Entropy of network ensembles

Ginestra Bianconi
The Abdus Salam International Center for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy
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In this paper we generalize the concept of random networks to describe network ensembles with nontrivial features by a statistical mechanics approach. This framework is able to describe undirected and directed network ensembles as well as weighted network ensembles. These networks might have nontrivial community structure or, in the case of networks embedded in a given space, they might have a link probability with a nontrivial dependence on the distance between the nodes. These ensembles are characterized by their entropy, which evaluates the cardinality of networks in the ensemble. In particular, in this paper we define and evaluate the *structural entropy*, i.e., the entropy of the ensembles of undirected uncorrelated simple networks with given degree sequence. We stress the apparent paradox that scale-free degree distributions are characterized by having small structural entropy while they are so widely encountered in natural, social, and technological complex systems. We propose a solution to the paradox by proving that scale-free degree distributions are the most likely degree distribution with the corresponding value of the structural entropy. Finally, the general framework we present in this paper is able to describe microcanonical ensembles of networks as well as canonical or hidden-variable network ensembles with significant implications for the formulation of network-constructing algorithms.

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I. INTRODUCTION

The quantitative measure of the order present in complex systems and the possibility to extract information from the complex of interactions in cellular, technological, and social networks is a topic of key interest in modern statistical mechanics. The field of complex networks [1,2] is having a rapid development and great success because the concepts coming from graph theory have wide applicability. Mainly, the characterization of the structure of different networks allows the scientific community to compare systems of very different nature. Different statistical mechanics tools have been devised to describe the different levels of organization of real networks. A description of the structure of a complex network is presently performed by measuring different quantities as (i) the density of the links, (ii) the degree sequence [3], (iii) the degree-degree correlations [4–6], (iv) the clustering coefficient [7,8], (v) the k-core structure [9–11], (vi) the community structure [2,12–14], and finally (vii) the nature of the embedding space [15–17]. Moreover, significant characteristics of the network are strength-degree correlations [18] if the network is weighted, and in-degree–out-degree correlations [19] if the network is directed. These phenomenological quantities describe the local or nonlocal topology of the network and do affect dynamical models defined on them [1].

While many different statistical mechanics models have been proposed [20–27] to describe how the power-law degree distribution can arise in complex networks, little work has been done on the problem of measuring the level of organization and "order" in the frame of theoretical statistical mechanics. Only recently in the field of complex networks has attention been addressed to the study of entropy measures [28–33] able to approach this problem. The *entropy of a given ensemble*, giving the normalized logarithm of the number of networks in the ensemble, has been introduced in Ref. [31]. This quantity can be used to assess the role that given structural characteristics have in shaping the network. In fact, given a real network, a subsequent series of randomized networks ensembles can be considered, where each subsequent ensemble shares one additional structural characteristic with the given network. The entropy of these subsequent network ensembles decreases as we proceed, adding constraints, and the difference between the entropies in two subsequent ensembles quantifies how restrictive is the introduced additional constraint. In the first part of this paper we construct a general statistical mechanics framework for generalized random network ensembles which satisfy given structural constraints. We call these ensembles "microcanonical." We also describe how to construct canonical network ensembles with generalized hidden variables following the lines of the papers [22–27]. Subsequently we make an account of most of the network ensembles that can be formulated: the ensemble of undirected networks with a given number of links and nodes, and the ensemble of undirected networks with given degree sequence, with given spatial embedding and community structure. Some of these network ensembles were already presented in [31] and we report their derivation here for completeness. This approach is further extended to weighted networks and directed networks. Finally we focus our attention on the structural entropy, i.e., the entropy of an ensemble of uncorrelated undirected simple networks of a given degree sequence. The structural entropy of a power-law network with constant average degree is monotonically decreasing as the power-law exponent $\gamma \to 2$. This result could appear in contradiction with the wide occurrence of power-law degree distribution in complex networks. Here we show by a statistical mechanics model that scale-free degree distributions are the most likely degree distributions with an associated small value of structural entropy while Poisson degree distributions are the most likely degree distributions of network ensembles with maximal structural entropy. This result indicates that the scale-free degree distribution emerges naturally when considering network ensembles with small structural entropy and therefore a larger amount of order.

The appearance of the power-law degree distribution reflects the tendency of social, technological, and especially biological networks toward "ordering." This tendency is at work regardless of the mechanism that is driving their evolution, which can be either off-equilibrium models with a preferential attachment mechanism [3], or a hidden-variable mechanism [22–27], or some other statistical mechanics mechanism [20,21].

II. STATISTICAL MECHANICS OF NETWORK ENSEMBLES

A network of N labeled nodes $i=1,2,\ldots,N$ is uniquely defined by its adjacency matrix $\mathbf{a}$ of matrix elements $a_{ij}\geqslant 0$ with $a_{ij}>0$ if and only if there is a link between node i and node j. Simple networks are networks without tadpoles or double links, i.e., $a_{ii}=0\ \forall\ i$ and $a_{ij}=0,1$. Weighted networks describe heterogeneous interactions between the nodes. In this paper we will consider weighted networks in which the matrix elements a_{ij} can take different integer null or positive values. Directed networks are described by nonsymmetric adjacency matrices $\mathbf{a}\neq\mathbf{a}^T$ where we have indicated by $\mathbf{a}^T$ the transpose of the matrix $\mathbf{a}$.

A structural constraint on a network can always be formulated as a constraint on the adjacency matrix of the graph, i.e.,

$$\vec{F}(\mathbf{a})=\vec{C}. \tag{1}$$

In order to describe microcanonical network ensembles with given structural constraints in [31] and in the following we will use a statistical mechanics perspective. Therefore we define a partition function Z of the ensemble as in the following, i.e.,

$$Z=\sum_{\mathbf{a}}\delta[\vec{F}(\mathbf{a})-\vec{C}]\exp\left(\sum_{ij}h_{ij}\Theta(a_{ij})+r_{ij}a_{ij}\right), \tag{2}$$

where we assume for simplicity that $\vec{F}(\mathbf{a})$ and a_{ij} takes only integer values, that $\delta[\cdot]$ indicates the Kronecker delta, and that $\Theta(x)$ is a step function with $\Theta(x)=1$ if $x>0$ and $\Theta(x)=0$ if $x=0$. Moreover, in Eq. (2), the auxiliary fields h_{ij},r_{ij} have been introduced as in classical statistical mechanics. The entropy per node Σ of the network ensemble is defined as

$$\Sigma=\frac{1}{N}\ln(Z)|_{h_{ij}=r_{ij}=0\ \forall\ (i,j)}. \tag{3}$$

The marginal probability for a certain value of the element a_{ij} of the adjacency matrix is given by

$$\pi_{ij}(A)=\frac{1}{Z}\sum_{\mathbf{a}}\delta(a_{ij}-A)\delta(\vec{F}(\mathbf{a})-\vec{C}). \tag{4}$$

