Dynamics of metastable breathers in nonlinear chains in acoustic vacuum

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The study of the dynamics of one-dimensional chains with both harmonic and nonlinear interactions, as in the Fermi-Pasta-Ulam and related problems, has played a central role in efforts to identify the broad consequences of nonlinearity in these systems. Nevertheless, little is known about the dynamical behavior of purely nonlinear chains where there is a complete absence of the harmonic term, and hence sound propagation is not admissible, i.e., under conditions of "acoustic vacuum." Here we study the dynamics of highly localized excitations, or breathers, which are known to be initiated by the quasistatic stretching of the bonds between adjacent particles. We show via detailed particle-dynamics-based studies that many low-energy pulses also form in the vicinity of the perturbation, and the breathers that form are "fragile" in the sense that they can be easily delocalized by scattering events in the system. We show that the localized excitations eventually disperse, allowing the system to attain an equilibrium-like state that is realizable in acoustic vacuum. We conclude with a discussion of how the dynamics is affected by the presence of acoustic oscillations.

I. INTRODUCTION

Fermi, Pasta, and Ulam's computational study of particle dynamics in a one-dimensional (1D) system of masses connected via nonlinear springs [1] ushered in a new era in the study of nonlinear many-particle systems [2,3]. We consider Hamiltonians of the general form

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{i=1}^{N} V(|x_{i+1} - x_i|), \qquad (1)$$

where henceforth we set $m_i \equiv m$, and we let x_i refer to the displacement of particle i from its equilibrium position. N refers to the number of particles in the system, and we set

$$V(|x_{i+1} - x_i|) \equiv \frac{\alpha}{2}(x_{i+1} - x_i)^2 + \frac{\beta}{n}(x_{i+1} - x_i)^n, \qquad (2)$$

where n is assumed to be an even number and more than 2 in this work [4–14].

Our focus here will be on the time evolution of the system for cases where the nonlinear term in Eq. (2) controls the dynamics, i.e., where $\alpha = 0$. Due to the absence of the harmonic term in Eq. (2) *no acoustic propagation* is allowed in the system, i.e., sustained particle oscillations about the original equilibrium position of any particle become inadmissible. Besides the fact that such systems have not been well explored in the absence of dissipation, there are three primary motivations for this study. (i) Earlier work by Nesterenko [15–18] and others [19–29] have shown that systems such as the one we consider, with $\alpha = 0$, which exhibit *acoustic vacuum*, are intrinsically nonlinear, and typically exhibit unusual dynamical properties. Much theoretical, simulational, and experimental work has been done in the context of unloaded, discrete granular systems in acoustic vacua [22–30]. In 1D, these systems admit solitary waves, allow the formation of secondary solitary waves when solitary waves collide [31–34], and exhibit the recently suggested quasiequilibrium phase in finite systems [35–38], which is typically characterized by large, persistent energy fluctuations [19]. (ii) In principle there is no reason why a comprehensive analysis of the dynamics of Hamiltonians addressed by Eqs. (1) and (2) should *exclude* the case where there are only those bonds that can sustain softer than harmonic interactions at small compressions and stiffer than harmonic interactions at larger compressions (for an existing treatment of the purely nonlinear case in a weakly dissipative system, see, for example, Refs. [10,39]). And (iii) the purely nonlinear regime is simply inaccessible at the present time using analytic tools and accurate dynamical simulations are needed to help in the development of physical insights into this regime.

Dynamics of purely nonlinear systems are usually highly sensitive to the details of the initial and boundary conditions. In an earlier study, we have reported on the dynamics of systems described by Eq. (2) when perturbed by a δ-function *velocity* imparted to any particle at an initial time $t=0$ [36]. As we shall see in Sec. III B below, velocity perturbations lead to very different system dynamics compared to that seen when the dynamics is initiated by position perturbations. We explored the system dynamics for both periodic boundary conditions and fixed boundary conditions [36]. In the former case, $v_1 = v_{N+1}$, while in the latter case, the particle velocities were simply reversed during collisions with a boundary. We reported on calculations for cases with $\alpha = 0$, $\beta = 1$ (purely nonlinear), $\alpha = 1$, $\beta = 0$ (purely linear), and $\alpha = 1$, $\beta = 1$ (both linear and nonlinear effects being of comparable strength) in Eq. (2). When $\alpha = 0$, $\beta = 1$, such a perturbation results in both a stable propagating compression pulse, or a solitary wave, and an identical and opposite propagating dilation pulse, or an antisolitary wave. The solitary and antisolitary waves interact among themselves and with each other, leading to the eventual formation of an *equilibrium-like* state with excitations made up of a Gaussian-energy-distributed collection of only solitary and antisolitary waves and hence with sustained energy fluctuations [35–37]. In all of our studies with $\alpha = 1$, $\beta = 0, 1$, the system eventually slipped into an equilibrium state. However, interesting dynamics arises as α/β is varied

