Small traveling clusters in attractive and repulsive Hamiltonian mean-field models

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Long-lasting small traveling clusters are studied in the Hamiltonian mean-field model by comparing between attractive and repulsive interactions. Nonlinear Landau damping theory predicts that a Gaussian momentum distribution on a spatially homogeneous background permits the existence of traveling clusters in the repulsive case, as in plasma systems, but not in the attractive case. Nevertheless, extending the analysis to a two-parameter family of momentum distributions of Fermi-Dirac type, we theoretically predict the existence of traveling clusters in the attractive case; these findings are confirmed by direct N-body numerical simulations. The parameter region with the traveling clusters is much reduced in the attractive case with respect to the repulsive case.

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I. INTRODUCTION

The Hamiltonian dynamics of discrete particle systems interacting at long range is usually approximated by the associated Vlasov equation, sometimes also called collisionless Boltzmann equation. The Vlasov approximation is used in various fields such as plasma physics and cosmology and is at the root of some peculiar dynamical properties of these Hamiltonian systems. Indeed, the Vlasov dynamics itself does not relax toward the discrete statistical equilibrium and instead often approaches one of its infinite numbers of stationary solutions. The time scale on which the Vlasov approximation is valid diverges with the number of particles N, and the original particle system thus remains trapped for a long time close to this stationary solution. The stationary solution is hence sometimes called a quasistationary state (QSS) for the particles dynamics.

Physical examples of such a scenario include galaxy formation [1,2], wave-particle interactions in plasmas [3], free-electron laser dynamics [4], and beam particles dynamics [5]. To avoid technical and computational difficulties, it is fruitful to introduce simplified toy models. One of the simplest is the Hamiltonian mean-field (HMF) model [6,7], in which particles are confined on the unit circle and have XY-type interactions. The model is widely used to study relaxation and QSS in long-range interacting systems [8–11].

In addition to the QSSs described above, some numerical results in the HMF model point to the alternative phenomenon of "traveling clusters" [11,12]. Due to the periodic boundary condition of the HMF model, the persistence of traveling clusters implies that the N-body dynamics approaches a periodic solution rather than a stationary solution and stays close to it for a long time before relaxing to the statistical equilibrium. A natural interpretation of this phenomenon would rely on an asymptotically stable periodic solution to the associated Vlasov equation, just like QSSs are associated with stable stationary solutions to the Vlasov equation. We thus refer to this N-body dynamical state with traveling clusters as a quasiasymptotic periodic state (QAPS). This general idea leaves many open questions: (i) Which initial conditions yield QAPS? (ii) How are the frequencies selected? (iii) Which type of interactions between attractive and repulsive is more favorable to observe QAPS? The purpose of this paper is to answer these questions in the context of the HMF model when initial conditions are perturbations of a homogeneous background.

In the field of plasma physics, a large body of work has been devoted to solutions to the Vlasov equations, including periodic ones (see [13] for a historical paper). Several studies, which often go under the name "nonlinear Landau damping," were devoted to the study of asymptotic periodic states consisting of traveling waves [14–16]; see also [17] for examples in two-dimensional fluid dynamics, where the term "quasimode" is often used. To our knowledge, previous numerical experiments were mainly restricted to plasmas and thus to systems with repulsive interactions [18–20]. In this paper, we make use of techniques developed in plasma physics in the context of the N-body HMF model both with attractive and repulsive interactions. We derive criteria to predict the presence of QAPS after a small to moderate perturbation of an initial homogeneous stationary state. We show that in the presence of attractive interactions, the favorable parameter zone to observe QAPS is drastically reduced with respect to the repulsive case but does not vanish. These predictions are then examined against N-body numerical simulations.