The probability of a link p_{ij} is given by

$$p_{ij}=\left.\frac{\partial\ln Z}{\partial h_{ij}}\right|_{h_{ij}=r_{ij}=0\ \forall\ (i,j)}. \tag{5}$$

In an ensemble of weighted networks we can define also the average weight w_{ij} of a link between node i and node j given by

$$w_{ij}=\left.\frac{\partial\ln Z}{\partial r_{ij}}\right|_{h_{ij}=r_{ij}=0\ \forall\ (i,j)}. \tag{6}$$

In a microcanonical network ensemble all the networks that satisfy given structural constraints have equal probability. Therefore the probability of a network G, described by the adjacency matrix $\mathbf{a}$, is given in the microcanonical ensemble by

$$P_M(\mathbf{a})=e^{-N\Sigma}\delta(\vec{F}(\mathbf{a})-\vec{C}). \tag{7}$$

If we allow for "soft" structural constraints in network ensembles we can describe canonical network ensembles. In a canonical network ensemble each network $\mathbf{a}$ has a different probability given by

$$P_C(\mathbf{a})=\prod_{ij}\pi_{ij}(a_{ij}). \tag{8}$$

For ensembles of simple networks Eq. (8) becomes

$$P_C(\mathbf{a})=\prod_{ij}p_{ij}^{a_{ij}}(1-p_{ij})^{1-a_{ij}}. \tag{9}$$

If the link probabilities $\pi_{ij}(a_{ij})$ are chosen equal to Eqs. (4) and (5), then the structural constraints $\vec{F}(\mathbf{a})=\vec{C}$ are satisfied on average, i.e.,

$$\langle\vec{F}(\mathbf{a})\rangle_{P_C(\mathbf{a})}=\vec{C}, \tag{10}$$

where the average $\langle\cdot\rangle_{P_C(\mathbf{a})}$ is performed over the canonical ensemble probability (8). For structural constraints that do not correspond to feasible networks [34], the entropy of the network ensemble is $\Sigma=-\infty$. Although the statistical mechanics problem is always well defined, the calculation of the partition function by saddle point approximation can be performed only if the number of constraints F_α with $\alpha=1,\ldots,M$ is at most extensive, i.e., $M=O(N)$; moreover phase transitions might emerge as condensation of loops [1] and first-order phase transitions [22]. In this paper we are going to consider only linear constraints of the adjacency matrix. Further developments of this statistical mechanics framework will involve perturbative approaches to solve nonlinear structural constraints.

III. UNDIRECTED SIMPLE NETWORKS

In an undirected simple network the adjacency matrix elements are zero or one ($a_{ij}=0,1$) and tadpoles are forbidden ($a_{ii}=0\ \forall\ i$). Therefore for undirected networks, without loss of generality, we can use the definition of Z given by (2) with $r_{ij}=0$, for all i,j. We can consider for these networks different types of structural constraints. In the following a few examples of particular interest are listed.

(i) The ensemble $G(N,L)$ of random networks with given number of nodes N and links $L=\Sigma_{i<j}a_{ij}$. In this case the statistical mechanics approach gives a "physics" derivation of the $G(N,L)$ random ensemble. In this case we have the structural constraint

$$\vec{F}(\mathbf{a})-\vec{C}=\sum_{i<j}a_{ij}-L=0. \quad (11)$$

(ii) The configuration model, i.e., the ensemble of networks with given degree sequence $\{k_1,\ldots,k_N\}$ with $k_i=\Sigma_j a_{ij}$. In this case the structural constraints are given by

$$F_\alpha(\mathbf{a})-C_\alpha=\sum_j a_{\alpha j}-k_\alpha=0 \quad (12)$$

for $\alpha=1,\ldots,N$.

(iii) The network with given degree sequence $\{k_1,\ldots,k_N\}$ and given average nearest neighbor connectivity $k_{\mathrm{NN}}(k)=[\Sigma_{i,j}\delta(k_i-k)a_{ij}k_j]/(kN_k)$ of nodes of degree k (with N_k indicating the number of nodes of degree k in the network). In this case the structural constraints are given by

$$F_\alpha(\mathbf{a})-C_\alpha=\sum_j a_{\alpha j}-k_\alpha=0 \quad (13)$$

for $\alpha=1,\ldots,N$ and

$$F_\alpha(\mathbf{a})-C_\alpha=\sum_{ij}\delta(k_i-k)a_{ij}k_j-kN_k k_{\mathrm{NN}}(k) \quad (14)$$

for $\alpha=N+1,\ldots,N+K$, with K indicating the maximal connectivity of the network.

(iv) The network ensemble with given degree sequence and given community structure. In this case we assume that each node is assigned a community $\{q_1,\ldots,q_N\}$ and we fix the number of links between nodes of different communities $A(q,q')=\Sigma_{i<j}\delta(\underline{q_{ij}}-q)\delta(\overline{q_{ij}}-q')a_{ij}$ with $\underline{q_{ij}}=\min(q_i,q_j)$ and $\overline{q_{ij}}=\max(q_i,q_j)$. In this case the structural constraints are given by

$$F_\alpha(\mathbf{a})-C_\alpha=\sum_j a_{\alpha j}-k_\alpha=0 \quad (15)$$

for $\alpha=1,\ldots,N$ and

$$F_\alpha(\mathbf{a})-C_\alpha=\sum_{i<j}\delta(\underline{q_{ij}}-q)\delta(\overline{q_{ij}}-q')a_{ij}-A(q,q') \quad (16)$$

for $\alpha=N+1,\ldots,N+Q(Q+1)/2$ with Q equal to the number of different features of the nodes. Here and in the following in order to have an extensive number of constraints we assume $Q=O(N^{1/2})$.