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and this will be discussed in future work [40]. Here we focus on the case $\alpha=0$. In contrast to earlier work, here we will consider the system dynamics when instead of a δ-function velocity perturbation at $t=0$, one or more quasistatically displaced particles are allowed to time evolve with the system *starting from rest* (i.e., from a zero-kinetic-energy state). As we shall see, for $\alpha=1$, $\beta=0$, the bond vibration efficiently disperses throughout the system, whereas for $\alpha=0$, $\beta=1$, the bond vibration(s) can be identified as "breathers" and are localized and sustained across many decades in time, with the system eventually beginning to slip into an equilibrium-like state with the qualification that the excitations in this state are comprised of solitary and antisolitary waves (as opposed to harmonic modes), as alluded to earlier. We have referred to this kind of state, which shows unusually large kinetic energy (and hence temperature) fluctuations, as the quasi-equilibrium state [19,26,34–38]. The well-studied problem of the dynamics of the $\alpha=1$, $\beta=1$ system [1,5,6,39] is discussed in closing with reference to the discussions in this study.

II. DETAILS OF THE CALCULATIONS

We solve the following coupled equations of motion for each of the N particles in our system:

$$m\ddot{x}_i = -\alpha[(x_{i+1}-x_i)-(x_i-x_{i-1})]$$
$$-\beta[(x_{i+1}-x_i)^{n-1}-(x_i-x_{i-1})^{n-1}], \qquad (3)$$

where we set $m=1$, $n=4$, and $N=100$ for most of the studies reported here. Some studies with $N=1000$ have also been performed. Since enlarging the system does not yield any special insights and is computationally challenging, we contend that such large system calculations at this stage are not warranted. We have also carried out studies with $n=6$ and 8. For steeper potentials the challenge to maintain high accuracy in the integration of the equations of motion becomes formidable. Preliminary results lead us to expect that the findings reported here for the $n=4$ case are broadly valid for larger even values of n.

We use the velocity-Verlet integration algorithm for carrying out the dynamical calculations [41]. The integration time step used was $\Delta t=10^{-5}$. The integrations were typically run for 10^5 time steps. Energy conservation was accurate to ten decimal places in our studies. We studied all the cases for both periodic and fixed boundary conditions. For fixed boundary condition studies the masses of the particles at the edges were set to be infinitely large, thereby allowing reflection of any propagating energy from the fixed boundaries (this is the same as what was reported in Ref. [36]). For periodic boundary conditions we imposed the condition that $x_1=x_{N+1}$ and $v_1=v_{N+1}$. As we shall see, the initial and the boundary conditions strongly influence the dynamics of these nonlinear systems, often leading to unexpected consequences.

III. RESULTS

We will now focus on the time evolution of one or more quasistatically displaced particles at rest at $t=0$ in a mono-

dispersed system described by the Hamiltonian in Eqs. (1) and (2) and where the dynamics of each particle is described by Eq. (3). We first make *three arguments* to describe the broad-brush-stroke picture of the dynamics of the perturbed bond. We next present the results from our dynamical simulations which are initiated by a single and two displaced particles at rest at $t=0$.