II. MODEL

The Hamiltonian of the HMF model [6,7] is

$$H=\sum_{i=1}^{N}\frac{p_i^2}{2}+\frac{1}{2N}\sum_{i,j=1}^{N}[1-c\cos(\theta_i-\theta_j)], \qquad (1)$$

where $(\theta_i)_{i=1}^N$ and $(p_i)_{i=1}^N$ denote angles of particles and their conjugate momenta and c is a coupling constant. We will use either $c=1$ (attractive case) or $c=-1$ (repulsive case). This model with $c=1$ ($c=-1$) retains only the first Fourier component of the gravitational (Coulomb) force in the one-dimensional self-gravitating (plasma) system. The role of the gravitational or electric field is played by the magnetization $\vec{M}(t)=[M_x(t),M_y(t)]$, defined by $M_x+iM_y=\sum_{j=1}^{N}e^{i\theta_j}/N$. In the continuous limit, the state of the system is described by the one-particle distribution $f(\theta,p,t)$ instead of the θ_i's and p_i's. The dynamics of the N-body system (1) is then given by the associated Vlasov equation,

$$\partial_t f + p\partial_\theta f - c\int\int \sin(\theta-\theta')f(\theta',p',t)d\theta' dp' \partial_p f = 0, \tag{2}$$

which is valid over a time scale diverging with N [21]. This time scale is at least ln N but may be much larger [10,22,23].

III. LINEAR THEORY

All homogeneous distributions $f_0(p)$ are stationary for Eq. (2). We consider in the following even single-humped homogeneous distributions $f_0(p)$ plus a small to moderate perturbation,

$$f(\theta,p,t=0) = f_0(p)(1 + a\cos\theta), \tag{3}$$

where a is the amplitude of the perturbation. We look for criteria predicting the observation of a QAPS on top of the homogeneous background $f_0(p)$ or on the contrary the complete relaxation to $f_0(p)$, that is a QSS.

The linear theory around a given $f_0(p)$ yields the Landau dispersion relation [6] for wave numbers $k=\pm 1$,

$$\epsilon(k,\omega) = 1 + c\pi P\int_{-\infty}^{+\infty}\frac{f_0'(p)}{p-\omega/k}dp + ci2\pi^2\eta(\omega)\frac{k}{|k|}f_0'(\omega/k) = 0, \tag{4}$$

where P denotes the principal value, and $\eta(\omega)=0$, 1/2, and 1 for positive, zero, and negative Im(ω), respectively. For $k \neq \pm 1$ no dispersion relation is obtained as the Fourier components of the potential vanish in the HMF model. Perturbation (3) is then roughly proportional to $e^{-i\omega t}$, where ω is a root of $\epsilon(k,\omega)$. We will express the criteria using the roots of $\epsilon(1,\omega)$.

First, we note that, at variance with the plasma or repulsive HMF cases, an even single-humped $f_0(p)$ may be unstable for the attractive HMF. A perturbation would then drive the system toward an inhomogeneous state, where $M = \|\vec{M}\| > 0$, which is beyond our scope. The first requirement is thus the stability of $f_0(p)$, that is, Im$(\omega)<0$, for all ω roots of dispersion relation (4).

IV. NONLINEAR LANDAU DAMPING AND BERNSTEIN-GREENE-KRUSKAL WAVES

Linear theory predicts that all perturbations of a stable $f_0(p)$ should asymptotically decay by Landau damping. Accordingly, the N particle dynamics should approach a QSS. However, it has been known since a long time in plasma physics that nonlinear effects may prevent the complete relaxation to $f_0(p)$ [14]. If the damping is sufficiently weak, some particles may be trapped in the resonance created by the perturbation. The trapping happens if the Landau relaxation time t_L is larger than the trapping time t_D. Calling $\omega^* = \omega_1 + i\omega_2$ $(\omega_1,\omega_2 \in \mathbb{R})$ the root of Eq. (4) closest to the real axis, the critical ratio $q_c = t_L/t_D = \sqrt{a}/|\omega_2|$ separating full damping from trapping was numerically found to be close to $q_c=1$ for a plasma [20]. The trapping condition then reads

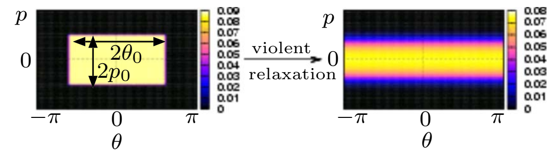

FIG. 1. (Color online) Schematic picture of the violent relaxation on μ space from the waterbag distribution (left) to the homogeneous Lynden-Bell distribution (right).

$$-\omega_2 \lessapprox \sqrt{a}, \tag{5}$$

where ω_2 is negative due to the stability requirement.