(v) The ensemble of networks with given degree sequence and dependence of the link probability on the distance of the nodes in an embedding geometrical space. In this ensemble we consider a fixed spatial distribution of nodes in space $\{\vec{r}_1,\ldots,\vec{r}_N\}$ and we consider all the networks compatible with the given degree sequence and the number of links linking nodes in a given distance interval. In this case we take Λ distance intervals $I_\ell=[d_\ell,d_\ell+(\Delta d)_\ell]$ with $\ell=1,\ldots,\Lambda$, and we fix the number of links linking nodes in a given distance interval. The structural constraints that we consider are therefore provide by the values of the vector $B(d_\ell)=\Sigma_{i<j}\chi_\ell(d_{i,j})a_{ij}$ where $d_{ij}=d(\vec{r}_i,\vec{r}_j)$ is the distance between nodes i and j in the embedding space and the characteristic function $\chi_\ell(x)=1$ if $x\in[d_\ell,d_\ell+(\Delta d)_\ell]$ and $\chi_d(x)=0$ otherwise. In this case the structural constraints can be expressed as

$$F_\alpha(\mathbf{a})-C_\alpha=\sum_j a_{\alpha j}-k_\alpha=0 \quad (17)$$

for $\alpha=1,\ldots,N$ and

$$F_\alpha(\mathbf{a})-C_\alpha=\sum_{i<j}\chi_\ell(d_{i,j})a_{ij}-B(d_\ell) \quad (18)$$

for $\alpha=N+1,\ldots,N+\Lambda$.

A. The entropy of the $G(N,L)$ ensemble

The networks in the $G(N,L)$ ensemble have given number of nodes N and links L. The entropy of this ensemble is given by the logarithm of the binomial

$$N\Sigma_0=\ln\left[\begin{pmatrix}\frac{N(N-1)}{2}\\ L\end{pmatrix}\right] \quad (19)$$

(we always assume distinguishable nodes in the networks [29]). The probability p_{ij} of a given link (i,j) is given by $p_{ij}^{(0)}=L/[N(N-1)/2]$ for every pair of nodes i,j. The ensemble $G(N,p)$ is the canonical ensemble corresponding to the microcanonical $G(N,L)$ ensemble.

B. The entropy of the configuration ensemble

In the configuration ensemble we consider all the networks with given degree sequence. Using (2) and (12) the partition function of the ensemble can be explicitly written as

$$Z_1=\sum_{\{a_{ij}\}}\prod_i\delta\left(k_i-\sum_j a_{ij}\right)\exp\left(\sum_{i<j}h_{ij}a_{ij}\right). \quad (20)$$

Expressing the δ functions in the integral form with Lagrangian multipliers ω_i for every $i=1,\ldots,N$, we get

$$Z_1=\int\mathcal{D}\omega\,\exp\left(-\sum_i\omega_i k_i\right)\prod_{i<j}(1+e^{\omega_i+\omega_j+h_{ij}}), \quad (21)$$

where $\mathcal{D}\omega=\Pi_i d\omega_i/(2\pi)$. We solve this integral by saddle point equations accounting also for second-order terms of the expansion. The entropy of this ensemble of networks can be approximated in the large-network limit $N\gg 1$ with

$$N\Sigma_1^{\mathrm{und}}\simeq-\sum_i\omega_i^* k_i+\sum_{i<j}\ln(1+e^{\omega_i^\star+\omega_j^\star})-\frac{1}{2}\sum_i\ln(2\pi\alpha_i) \quad (22)$$

with $N\Sigma_1^{\mathrm{und}}=\ln(Z_1)$ and the Lagrangian multipliers ω_i satisfying the saddle point equations

$$k_i=\sum_{j\neq i}\frac{e^{\omega_i^\star+\omega_j^\star}}{1+e^{\omega_i^\star+\omega_j^\star}}, \quad (23)$$

and the coefficients α_i approximated with

$$\alpha_i \simeq \sum_j \frac{e^{\omega_i^\star+\omega_j^\star}}{(1+e^{\omega_i^\star+\omega_j^\star})^2}. \quad (24)$$

The probability of a link i,j in this ensemble is given by

$$p_{ij}^{(1)} = \frac{e^{\omega_i^\star+\omega_j^\star}}{1+e^{\omega_i^\star+\omega_j^\star}}. \quad (25)$$

In this ensemble $p_{ij} \neq f(\omega_i)f(\omega_j)$; consequently *the model retains some "natural" correlations* [27] given by the degree sequence. In fact these are nothing else than the correlations of the configuration model [35].

The canonical model corresponding to the configuration model is then a hidden-variable model where each node i is assigned a hidden variable ω_i and the probability for each link follows (25). Similar expressions were already derived in different papers [22,25,27] but with a different interpretation. Here the hidden variables ω_i are simply fixing the average degree of each node. Moreover we note that the derivation of [26] guarantees that in the canonical model the connectivity of each node is distributed according to a Poisson distribution with average $\Sigma_j p_{ij}$.

The form of the probability p_{ij} is such that, when inferring the values of the hidden variables ω_i for a canonical network in this ensemble by maximum likelihood methods, we obtain $\omega_i' = \omega_i$ in the large-network limit [27].

The case in which there is a structural cutoff in the network $k_i < \sqrt{\langle k\rangle N}$ is of particular interest. In this case we can approximate Eq. (23) by $e^{\omega_i^\star} \simeq k_i/\sqrt{\langle k\rangle N}$, $\alpha_i \simeq k_i$. In this limit the network is *uncorrelated*. and the probabilities of a link are given by $p_{ij}^{(1),\mathrm{uncorr}} = k_i k_j/(\langle k\rangle N)$, since $\omega_i^* < 0$. We call the entropy of these uncorrelated ensembles the *structural entropy* Σ_S, and we can evaluate it from the explicit expression

$$\begin{aligned} N\Sigma_S \simeq & -\sum_i \ln[k_i/\sqrt{\langle k\rangle N}]k_i - \frac{1}{2}\sum_i \ln(2\pi k_i) + \frac{1}{2}\sum_{ij}\frac{k_i k_j}{\langle k\rangle N} \\ & - \sum_{ij}\frac{1}{4}\frac{k_i^2 k_j^2}{(\langle k\rangle N)^2} + \cdots = -\sum_i (\ln k_i - 1)k_i \\ & - \frac{1}{2}\sum_i \ln(2\pi k_i) + \frac{1}{2}\langle k\rangle N[\ln(\langle k\rangle N) - 1] - \frac{1}{4}\left(\frac{\langle k^2\rangle}{\langle k\rangle}\right)^2 \\ & + \cdots . \end{aligned} \quad (26)$$