A. Argument 1: Bond energy must delocalize

Let us write some time-dependent collection of bond lengths as $L(t) \equiv \Sigma_j[x_{j+1}(t)-x_j(t)]$. $L(t)$ can now be thought of as a dynamical variable. Recall now that the Liouville equation, which describes the time evolution of a dynamical variable for a classical system, can be written as $dL(t)/dt = \{L(t),H\}$, where

$$\{L,H\} \equiv \sum_{i=1}^{N} \left[\left(\frac{\partial L}{\partial x_i}\right)\left(\frac{\partial H}{\partial p_i}\right) - \left(\frac{\partial L}{\partial p_i}\right)\left(\frac{\partial H}{\partial x_i}\right) \right].$$

Observe that the Poisson bracket $\{L,H\} \neq 0$ when $t > 0$ for H given by Eqs. (1) and (2). It is clear then that $L(t)$ is *not* a constant of motion. Hence, one would expect that, as $t \to \infty$, the energy imparted at $t=0$ to one or more perturbed bond(s) would indeed be eventually delocalized from the perturbed bond(s) and shared by more particles [42]. Further, given that the evaluation of the above Poisson bracket leads to velocity terms, there would be some spatial oscillations and/or diffusion associated with any long-lived vibration. The calculation of the Poisson bracket (not shown here but it can be easily done) reveals that this energy propagation will happen because of momentum transfer between particles. Thus, an initially oscillating bond (often called a breather) must eventually delocalize in time, a behavior seen here and in many other studies [43]. This delocalization should transpire regardless of the values that α, β might take (which is evident from performing the Poisson bracket computation above). It may be noted, however, that the rate at which momentum transfer happens along the chain may be highly nonuniform and dependent on the Hamiltonian and the nature of the initial and boundary conditions, and of course this is why the initial and boundary conditions play such an important role in these strongly nonlinear systems.

B. Argument 2: The virial theorem, system kinetic energy, and system dynamics

Since the system is assumed to have been perturbed quasistatically, the initial kinetic energy of the system is zero. Thus, initially all the energy is stored as potential energy. Given the virial theorem [44], which allows us to write a conservative system's kinetic and potential energy in terms of its total energy, it immediately follows (see Ref. [44]) that for a system with a potential energy function of the form $V(|x_{i+1}-x_i|)=(x_{i+1}-x_i)^n$, the average total kinetic energy $\langle K \rangle=[n/(n+2)]E$, where E is the total energies of the system. For our system with $\alpha=1$, $\beta=0$, $n=2$ and hence $\langle K \rangle = \frac{1}{2}E$, whereas for $\alpha=0$, $\beta=1$, $n=4$ and hence $\langle K \rangle = \frac{2}{3}E$ [19]. So the magnitude of $\langle K \rangle$ to be eventually attained through the energy distribution process is *significantly higher* when

$\alpha=0$, $\beta=1$ compared to the same in the harmonic case.

Of course, the virial theorem alone does not imply that this kinetic energy is for every particle to share, i.e., the system is ergodic. In principle, it is possible for the system to have all that kinetic energy reside on one or two or a few particles, and hence be nonergodic. However, in this system, our calculations suggest that energy does eventually delocalize and so in the arguments below we try to establish why such delocalization is to be expected.

The dispersion of energy from the perturbed *bonds* proceeds very differently for the harmonic and quartic systems. Solution of the equations of motion for $\beta=0$ in Eq. (2), i.e., for the harmonic case, shows that long-lived and slowly decorrelating oscillations of the particles can be used to characterize the energy transport in real time and real space from one particle to the next through harmonic bonds (the canonical ensemble statistical mechanics of this problem has been addressed comprehensively and rigorously in Ref. [45]). Regarding quartic systems, we first note that recent numerical and some analytical work on Hertz and Hertz-like potentials (see the comprehensive review in Ref. [19]), i.e., one-sided algebraic nonlinear potentials, has shown that the energy transport from one particle to the next starts off slowly when the interparticle bond is slightly compressed, since in all these potentials the repulsive potential is weaker than the harmonic potential at small enough compressions. Similar physics applies to the two-sided potential also. So, as the bond compression progresses, because of the nonlinear nature of the potential, the bond becomes energetically progressively expensive to compress and hence the compression must stop [19,28,29]. In the absence of a rigorous solution, we are unable to develop a theoretical description of what intuition suggests at this stage. The energy transport from one particle to the next is not effected via continued oscillations in the purely anharmonic potential case but rather solely via lumps of energy. These energy lumps turn out to be solitary waves when they arise from compression of anharmonic bonds or antisolitary waves when they come from dilation of anharmonic bonds [19,36]. A crude way of picturing this energy transport process due to the quartic term can be thought of as energy transport via *hopping* from one site to the next.