Under the trapping condition, we may expect a wave with velocity set by ω_1 and amplitude a' as asymptotic state of the system. From symmetry of $f_0(p)$, there must actually be two waves, with velocities $\pm\omega_1$. Despite these waves being nonlinear in essence, a superposition is possible if the resonances created by the different waves do not overlap [15]. The nonoverlapping condition yields another criterion, $\sqrt{a'} \lessapprox |\omega_1|$, since the width of the resonance is proportional to $\sqrt{a'}$. If a and a' are of the same order of magnitude, the nonoverlapping condition is

$$\sqrt{a} \lessapprox |\omega_1|. \tag{6}$$

Whereas criterion (5) is widely used in plasma physics, we are not aware of a previous appearance of criterion (6) in this context. The above criteria are summarized as

$$0 < -\omega_2 \lessapprox \sqrt{a} \lessapprox |\omega_1|. \tag{7}$$

We now examine criteria (7) on concrete examples. For this purpose, we choose as $f_0(p)$ a two-parameter family of homogeneous Lynden-Bell distributions $f_{\rm LB}(p)$,

$$f_{\rm LB}(p) = \frac{n_0}{e^{\alpha+\beta p^2/2}+1}. \tag{8}$$

These distributions appear as the result of the violent relaxation of a waterbag initial condition [1,4,11,24–26]. The Lynden-Bell distributions are parametrized by magnetization $M_0=\sin\theta_0/\theta_0$ and energy $U=p_0^2/6+(1-cM_0^2)/2$ of the corresponding rectangle waterbag distributions shown in Fig. 1 [24]. The magnetization M_0 controls the crossover of the Lynden-Bell distributions from a waterbag $(M_0=0,\ \theta_0=\pi)$ to a Gaussian $(M_0=1,\ \theta_0=0)$ [see Fig. 2 for examples of Lynden-Bell distributions]. We prepare the two-parameter family for the attractive case and use the same family for the repulsive case to compare the two types of interactions.

The constraints imposed by criteria (7) are summarized in Figs. 3 and 4 for the attractive and the repulsive cases, respectively. Clearly, the favorable zone for the appearance of QAPS is much wider in the repulsive case, but the attractive HMF model does present a parameter region where QAPS should be observable.

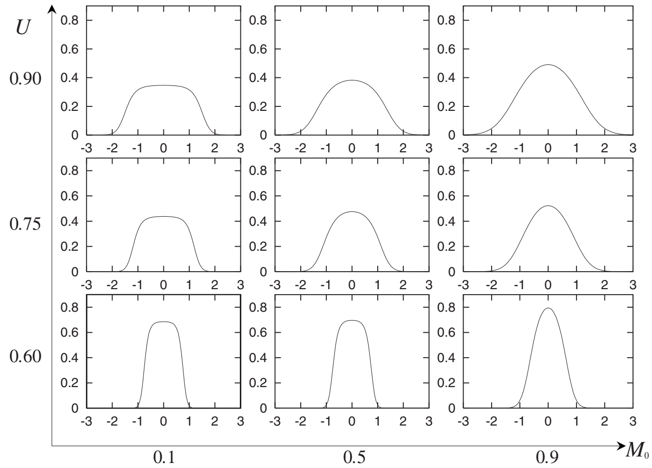

FIG. 2. Examples of the Lynden-Bell distributions (8) on the parameter plane (M_0, U). In each of nine panels, the horizontal axis represents p and the vertical $f_{LB}(p)$.

V. NUMERICAL TESTS

We perform N-body numerical simulations with $N=10^6$ particles whose positions and momenta are randomly drawn according to Eq. (3). The perturbation amplitudes are $a=0.01$, 0.05, and 0.1. We observe whether the system relaxes to a QSS close to $f_0(p)$ or evolves toward a QAPS, manifested by long lasting oscillations in the magnetization $[M_x(t), M_y(t)]$. The attractive and the repulsive cases are reported in Figs. 5 and 6 for the three points marked on Figs. 3 and 4, respectively.