Expression (26) gives for the number of networks $\mathcal{N}_S$ in the ensemble

$$\mathcal{N}_S^{\mathrm{uncorr}} \simeq \frac{(\langle k\rangle N)!!}{\prod_i k_i!}\exp\left[-\frac{1}{4}\left(\frac{\langle k^2\rangle}{\langle k\rangle}\right)^2 + O(\ln N)\right]. \quad (27)$$

From combinatorial arguments we can derive an expression $\mathcal{N}_C^{\mathrm{uncorr}}$ for the number of uncorrelated networks with a given degree sequence which agrees with the above estimate (27) in the limit $N \gg 1$, i.e.,

$$\ln \mathcal{N}_c^{\mathrm{uncorr}} = \ln \mathcal{N}_S^{\mathrm{uncorr}} + O(\ln N). \quad (28)$$

In fact by combinatorial arguments we can show that the number of networks with given degree sequence is given by the following expression in the large-N limit, i.e.,

$$\mathcal{N}_c^{\mathrm{uncorr}} \propto \frac{(2L-1)!!e^{-(\langle k^2\rangle/\langle k\rangle)^2/4}}{\prod_i k_i!}. \quad (29)$$

The factor $(2L-1)!!$ accounts for the total number of wirings of the links. In fact if we want to construct a network, given a certain distribution of half edges through the N nodes of the network, as a first step we take a half edge and we match it with one of the $2L-1$ other half edges of the network. Second, we match a new half edge with one of the $2L-3$ remaining half edges. Repeating this procedure, we get one out of $(2L-1)!!$ possible wirings of the links. This number includes also the wiring of the links which gives rise to networks with double links. To estimate the number of such undesired wirings we assume that the network is random, i.e., that the probability that a node with k_i half edges connects to a node with k_j half edges is a Poisson variable with average $k_i k_j/(\langle k\rangle N)$. In this hypothesis the probability Π that the network does not contain double links is equal to [36]

$$\Pi = \prod_{i<j}\left(1 + \frac{k_i k_j}{\langle k\rangle N}\right)e^{-k_i k_j/\langle k\rangle N} \sim e^{-(\langle k^2\rangle/\langle k\rangle)^2/4}. \quad (30)$$

Finally in the expression (29) for $\mathcal{N}_c$ there is an additional term which takes into account the number of wirings of the links giving rise to equivalent networks without double links. This term is given by the number of possible permutations of the half edges at each node, i.e., $\Pi_i k_i!$. We note here that a similar result was derived by mathematicians for the case in which the maximal connectivity $K < N^{1/3}$ [37] and an inequality was proved for the case $K > N^{1/3}$ [38]. Now we extend these results by statistical mechanics methods to uncorrelated networks with maximal connectivity $K < \sqrt{\langle k\rangle N}$.

C. The entropy of network ensembles with fixed degree correlations

We consider now network ensembles, with given degree correlations and given average degree of neighboring nodes, that satisfy the constraints defined in Eqs. (13) and (14). We can proceed to the evaluation of the probability of a link $p_{ij}^{(2)}$ and the calculation of the entropy of the ensemble as in the configuration model. In this case we have to introduce the Lagrangian multipliers ω_i fixing the degree of node i and the Lagrangian multipliers A_k fixing the average degree of nodes of degree k.

The partition function of this ensemble can be evaluated at the saddle point, giving for the entropy of the ensemble, in the thermodynamic limit,

$$\begin{aligned} N\Sigma_2^{\mathrm{und}} \simeq & -\sum_i \omega_i^* k_i - \sum_k A_k^* k_{nn}(k) k N_k \\ & + \sum_{i<j}\ln(1 + e^{\omega_i^*+\omega_j^*+k_i A_{k_j}^*+k_j A_{k_i}^*}) - \frac{1}{2}\sum_i \ln(2\pi\alpha_i) \\ & - \frac{1}{2}\sum_k \ln(2\pi\alpha_k) \end{aligned} \quad (31)$$

where $\omega_i^\star$ and $A_k^\star$ satisfy the saddle point equations

$$k_i = \sum_{j\neq i} \frac{e^{\omega_i^* + \omega_j^* + k_j A^*_{k_i} + k_i A_{k_j}}}{1 + e^{\omega_i^* + \omega_j^* + k_j A^*_{k_i} + k_i A^*_{k_j}}},$$

$$k_{\mathrm{NN}}(k) = \frac{1}{kN_k} \sum_i \delta(k_i - k) \sum_{j\neq i} k_j \frac{e^{\omega_i^* + \omega_j^* + k_j A^*_{k_i} + k_i A^*_{k_j}}}{1 + e^{\omega_i^* + \omega_j^* + k_j A^*_{k_i} + k_i A^*_{k_j}}}, \tag{32}$$

and where α_i, α_k are approximately equal to the expressions

$$\alpha_i \simeq \sum_j \frac{e^{\omega_i^* + \omega_j^* + k_j A^*_{k_i} + k_i A^*_{k_j}}}{(1 + e^{\omega_i^* + \omega_j^* + k_j A^*_{k_i} + k_i A^*_{k_j}})^2},$$

$$\alpha_k \simeq \sum_i \delta(k_i - k) \sum_{j\neq i} k_j^2 \frac{e^{\omega_i^* + \omega_j^* + k_j A^*_{k_i} + k_i A^*_{k_j}}}{(1 + e^{\omega_i^* + \omega_j^* + k_j A^*_{k_i} + k_i A^*_{k_j}})^2}. \tag{33}$$

The probability $p_{ij}^{(2)}$ of the link (i,j) in this ensemble is given by

$$p_{ij}^{(2)} = \frac{e^{\omega_i^* + \omega_j^* + k_j A^*_{k_i} + k_i A_{k_j}}}{1 + e^{\omega_i^* + \omega_j^* + k_j A^*_{k_i} + k_i A^*_{k_j}}}. \tag{34}$$

This formula generalizes the hidden-variable formula of the configuration model to networks with strong degree-degree correlations. In particular, in order to build a canonical network with strong degree-degree correlation we can consider nodes with hidden variables θ_i and G_θ and a probability p_{ij} to have a link between a node i and a node j given by

$$p_{ij} = \frac{\theta_i \theta_j (G_{\theta_i})^{\theta_j} (G_{\theta_j})^{\theta_i}}{1 + \theta_i \theta_j (G_{\theta_i})^{\theta_j} (G_{\theta_j})^{\theta_i}}. \tag{35}$$