Given that the kinetic energy is $\frac{2}{3}E$, much of the potential energy in the perturbed bond must eventually be distributed among the system masses. Assuming that energy delocalization from a perturbed bond proceeds at *comparable rates* in time for the $\alpha=1$, $\beta=0$ and $\alpha=0$, $\beta=1$ cases, the harmonic chain would disperse $\frac{1}{2}E$ energy more rapidly than the $\frac{2}{3}E$ required for the quartic chain. As we shall see, this difference is evident from our calculations. But the real problem is more complex—and as the simulations below show, the hopping process proceeds at a fluctuating speed and kinetic energy transport has the propensity to stop often in the bond perturbation case. We have found it difficult to construct a *simple* theoretical argument to predict the sluggish nature of the energy hopping process.

C. Argument 3: Duffing oscillatorlike dynamics versus energy hopping

We offer below a way of looking at how the energy would leak out of a squeezed or stretched bond, with the perturbation being effected by the stretching or squeezing of adjacent masses in the purely nonlinear case. Now, in a mean-field sense, the dynamics of each particle in the perturbed quartic chain can be thought of as roughly equivalent to the dynamics of a Duffing oscillator, i.e., of a single particle in a purely quartic potential well [46]. The Duffing oscillator exhibits a weighted infinite set of odd frequencies, $\{\Omega(E), 3\Omega(E), 5\Omega(E), \ldots\}$, where the weighting depends on the energy E and $\Omega(E)=1+\gamma(\beta)E+O(E^2)$, with γ as some coefficient [46]. By performing simple simulations one can show that the damping-free Duffing oscillator shows sustained oscillations with symmetrically located uniform positive and negative velocities along the x-p plane accompanied by a quick crossover from one velocity to the other as it turns around—another way to think of a Duffing oscillator is by picturing the swaying motion of a pendulum with a bob at the end of a floppy wire (as opposed to the familiar example of a rigid rod). The most detailed and visually meaningful treatment of the dynamics of the Duffing oscillator may be readily found online in applets developed by Hsiao. However, when such an oscillator is coupled with the other degrees of freedom, in this case with the nearest neighbors, the resultant dynamics can be quite rich [47–50]. Initially, all the energy in the perturbed bond is potential energy. The virial theorem (Sec. III B) requires that the average potential energy per particle be $\frac{1}{3}E/N$. To attain this average, energy must leak out from the stretched or squeezed bond. But such leakage is possible only via the formation of multiple solitary and antisolitary waves in the vicinity of the squeezed or stretched bond as time progresses. These waves would be the vehicles needed to gradually export energy from the perturbed bond. The solitary and antisolitary waves thus formed will interact with each other. Since no acoustic propagation is allowed, the by-product of the interactions must always be solitary and antisolitary waves. While the details of these interactions are poorly understood at an analytic level at present, our simulations show the slow onset of an equilibriumlike phase. We conjecture that, until the average particle energies are attained, localized oscillations must remain in the perturbed bond or bonds. This is possibly why the breathers remain stable for extended times in these systems and then eventually abruptly lose their identity as the system slips into the quasiequilibrium state.

In this eventual phase, our calculations suggest that the system may not have significant memory of initial conditions in most instances, and shows a Gaussian distribution of velocities (as expected based on the central limit theorem). The typical kinetic energy fluctuations per particle seem somewhat large, at $\pm 10\%$ or more, from the mean value of $\frac{2}{3}E/N$ [19,36]. The kinetic energy fluctuations will of course vanish as $N\rightarrow\infty$. However, in these systems, the decay of these fluctuations with system size tends to be very slow compared to those which support acoustic propagation [19]. A point to be made, however, is that for certain initial conditions, it is possible that the system dynamics may be distinct. However, these cases are special due to the existence of high symmetries and will be addressed in a separate study [51].
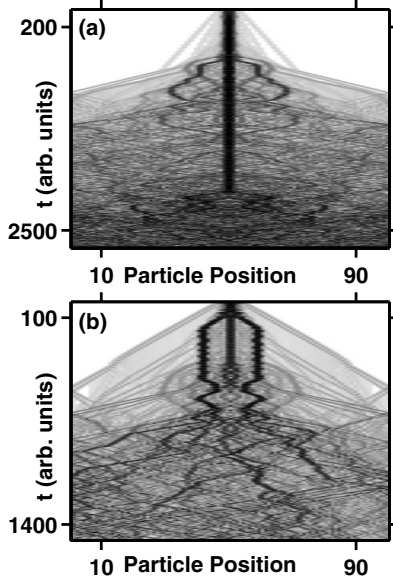
FIG. 1. (a) Formation of an even-parity breather with fixed boundary condition at time $t=0$ (energy in gray scale versus space versus time data) in a purely quartic chain where acoustic propagation is not admissible. (b) Formation of odd-parity breathers with fixed boundary conditions. Both (a) and (b) are for systems with fixed boundary conditions, as is evident from boundary reflection.