In the attractive case, the point with $M_0=0.02$ satisfies criteria (7), but the points with $M_0=0.5$ and $M_0=1$ break criteria (5) and (6), respectively. Accordingly, the power spectrum of $M_x(t)$ shows a peak around $|\omega_1|$ for $M_0=0.02$ and no peak for $M_0=0.5$ and 1. The presence of two traveling clusters in the first case creates bumps in the momentum

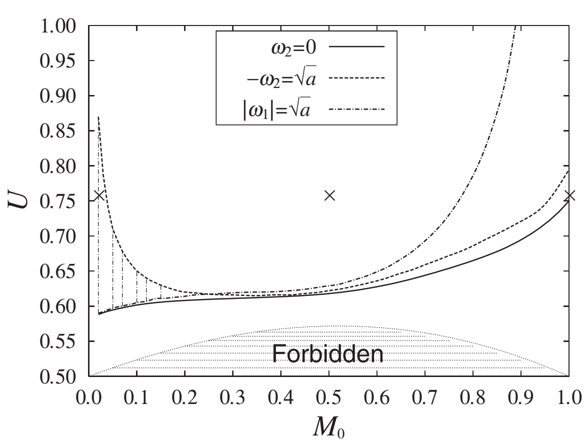

FIG. 3. Graphical representation of criteria (7) for $a=0.01$ in the attractive case. The favorable zone to observe QAPS is represented as the vertically hatched area. The three crosses ($\times$) on the line $U=0.76$ mark the sample points for N-body simulations (see Fig. 5). The horizontally hatched area is forbidden due to energy constraint; its boundary is expressed as $2U=1+(M_0\theta_0)^2/(\pi^2-\theta_0^2)$, where θ_0 solves $M_0\theta_0=\sin\theta_0$.

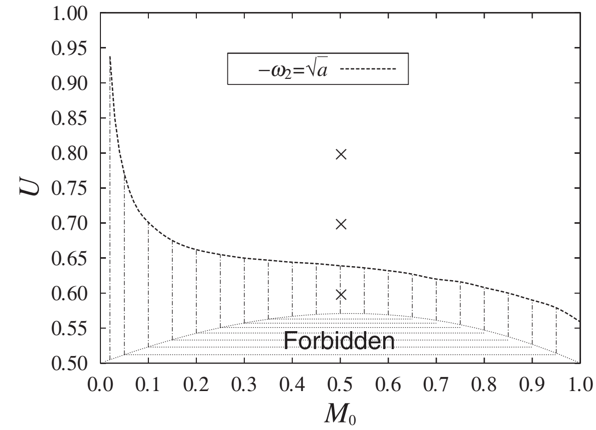

FIG. 4. The same as Fig. 3 but in the repulsive case. The two criteria $\omega_2<0$ and $|\omega_1|>\sqrt{a}$ are satisfied in the whole area, and hence the corresponding curves do not appear. The three crosses ($\times$) on the line $M_0=0.5$ mark the sample points for N-body simulations (see Fig. 6).

distribution, as shown on Fig. 7. The $M_0=1$ point satisfies Eq. (5) and not Eq. (6) but is somewhat peculiar as $\omega_1=0$; we note that for another point in this zone with $\omega_1\neq 0$, the pictures are similar to Figs. 5(c) and 5(f) (not reported). The repulsive case is in Fig. 6. The power spectrum shows a clear peak around $|\omega_1|$ for $U=0.6$ (inside the favorable zone) and much smaller ones for $U=0.7$ and 0.8 (outside the favorable zone). This is in accordance with the analytical predictions; the presence of small peaks in *a priori* unfavorable cases indicates that the repulsive HMF sustains the oscillations much more easily than the attractive one.