D. The entropy of network ensembles with given degree sequence and given community structure

The partition function of network ensembles with given degree sequence (15) and given community structure (16) can be evaluated by saddle point approximation in the large-network limit as long as $Q = O(N^{1/2})$. Following the same steps as in the previous case we find that the entropy for this ensemble is given by

$$\begin{aligned} N\Sigma_c \simeq & - \sum_i k_i \omega_i^\star - \sum_{q \leq q'} A(q,q') w(q,q')^\star \\ & + \sum_{i<j} \ln(1 + e^{\omega_i^\star + \omega_j^\star + w^\star(\underline{q}_{ij}, \bar{q}_{ij})}) - \frac{1}{2} \sum_i \ln(2\pi\alpha_i) \\ & - \frac{1}{2} \sum_{q<q'} \ln(2\pi \alpha_{q,q'}) \end{aligned} \tag{36}$$

with the Lagrangian multipliers $\{\omega_i^\star\}, \{w^\star_{q,q'}\}$ satisfying the saddle point equations

$$k_i = \sum_{j\neq i} \frac{e^{\omega_i^\star + \omega_j^\star + w^\star(\underline{q}_{ij}, \bar{q}_{ij})}}{1 + e^{\omega_i^\star + \omega_j^\star + w^\star(\underline{q}_{ij}, \bar{q}_{ij})}},$$

$$A(q,q') = \sum_{i<j} \delta(\underline{q}_{ij} - q) \delta(\bar{q}_{ij} - q') \frac{e^{\omega_i^\star + \omega_j + w^\star(q,q')}}{1 + e^{\omega_i^\star + \omega_j^\star + w^\star(q,q')}}, \tag{37}$$

and with $\alpha_i, \alpha_{q,q'}$ that can be approximated by

$$\alpha_i \simeq \sum_j \frac{e^{\omega_i^\star + \omega_j^\star + w^\star(\underline{q}_{ij}, \bar{q}_{ij})}}{(1 + e^{\omega_i^\star + \omega_j^\star + w^\star(\underline{q}_{ij}, \bar{q}_{ij})})^2},$$

$$\alpha_{q,q'} \simeq \sum_{i<j} \delta(\underline{q}_{ij} - q) \delta(\bar{q}_{ij} - q') \frac{e^{\omega_i^\star + \omega_j^\star + w^\star(q,q')}}{(1 + e^{\omega_i^\star + \omega_j^\star + w^\star(q,q')})^2}. \tag{38}$$

In this ensemble the probability for a link $p_{ij}^{(c)}$ between a node i and a node j is equal to

$$p_{ij}^{(c)} = \frac{e^{\omega_i^\star + \omega_j^\star + w^\star(\underline{q}_{ij}, \bar{q}_{ij})}}{1 + e^{\omega_i^\star + \omega_j^\star + w^\star(\underline{q}_{ij}, \bar{q}_{ij})}}. \tag{39}$$

Assigning each node a hidden variable θ_i and to each pair of communities the symmetric matrix $V(q,q')$, we can construct the hidden variable or canonical ensemble by extracting each link with probability

$$p_{ij} = \frac{\theta_i \theta_j V(q_i, q_j)}{1 + \theta_i \theta_j V(q_i, q_j)}. \tag{40}$$

E. The entropy of network ensembles with given distance between the nodes

Finally, we consider the ensemble of undirected networks living in a generic embedding space and with structural constraints described by Eqs. (17) and (18). Following the same steps as in the previous cases, we find that the entropy for such an ensemble in the large network limit is given by

$$\begin{aligned} N\Sigma_d \simeq & - \sum_i k_i \omega_i^\star - \sum_{\ell=1}^{\Lambda} B(d_\ell) g^\star(d_\ell) \\ & + \sum_{i<j} \ln\left[1 + \exp\left(\omega_i^\star + \omega_j^\star + \sum_\ell \chi_\ell(d_{ij}) g^\star(d_\ell)\right)\right] \\ & - \frac{1}{2} \sum_i \ln(2\pi\alpha_i) - \frac{1}{2} \sum_{\ell=1}^{\Lambda} \ln(2\pi\alpha_\ell) \end{aligned} \tag{41}$$

with the Lagrangian multipliers $\{\omega_i^\star\}, \{g_d^\star\}$ satisfying the saddle point equations

$$k_i = \sum_{j\neq i} \frac{\exp\left(\omega_i^\star + \omega_j^\star + \sum_\ell \chi_\ell(d_{ij}) g^\star(d_\ell)\right)}{1 + \exp\left(\omega_i^\star + \omega_j^\star + \sum_\ell \chi_\ell(d_{ij}) g^\star(d_\ell)\right)},$$

$$B(d_\ell) = \sum_{i<j} \chi_\ell(d_{ij}) \frac{e^{\omega_i^\star + \omega_j + g^\star(d_\ell)}}{1 + e^{\omega_i^\star + \omega_j^\star + g^\star(d_\ell)}}, \tag{42}$$

and the variables $\alpha_i, \alpha_{q,q'}$ approximated by the expressions

$$\alpha_i \simeq \sum_j \frac{\exp\left(\omega_i^\star + \omega_j^\star + \sum_\ell \chi_\ell(d_{ij}) g^\star(d_\ell)\right)}{\left[1+\exp\left(\omega_i^\star + \omega_j^\star + \sum_\ell \chi_\ell(d_{ij}) g^\star(d_\ell)\right)\right]^2},$$

$$\alpha_\ell \simeq \sum_{i,j} \chi_d(d_{ij}) \frac{e^{\omega_i^\star+\omega_j^\star+g^\star(d_\ell)}}{(1+e^{\omega_i^\star+\omega_j^\star+g^\star(d_\ell)})^2}. \tag{43}$$

The probability for a link between node i and j is equal to

$$p_{ij}^{(d)} = \sum_\ell \chi_\ell(d_{ij}) \frac{e^{\omega_i^\star+\omega_j^\star+g^\star(d_\ell)}}{1+e^{\omega_i^\star+\omega_j^\star+g^\star d_\ell}}. \tag{44}$$