D. Results from dynamical simulations: Unstable energy localizations and (anti)solitary wave collisions in acoustic vacuum

We have performed extensive dynamical simulations to probe the time-dependent behavior of the coupled Newton equations in Eq. (3) for three cases, when $\alpha=0$, $\beta=1$, $\alpha=1$, $\beta=0$, and $\alpha=1$, $\beta=1$. The results of our studies can be presented in a variety of ways and indeed in a previous publication [36] some details such as the confirmation of the Gaussian distribution of velocities at late times (an expected result based on the central limit theorem) and the existence of sustained fluctuations in the average kinetic energy per particle have been discussed [19,35–38]. Here we focus on how the potential energy imparted for an even-parity perturbation (because an even number of particles are being perturbed) or in an odd-parity perturbation (where one particle has been perturbed) gradually gets dispersed into the system. Both of these cases lead to the emergence of long-lived localized excitations—the so-called breathers. Our studies suggest that an appropriate way of studying this energy dispersion process may be via contour plots with the magnitude of energy plotted against particle position and time. This is how we have depicted our results in Figs. 1–5 below.

In Fig. 1(a), we show the system dynamics when two adjacent particles are quasistatically stretched (or squeezed) and let go, resulting in an even-parity perturbation. Fixed boundary conditions are employed in the studies reported in Figs. 1(a) and 1(b). Such a perturbation initially stores all energy as potential energy and hence in the perturbed bonds that connect these particles at time $t=0$. The dynamics that follows has two essential features. Argument 1 implies that the energy in the bond must spread out into the chain,
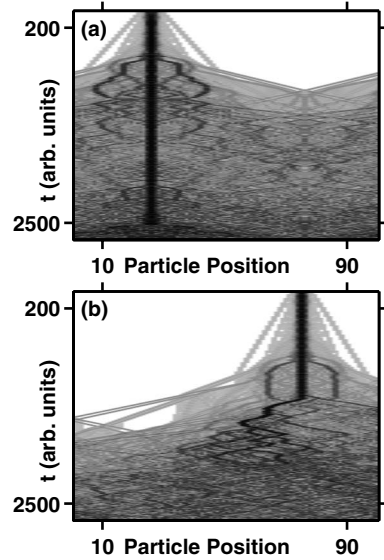

FIG. 2. (a) Energy in gray scale versus space versus time for an off-centered even-parity perturbation in a quartic chain with periodic boundary condition. (b) Effects of off-centered even-parity perturbation is shown here for a chain with fixed boundaries. Observe the diffusion of the breathers away from the nearest boundary.

whereas argument 2 demands that the average kinetic and potential energies per particle be $\frac{2}{3}E/N$ and $\frac{1}{3}E/N$, respectively. Because acoustic oscillations are prohibited, the only way to satisfy arguments 1 and 2 would be to emit solitary and antisolitary waves from the vibrating bond(s) until argument 2 is satisfied. In Fig. 1(a) one can see these solitary and antisolitary waves emanating symmetrically from the central breather.

As alluded to above, dynamical calculations show energy leakage. Attendant to this leakage are small-scale movements of the oscillating bonds that constitute any breather. In addi-
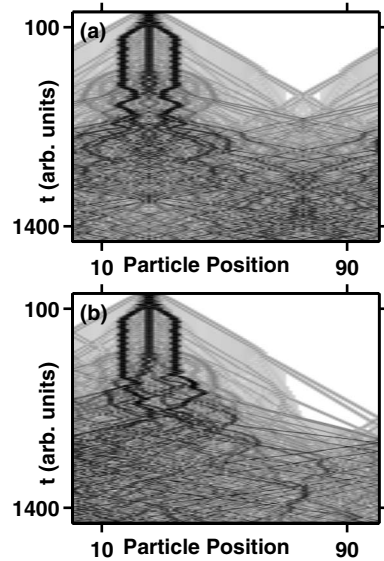