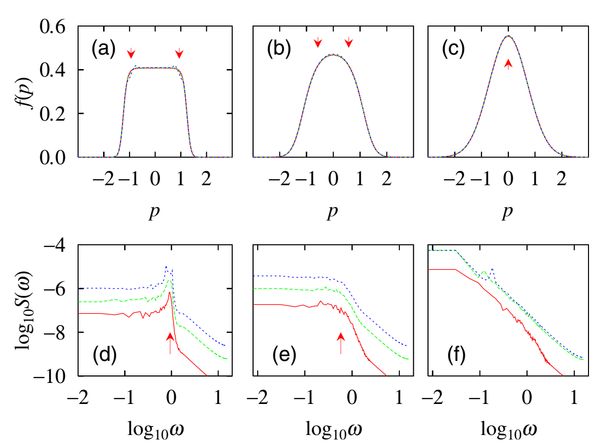

FIG. 5. (Color online) Attractive case. Momentum distributions at $t=205$ (upper panels) and power spectra of $M_x(t)$ (lower panels). $U=0.76$ and $M_0=0.02$, 0.5, and 1 from left to right. The amplitude of perturbation is $a=0.01$ (red solid), 0.05 (green dashed), and 0.1 (blue dotted) from bottom to top in lower panels; the corresponding momentum distributions almost collapse onto the initial Lynden-Bell distribution. The normalization condition for momentum distributions is $\int_{\mathbb{R}} f(p)dp=1$. Arrows mark the positions of $\pm\omega_1$. Figures are produced as averages over 50 realizations with $N=10^6$ and in the interval $0\leq t\leq 205$ for the power spectra. Note that each of $a=0.05$ and 0.1 for $M_0=1$ yields a drifting inhomogeneous background, and hence the small peaks on panel (f) are of no interest.

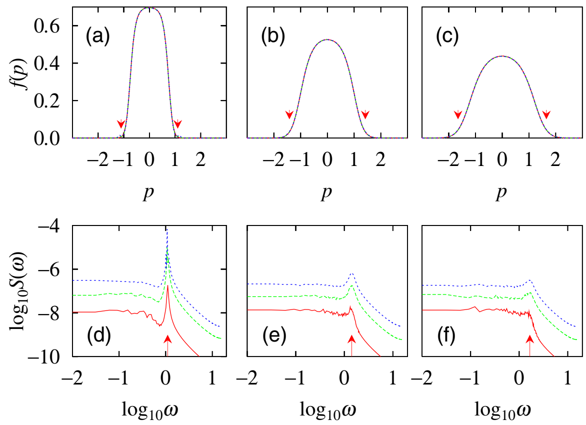

FIG. 6. (Color online) The same as Fig. 5 but repulsive case. $M_0=0.5$ and $U=0.6$, 0.7, and 0.8 from left to right.

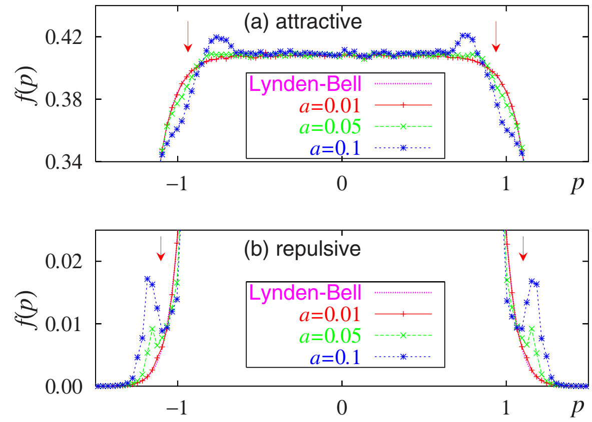

FIG. 7. (Color online) Magnification of Figs. 5(a) and 6(a). In both cases, the presence of traveling clusters creates dips and bumps in the momentum distribution.

VI. DISCUSSION

We asked in Sec. I the three questions [(i)–(iii)] on the appearance of traveling clusters or QAPS. From the N-body numerical simulations, we conclude that the answers are as follows. (i) Criteria (7) predict the initial conditions yielding QAPS under small to moderate perturbations like Eq. (3); (ii) frequencies of the traveling clusters are estimated by the root of the Landau dispersion relation closest to the real axis; and (iii) both attractive and repulsive interactions may sustain traveling clusters, but repulsive ones are more favorable. In particular and in contrast with the repulsive case, there can be no traveling clusters close to Gaussian distribution in the attractive HMF.

The criteria derived in this paper are based on the Landau dispersion relation. Thus, for initial conditions very far from a stationary solution of the associated Vlasov equation, like those used in [11,27], it breaks down. We note however that even in this case, the general idea of seeing QAPS as being close to periodic solutions to the Vlasov equation should still be valid.