Therefore the hidden-variable model associated with this ensemble corresponds to a model where we fix the hidden variables θ_i and $\vec{r}_i$ and and the vector $W(d_\ell)$ is given and we draw a link between node i and node j according to

$$p_{ij} = \sum_\ell \chi_\ell(d_{ij}) \frac{\theta_i\theta_j W(d_\ell)}{1+\theta_i\theta_j W(d_\ell)}. \tag{45}$$

IV. WEIGHTED NETWORKS

Many networks have, in addition to a nontrivial topological structure, weighted links. We will assume in this paper that the weight of a link can take only integer values and consequently a link between a node i and node j is characterized by an integer number $a_{ij} \geqslant 1$. This is not a very stringent constraint since any finite network with weights of the links taking rational numbers can be easily reduced to a network of integer weights. In a weighted network the degree k_i and the strength s_i of the node i are defined as

$$k_i = \sum_{j\neq i} \Theta(a_{ij}),$$

$$s_i = \sum_{j \neq i} a_{ij}, \tag{46}$$

where $\Theta(x)=0$ if $x=0$ and $\Theta(x)=1$ is $x>0$. It is possible to define series of weighted networks by considering networks with fixed total strength, with given strength sequence, with given strength and degree sequence and proceeding by adding additional features as in the unweighted case. Here and in the following we study the most relevant cases.

(i) The network ensemble with given total strength S. The structural constraint in this case is equal to

$$F(\mathbf{a}) - C = \sum_{i<j} a_{ij} - S = 0. \tag{47}$$

(ii) The network ensemble with given strength sequence $s_1, \ldots, s_N$. The structural constraints are for this ensemble given by

$$F(\mathbf{a})_\alpha - C_\alpha = \sum_j a_{\alpha j} - s_\alpha = 0, \tag{48}$$

for $\alpha = 1, \ldots, N$.

(iii) The network ensemble with given degree sequence $\{k_1, \ldots, k_n\}$ and strength sequence $\{s_1, \ldots, s_N\}$. For this ensemble the structural constraints are given by

$$F(\mathbf{a})_\alpha - C_\alpha = \sum_j \Theta(a_{\alpha j}) - k_\alpha = 0 \tag{49}$$

for $\alpha = 1, \ldots, N$ and

$$F(\mathbf{a})_\alpha - C_\alpha = \sum_j a_{\alpha j} - s_\alpha = 0 \tag{50}$$

for $\alpha = N+1, \ldots, 2N$.

A. The entropy of weighted network ensembles with given total strength S

The entropy of this ensemble is given by

$$N\Sigma_1^W = \ln\left[\begin{pmatrix} \frac{N(N-1)}{2} + S \\ \frac{N(N-1)}{2} \end{pmatrix}\right].$$

The average value of the weight of the link from i to j is given by

$$w_{ij} = \langle a_{ij} \rangle_1^W = \frac{S}{\frac{N(N-1)}{2}} \tag{51}$$

and the probability of a link between node i and j is equal to

$$p_{ij}^{W,1} = \frac{S}{S + \frac{N(N-1)}{2}}. \tag{52}$$

Therefore the simple networks with adjacency matrix $((A_{ij}))$ that can be constructed starting from the weighted networks with adjacency matrix $((a_{ij}))$, by putting $A_{ij} = \Theta(a_{ij})\ \forall\ i,j$, are uncorrelated. The canonical ensemble is given by Eq. (8) with

$$\pi_{ij}(a_{ij}) = \frac{e^{\omega a_{ij}}}{1-e^\omega} \tag{53}$$

and $\omega = -\ln[1 + N(N-1)/(2S)]$.

B. The entropy of weighted network ensembles with given strength sequence

To calculate the entropy of undirected networks with a given strength sequence of degrees $\{s_i\}$ we proceed by the saddle point approximation as in previous cases. We find that the entropy of this ensemble of networks is given by

$$N\Sigma_1^W \simeq -\sum_i \omega_i^\star s_i - \sum_{i<j} \ln(1 - e^{\omega_i^\star + \omega_j^\star}) - \frac{1}{2}\sum_i \ln(2\pi\lambda_i) \tag{54}$$

with the Lagrangian multipliers $\omega_i^\star$ satisfying the saddle point equations

$$s_i = \sum_{j\neq i} \frac{e^{\omega_i^\star+\omega_j^\star}}{1-e^{\omega_i^\star+\omega_j^\star}} \tag{55}$$

and with λ_i being the eigenvectors of the Jacobian of the function

$$\mathcal{F} = \sum_{i<j} \ln(1-e^{-\omega_i-\omega_j}). \tag{56}$$

The average value of the weight of the link from i to j is given by

$$\langle a_{ij}\rangle_1^W = \frac{e^{\omega_i^\star+\omega_j^\star}}{1-e^{\omega_i^\star+\omega_j^\star}}. \tag{57}$$

and the probability of a link between node i and j is equal to

$$p_{ij}^{W,1} = e^{\omega_i^\star+\omega_j^\star}. \tag{58}$$

The canonical ensemble (8) in this case can be constructed by assigning to every possible link (i,j) the weight a_{ij} with the probability

$$\pi_{ij}(a_{ij}) = \frac{e^{(\omega_i^\star+\omega_j^\star)a_{ij}}}{1-e^{\omega_i^\star+\omega_j^\star}}. \tag{59}$$

C. The entropy of weighted network ensembles with given strength and degree sequence

The entropy of weighted networks with a given strength and degree sequence $\{s_i,k_i\}$ in the large-size network limit is given by

$$\begin{aligned} N\Sigma_2^W \simeq & -\sum_i \omega_i^\star s_i - \sum_i \psi_i^\star k_i - \sum_i \\ & + \sum_{i<j} \ln\left(1+e^{\psi_i^\star+\psi_j^\star}\frac{1}{e^{-\omega_i^\star-\omega_j^\star}-1}\right) \\ & + \frac{1}{2}\sum_{\ell=1} 2N\sum_i \ln(2\pi\lambda_\ell) \end{aligned} \tag{60}$$

with the Lagrangian multipliers satisfying the saddle point equations

$$k_i = \sum_{j\neq i} \frac{e^{\psi_i^\star+\psi_j^\star}}{e^{\psi_i^\star+\psi_j^\star}+e^{-(\omega_i^\star+\omega_j^\star)}-1}, \tag{61}$$