FIG. 3. (a) Energy in gray scale versus space versus time for a periodic quartic chain with an odd parity perturbation, resulting in three unstable breathers and solitary and antisolitary waves. (b) Here we show the same data as in (a) for the case of fixed boundary.
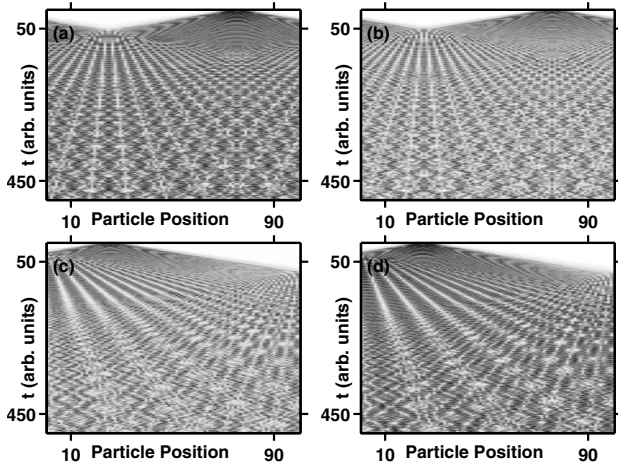
FIG. 4. (a) Energy versus space versus time for a harmonic chain subjected to an even-parity perturbation. (b) The same system for an odd-parity perturbation. In both cases the perturbations are off centered. Our study suggests that the system retains no memory of the initial perturbation and efficiently equipartitions its energy. In (c) and (d) we show the cases of evolution of odd- and even-parity perturbations, respectively, for a system with periodic boundary condition.

tion, we find a peculiar kind of dynamics that is generic to the breather modes. When the initial or main breather in Fig. 1(a) has weakened sufficiently as a result of small-scale movements or oscillation type dynamics, it produces two more breathers, each on one side. The two new breathers thus produced drift but only to get localized after moving some ten or so bond lengths. Breathers such as the central
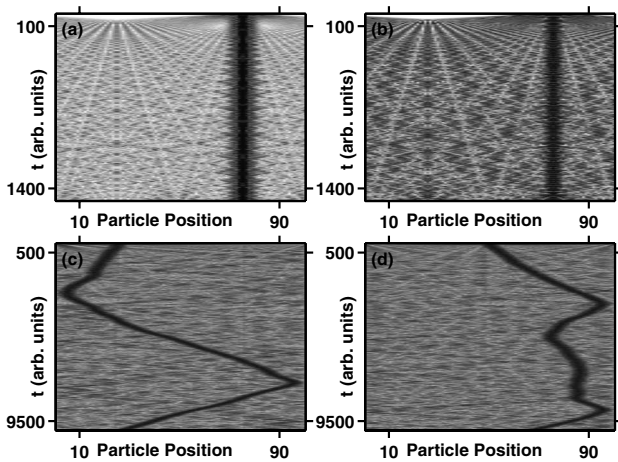

FIG. 5. Energy gray scale results versus space versus time for a mixed (harmonic+quartic) chain with periodic boundary condition subjected to (a) even-parity and (b) odd-parity perturbation. In both cases the system yields one breather although the spatial extents of the breathers are unequal. In (c) and (d) we show odd-parity breathers in a mixed system with fixed boundary conditions. Off-centered breathers such as the one in (c) and centered breathers such as the one in (d) are highly diffusive. The time data are shown over nearly two decades. The spatiotemporal energy landscape is devoid of patterns, which suggests that systematic energy minima and maxima may not exist in the landscape shown in (c) and (d).

breather that carry more energy [and hence are darker in Fig. 1(a)] tend to hold their identities in spite of movements. Those that carry less energy are more mobile. Our studies suggest that the length scale associated with these drifts depends upon the nature of the potential energy function in Eq. (2). The breathers possess too much potential energy compared to kinetic energy and it is hence energetically expensive to move significant amounts of energy over distances of several bond lengths.