It would be very interesting to generalize this study close to inhomogeneous stationary states: this could help us to understand the observations in [12]. Beyond the HMF model, the search for quasiasymptotic periodic states in other long-range interacting systems would be of course a very desirable future work, for which the general techniques used in this paper may be useful.

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[1] D. Lynden-Bell, Mon. Not. R. Astron. Soc. **136**, 101 (1967).
[2] T. Tsuchiya, T. Konishi, and N. Gouda, Phys. Rev. E **50**, 2607 (1994).
[3] Y. Elskens and D. Escande, *Microscopic Dynamics of Plasmas and Chaos* (Institute of Physics, Bristol, 2003).
[4] J. Barré, T. Dauxois, G. De Ninno, D. Fanelli, and S. Ruffo, Phys. Rev. E **69**, 045501(R) (2004).
[5] C. Benedetti, S. Rambaldi, and G. Turchetti, Physica A **364**, 197 (2006).
[6] S. Inagaki and T. Konishi, Publ. Astron. Soc. Jpn. **45**, 733 (1993).
[7] M. Antoni and S. Ruffo, Phys. Rev. E **52**, 2361 (1995).
[8] V. Latora, A. Rapisarda, and C. Tsallis, Phys. Rev. E **64**, 056134 (2001).
[9] F. Baldovin and E. Orlandini, Phys. Rev. Lett. **96**, 240602 (2006).
[10] Y. Y. Yamaguchi, J. Barré, F. Bouchet, T. Dauxois, and S. Ruffo, Physica A **337**, 36 (2004).
[11] A. Antoniazzi, F. Califano, D. Fanelli, and S. Ruffo, Phys. Rev. Lett. **98**, 150602 (2007).
[12] H. Morita and K. Kaneko, Phys. Rev. Lett. **96**, 050602 (2006).
[13] I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. **108**, 546 (1957).
[14] T. O'Neil, Phys. Fluids **8**, 2255 (1965).
[15] M. Buchanan and J. J. Dorning, Phys. Rev. Lett. **70**, 3732 (1993).
[16] J. P. Holloway and J. J. Dorning, Phys. Rev. A **44**, 3856 (1991); M. Buchanan and J. J. Dorning, Phys. Rev. E **52**, 3015 (1995); C. Lancellotti and J. J. Dorning, Phys. Rev. Lett. **80**, 5236 (1998); M. C. Firpo and Y. Elskens, *ibid.* **84**, 3318 (2000); C. Lancellotti and J. J. Dorning, Phys. Rev. E **68**, 026406 (2003).
[17] R. J. Briggs, J. D. Daugherty, and R. H. Levy, Phys. Fluids **13**, 421 (1970); D. A. Schecter, D. H. E. Dubin, A. C. Cass, C. F. Driscoll, I. M. Lansky, and T. M. O'Neil, *ibid.* **12**, 2397 (2000).
[18] G. Manfredi, Phys. Rev. Lett. **79**, 2815 (1997).
[19] M. Brunetti, F. Califano, and F. Pegoraro, Phys. Rev. E **62**,

4109 (2000).
[20] A. V. Ivanov, I. H. Cairns, and P. A. Robinson, Phys. Plasmas **11**, 4649 (2004).
[21] W. Braun and K. Hepp, Commun. Math. Phys. **56**, 101 (1977).
[22] Y. Y. Yamaguchi, Phys. Rev. E **68**, 066210 (2003).
[23] E. Caglioti and F. Rousset, Arch. Ration. Mech. Anal. **190**, 517 (2008).
[24] A. Antoniazzi, D. Fanelli, S. Ruffo, and Y. Y. Yamaguchi, Phys. Rev. Lett. **99**, 040601 (2007).
[25] P.-H. Chavanis, Eur. Phys. J. B **53**, 487 (2006).
[26] Y. Y. Yamaguchi, Phys. Rev. E **78**, 041114 (2008).
[27] R. Bachelard, C. Chandre, D. Fanelli, X. Leoncini, and S. Ruffo, Phys. Rev. Lett. **101**, 260603 (2008).