$$s_i = \sum_{j\neq i} \frac{e^{-(\omega_i^\star+\omega_j^\star)+(\psi_i^\star+\psi_j^\star)}}{(e^{si_i^\star+\psi_j^\star}+e^{-(\omega_i^\star+\omega_j^\star)}-1)(e^{-\omega_i^\star-\omega_j^\star}-1)}, \tag{62}$$

and with λ_ℓ being the eigenvectors of the Jacobian of the function

$$\mathcal{F} = \sum_{i<j} \ln\left(1+e^{\psi_i+\psi_j}\frac{1}{e^{-\omega_i-\omega_j}-1}\right) \tag{63}$$

calculated at the values $\{\omega_i^\star,\psi_i^\star\}$. The average weight of the link (ij) is given by

$$\langle a_{ij}\rangle_2^W = \frac{e^{-(\omega_i^\star+\omega_j^\star)+(\psi_i^\star+\psi_j^\star)}}{(e^{si_i^\star+\psi_j^\star}+e^{-(\omega_i^\star+\omega_j^\star)}-1)(e^{-\omega_i^\star-\omega_j^\star}-1)} \tag{64}$$

and the probability of a link between node i and j is equal to

$$p_{ij}^{W,2} = \frac{e^{\psi_i^\star+\psi_j^\star}}{e^{\psi_i^\star+\psi_j^\star}+e^{-(\omega_i^\star+\omega_j^\star)}-1} \tag{65}$$

The canonical ensemble (8) in this case can be constructed by assigning to every possible link (i,j) the weight a_{ij} with the probability

$$\pi_{ij}(a_{ij}) = \frac{e^{(\psi_i^\star+\psi_j^\star)\Theta(a_{ij})}e^{(\omega_i^\star+\omega_j^\star)a_{ij}}}{1+e^{\psi_i^\star+\psi_j^\star}\dfrac{1}{e^{-\omega_i^\star-\omega_j^\star}-1}}. \tag{66}$$

V. DIRECTED NETWORKS

An undirected network is determined by a symmetric adjacency matrix, while the matrix of a directed network is in general nonsymmetric. Consequently the degrees of freedom of a directed network are more than the degrees of freedom of an undirected network. In the following we only consider the following network ensembles.

(i) Network ensemble with fixed total number of directed links. The structural constraint in this case is equal to

$$F(\mathbf{a}) - C = \sum_{ij} a_{ij} - S = 0. \tag{67}$$

(ii) Network ensemble with given directed degree sequence $\{k_1^{\text{in}},k_1^{\text{out}},\ldots,k_N^{\text{in}},k_N^{\text{out}}\}$. The structural constraints in this case are

$$F(\mathbf{a})_\alpha - C_\alpha = \sum_j a_{\alpha j} - k_\alpha^{\text{out}} = 0 \tag{68}$$

for $\alpha=1,\ldots,N$ and

$$F(\mathbf{a})_\alpha - C_\alpha = \sum_j a_{j\alpha} - k_\alpha^{\text{in}} = 0 \tag{69}$$

for $\alpha=N+1,\ldots,2N$.

A. The entropy of directed network ensembles with fixed number of directed links

If we consider the number of directed networks $\mathcal{N}_0^{\text{dir}}$ with given number of nodes and fixed number of directed links we find

$$N\Sigma_0^{\text{dir}} = \ln\left[\binom{N(N-1)}{L^{\text{dir}}}\right]. \tag{70}$$

In this case the probability of a directed link is given by

$$p_{ij} = \frac{L}{N(N-1)}. \tag{71}$$

B. The entropy of directed network ensembles with given degree sequence

To calculate the entropy of directed networks with a given degree sequence of in- and out-degrees $\{k_i^{\text{out}},k_i^{\text{in}}\}$ we just have to impose the constraints on the incoming and outgoing connectivity,

$$Z_1^{\rm dir} = \sum_{\{a_{ij}\}} \prod_i \delta\left(k_i^{\rm out} - \sum_j a_{ij}\right) \prod_i \delta \times \left(k_i^{\rm in} - \sum_j a_{ji}\right) \exp\left(\sum_{ij} h_{i,j} a_{ij}\right). \quad (72)$$

Following the same approach as for the undirected case, we find that the entropy of this ensemble of networks is given by

$$N\Sigma_1^{\rm dir} \simeq -\sum_i \omega_i^\star k_i^{\rm out} - \sum_i k_i^{\rm in} \hat{\omega}_i^\star + \sum_{i\neq j} \ln(1 + e^{\omega_i^\star + \hat{\omega}_j^\star}) - \frac{1}{2}\sum_i \ln[(2\pi)^2 \alpha_i^{\rm in} \alpha_i^{\rm out}] \quad (73)$$

with the Lagrangian multipliers satisfying the saddle point equations

$$k_i^{\rm out} = \sum_{j\neq i} \frac{e^{\omega_i^\star + \hat{\omega}_j^\star}}{1 + e^{\omega_i^\star + \hat{\omega}_j^\star}},$$

$$k_i^{\rm in} = \sum_{j\neq i} \frac{e^{\omega_j^\star + \hat{\omega}_i^\star}}{1 + e^{\omega_j^\star + \hat{\omega}_i^\star}}, \quad (74)$$

with

$$\alpha_i^{\rm out} \simeq \sum_{j\neq i} \frac{e^{\omega_i^\star + \hat{\omega}_j^\star}}{(1 + e^{\omega_i^\star + \hat{\omega}_j^\star})^2},$$

$$\alpha_i^{\rm in} \simeq \sum_{j\neq i} \frac{e^{\omega_j^\star + \hat{\omega}_i^\star}}{(1 + e^{\omega_j^\star + \hat{\omega}_i^\star})^2}. \quad (75)$$

The probability for a directed link from i to j is given by

$$p_{ij}^{1,\rm dir} = \frac{e^{\omega_i^\star + \hat{\omega}_j^\star}}{1 + e^{\omega_i^\star + \hat{\omega}_j^\star}}. \quad (76)$$

If $\omega_i + \hat{\omega}_j < 0\ \forall\ i, j = 1, \ldots, N$, the directed network becomes uncorrelated and we have $p_{ij}^{1,\rm dir} = k_i^{\rm out} k_j^{\rm in} / \sqrt{\langle k_{\rm in}\rangle N}$. Given this solution the condition for having uncorrelated directed networks is that the maximal in-degree $K^{\rm in}$ and the maximal out-degree $K^{\rm out}$ should satisfy $K^{\rm in} K^{\rm out} / \sqrt{\langle k_{\rm in}\rangle N} < 1$. The entropy of the directed uncorrelated network is then given by

$$N\Sigma_{1,\rm dir}^{\rm uncorr} \simeq \ln(\langle k_{\rm in}\rangle N)! - \sum_i \ln(k_i^{\rm in}! k_i^{\rm out}!) - \frac{1}{2} \frac{\langle k_{\rm in}^2\rangle}{\langle k_{\rm in}\rangle} \frac{\langle k_{\rm out}^2\rangle}{\langle k_{\rm out}\rangle}, \quad (77)$$

which has a clear combinatorial interpretation as it happens also for the undirected case.