Similar features are seen in Fig. 1(b), where a single particle is quasistatically perturbed to produce a compressed and a stretched bond, resulting in an odd-parity perturbation. The central particle's oscillations lead to complex short-term dynamics, leading to the formation of a solitary and an antisolitary wave, which carry much of the kinetic energy that is initially available to the system. The system quickly forms three breathers, one central breather that is centered in the vicinity of the perturbed particle and two symmetrically positioned ones that are adjacent to the central breather again at a distance of about ten or so bond lengths. Solitary and antisolitary waves continue to form from the two side breathers, which oscillate back and forth continuously during the process of creation of the solitary and antisolitary waves. The production of the waves toward the central breather destabilizes the same, and eventually the dissipation of the central breather destabilizes the two side breathers. At such late times, interactions between the breathers and the solitary and antisolitary waves end up making the breathers more mobile, and eventually the system cascades into a quasiequilibrium phase. The odd-parity perturbation produces a system that is significantly less stable than in the even-parity case. We are presently unable to offer more detailed theoretical arguments to justify the emergence of the three breathers seen in Figs 1(a) and 1(b) and also later [see Figs. 2(a), 2(b), 3(a), and 3(b)].

Figures 2(a) and 2(b) record the dynamics of the even-parity perturbation in which a single breather is initially produced as in Fig. 1(a). In Fig. 2(a), we use periodic boundary conditions and show that a perturbation initiated at any position results in identical dynamics as in Fig. 1(a). In Fig. 2(b) we show results from the same even-parity perturbation but now initiated near a boundary in a fixed boundary system. The dynamics in this case turns out to be markedly different than in the periodic boundary condition case. At early enough times the dynamics remains identical to the periodic boundary condition problem shown in Fig. 2(a). However, the reflected waves from the nearest fixed boundary destabilizes the central breather, making it drift in a direction that is opposite to that of the nearest wall. We have observed similar effects (not shown) when we study the case where the even-parity perturbation is effected near the other well, i.e., the breather eventually drifts in the direction opposite to that of the nearest wall. Identical effects are shown in Figs 3(a) and 3(b) where in Fig. 3(a) we show that the dynamics associated with an odd-parity perturbation is position independent in a periodic chain but is less stable and suffers a marked repulsion by the boundary wall in a fixed boundary system [see Fig. 3(b)].

E. Results from dynamical simulations: Harmonic case and the mixed chain

To understand the peculiar dynamics of the fully nonlinear system in terms of breathers and solitary and antisolitary waves in described in Secs. III A–III D, it is instructive to contrast with the dynamics of the system as described by Eq. (2) when sound propagation is possible. We first describe the dynamics of the purely harmonic chain, i.e., when $\alpha=1$, $\beta=0$ in Eq. (2).

Figures 4(a) and 4(b) show the spatiotemporal description of the system dynamics for a harmonic chain with periodic boundary condition when the dynamics is initiated by perturbing two adjacent particles and when the same is initiated by a single-particle perturbation that has been stretched out of equilibrium. Figures 4(a) and 4(b) show that in a harmonic system, irrespective of the nature of the initial perturbation, the system undergoes *identical* time evolution. Figures 4(c) and 4(d) show the time evolution of the system when a single-particle and a two-particle displacement, respectively, initiate the excitation at $t=0$. These studies are carried out in the presence of fixed boundary conditions. As expected, for this harmonic system, the dynamics is that of a linear system and is completely independent of initial conditions.

The dynamics of the $\alpha=1$, $\beta=1$ system in Eq. (2) turns out to be very different from the dynamics under conditions of acoustic vacuum and that of the purely harmonic chain. Like the case of the harmonic chain, the $\alpha=1$, $\beta=1$ and equivalent cases have been extensively discussed in the literature and continue to be of much current interest. We discuss this case in the present context briefly below.

