures at a time $t\gg 1$. Hence $p_{X_t}(m)=(1-e^{-f/n})e^{-fm/n}$ and the jumping particle is at the right-hand side of the next barrier with probability $\sum_{m>n}p_{X_t}(m)=e^{-f}$. On the other hand, when the barrier is raised the system gains a potential energy f. Thus, the entropy reduction due to information is $H_b(e^{-f})$, while the potential energy gained in one step is fe^{-f}. Therefore the maximum efficiency for the quasistatic operation of the Markovian particle pump is $\eta_q=fe^{-f}/H_b(e^{-f})$. We note that $0<\eta_{\text{nq}}<\eta_q<1$, as expected.

In order to compare with results in Fig. 2 note that for the same parameter values, a measurement step in the quasistatic regime reduces the entropy on average an amount $H_b(e^{-1})\approx 0.66$. However, a measurement step in the quasistatic regime requires many evolution time steps, resulting in a very low entropy reduction per time step.

V. CONCLUSIONS

In this paper we have addressed the thermodynamics of closed-loop controlled systems, focusing on what characterizes them, namely, the use of information. Our results show explicitly how to calculate the entropy reduction due to information, Eq. (7) or (8). Therefore, they allow one to compute the thermodynamic quantities and their relations for feedback controlled systems. In particular, we have calculated the thermodynamic relations for isothermal feedback controlled systems, Eqs. (12)–(14), and also the maximum efficiency attainable, Eqs. (15) and (16). As a case example, we have shown how to apply our general results to a simple system that performs an isothermal information-fueled particle pumping for both a maximum measurement regime and a quasistatic regime. The results presented in this paper allow one to study the thermodynamics of many other feedback controlled systems. It will be particularly interesting to obtain the thermodynamics of feedback flashing ratchets that have been studied theoretically [12] and recently realized experimentally [13].

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