In Fig. 5(a) we present the time evolution of the *periodic* system when two adjacent particles (in this case particles 74 and 75) are quasistatically pulled closer by equal and opposite displacements and released at $t=0$. In Fig. 5(b) only one particle (particle 75) is quasistatically displaced and released at $t=0$. The total energies in both cases are kept the same. We recall that, in acoustic vacuum, these two cases lead to very different dynamics. The former case forms one metastable breather while the latter produces three metastable breathers. Interestingly, here we see that in both cases the system produces a *single* breather. It should be noted though that the spatial extent of these breathers are different, with the one in the even-parity case in Fig. 5(a) being wider than the one in the odd-parity case in Fig. 5(b). However, the striking differences in the system dynamics between the cases when there is an acoustic vacuum versus the case when there is acoustic propagation suggest that there are significant differences between the ways in which the system shares its energy in the two cases. Early studies on this problem may be found in Refs. [52–54], where a mode-analysis-based theoretical approach was advanced. Our simulations confirm that sound waves play a crucial role in breather formation and breather diffusion in these systems. The remarkable stability of these breathers in Figs. 5(a) and 5(b) suggests that in periodic systems the energy carried by the breathers remain approximately localized due to the existence of periodic oscillations arising from the solitary and antisolitary waves and the decorrelating acoustic vibrations at each site [45]. Although our data are shown only up to t

$=1500$ here, these calculations have been performed up to t $=10\,000$ and we do not see *any* diffusive behavior of the breathers for the data shown in Figs. 5(a) and 5(b).

The introduction of the *fixed* boundary condition causes significant changes in the time evolution of the system. The results shown in Figs. 5(c) and 5(d) are shown for the entire length of the simulations. Clearly, the spatial extents of the breathers are not fixed. In Fig. 5(c), we show the evolution starting with particle 25 being quasistatically stretched. In Fig. 5(d), particle 50 has been quasistatically stretched. What is remarkable is how mobile these breathers are and how stable they remain as they diffuse. The widths of the breathers also vary as they diffuse, suggesting that the breather continues to strongly interact with the rest of the system during diffusion. Our studies suggest that the diffusive nature of the breathers in the fixed boundary case is likely due the lack of stable regions in the chain where the breathers can position themselves without having to exchange significant amounts of kinetic energy or transfer momentum with passing waves. It turns out that the breathers not only give momentum to the passing waves (solitary, antisolitary, acoustic and combinations thereof) but also gain momentum from them. An extensive study to explore boundary effects will be discussed in a separate investigation [55].

V. SUMMARY AND CONCLUSION

We have considered the problem of time evolution of a quasistatically perturbed bond or bonds in a chain of masses connected by springs. We have focused on the dynamical problem in which the springs are quartic in this work. It turns out that in such a system, the absence of the harmonic term implies that sound waves cannot propagate through the chain—and hence it is in acoustic vacuum. To propagate energy through acoustic vacuum in the presence of a quartic or similar power law potential the system resorts to making compression pulses, which are solitary waves, and dilation pulses, which are antisolitary waves. Localized vibrations that would arise from positional perturbation with zero velocities at $t=0$ manifest themselves as long-lived stable oscillations or breathers. In the presence of acoustic vacuum, the breathers relax by emitting solitary and antisolitary waves. The breathers also end up diffusing slowly in these systems, with the diffusion typically happening because the breather is unstable after the emission of a solitary-antisolitary wave pair or because the breather collided with a solitary or antisolitary wave. In systems with fixed boundary conditions, one sees that the breathers tend to move away from the nearest boundary due to such collisions. The diffusion process is discontinuous in time, resulting in regions where breathers localize for a while before moving on. The exact reasons for breather localization are not known, and perhaps the level of clarity needed for constructing an analytic description of the nonlinear dynamics is yet to emerge. The dynamics of systems with quasistatic bond perturbations hence turns out to be entirely different from what is seen when a velocity perturbation initiates the system dynamics [36].

To contrast the dynamics in acoustic vacuum with dynamics in the presence of acoustic propagation, we have pre-

sented a brief study of the purely linear chain, which has also been studied elsewhere via exact analysis [45]. We show that in the purely linear chain the dynamics is independent of initial conditions and the system quickly equipartitions its energy. No localized modes are found. We have also contrasted the acoustic vacuum system with a study of time evolution when both linear and nonlinear terms are present. Here we find that boundary conditions play a critical role. Under periodic boundary conditions, the system produces highly stable breathers which are not perturbed by waves passing through it. Further, in all the cases we investigated, the system prefers to make a single breather. In the presence of a fixed boundary the breathers are found to be remarkably diffusive, and we contend that the acoustic, solitary, and antisolitary waves can exchange sufficient momentum with the breather to make it move. These effects have been observed before but there is no dynamical analysis that is available to explain the origins of such behavior at this time.

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