VI. NATURAL DEGREE DISTRIBUTION CORRESPONDING TO A GIVEN STRUCTURAL ENTROPY

For power-law networks with power-law exponent $\gamma \in (2,3)$ the entropy of the networks with fixed degree sequence $\Sigma_1^{\rm und}$ given by Eq. (22) decreases with the value of the power-law exponent γ when we compare network ensembles with the same average degree [31]. Therefore scale-free networks have much smaller entropy than homogeneous networks. This fact seems to be in contrast with the fact that scale-free networks are the underlying structure of a large class of complex systems. The apparent paradox can be easily resolved if we consider that many networks are the result of a nonequilibrium dynamics. Therefore they do not have to satisfy the maximum entropy principle. Nevertheless, in order to give more insight and comment on the universal occurrence of power-law networks in this section we derive the most likely degree distribution of given *structural entropy* when the total number of nodes and links is kept fixed. By structural entropy we define the entropy Σ_S given by Eq. (26) of uncorrelated networks with fixed degree distribution. In particular we construct a statistical model very closely related to the urn or "ball in the box" models [21,39].

We consider degree distributions $\{N_k\} = \Sigma_i \delta(k - k_i)$ which arise from the random distribution of the $2L$ half edges through the N nodes of the network. The number of ways $\mathcal{N}_{\{N_k\}}$ in which we can distribute the $(2L)$ half edges in order to have an $\{N_k\}$ degree distribution is

$$\mathcal{N}_{\{N_k\}} = \frac{(2L)!}{\prod_k (kN_k)!}. \quad (78)$$

We want to find the most likely degree distribution that corresponds to a given value of the structural entropy.

Using the tools of statistical mechanics, we define a normalized partition function $\mathcal{Z}$ as

$$\mathcal{Z} = \frac{1}{C} \sum_{\{N_k\}}{}' \mathcal{N}_{\{N_k\}} e^{\beta N \Sigma_S(\{N_k\})} \quad (79)$$

with $C = (2L)! \exp[\beta(2L)!!]$. The role of the parameter β in Eq. (79) is to fix the average value of the structural entropy Σ_S. When $\beta \to \infty$ the structural entropy Σ_S is maximized and when $\beta \to \beta_{\rm min} = 1$ the structural entropy Σ_S is minimized.

In Eq. (79) the sum Σ' over the $\{N_k\}$ distributions is extended only to $\{N_k\}$ for which the total number of nodes N and the total number of links L in the network is fixed, i.e.,

$$\sum_k N_k = N,$$

$$\sum_k kN_k = 2L. \quad (80)$$

To enforce these conditions we introduce in Eq. (79) the δ functions in the integral form, providing the expression

$$\mathcal{Z} = \frac{1}{(2L)!} \int \frac{d\lambda}{2\pi} \int dS \int \frac{d\mu}{2\pi} \int \frac{d\nu}{2\pi} \sum_{\{N_k\}} \exp\Bigg[-\beta \sum_k N_k \ln k! - \frac{\beta}{4}\left(\frac{S}{\langle k\rangle}\right)^2 - \sum_k \ln[(kN_k)!] - i\lambda\left(2L - \sum_k N_k k\right) - i\mu\left(N - \sum_k N_k\right) - i\nu\left(NS - \sum_k k^2 N_k\right)\Bigg]. \quad (81)$$

This last expression can be further simplified as in the following, i.e.,

$$\mathcal{Z}=\int dS\int\frac{d\lambda}{2\pi}\int\frac{d\mu}{2\pi}\int\frac{d\nu}{2\pi}\times\exp\left[-i\lambda 2L-i\mu N-i\nu NS-\frac{\beta}{4}\left(\frac{S}{\langle k\rangle}\right)^2+\sum_k \ln G_k(\lambda,\mu,\nu)\right]$$
$$=\int dS\int\frac{d\lambda}{2\pi}\int\frac{d\mu}{2\pi}\int\frac{d\nu}{2\pi}\exp[Nf(\lambda,\mu,\nu,S)],\quad(82)$$

where

$$G_k(\lambda,\mu,\nu)=\sum_{N_k}\frac{1}{(kN_k)!}\left[kN_k\left(i\lambda+i\frac{\mu}{k}+i\nu k-\frac{\beta}{k}\ln(k!)\right)\right].\quad(83)$$

Assuming that the sum over all N_k can be approximated by the sum over all $L_k=kN_k=1,2,\ldots,\infty$ we get $\ln G_k(\lambda,\mu,\nu)=\exp[i\lambda+i\mu/k-(\beta/k)\ln(k!)+i\nu k]$ and

$$f(\lambda,\mu,\nu,S)=-i\langle k\rangle\lambda-i\mu-i\nu S-\frac{\beta}{4}\left(\frac{S}{\langle k\rangle}\right)^2+\frac{1}{N}\sum_k e^{i\lambda+i\mu/k-(\beta/k)\ln(k!)+i\nu k}\quad(84)$$

where $\langle k\rangle=2L/N$ indicates the average degree of the network. By evaluating (82) at the saddle point, deriving the argument of the exponential respect to λ and ν, we obtain

$$1=\frac{1}{N}\sum_k\frac{1}{k}e^{i\lambda+i\mu/k-(\beta/k)\ln(k!)+i\nu k},$$

$$\langle k\rangle=\frac{1}{N}\sum_k e^{i\lambda+i\mu/k-(\beta/k)\ln(k!)+i\nu k},$$

$$S=\frac{1}{N}\sum_k k^2 e^{i\lambda+i\mu/k-(\beta/k)\ln(k!)+i\nu k},$$

$$i\nu N=-\beta\frac{S}{2\langle k\rangle^2}.\quad(85)$$

These equations always have a solution for sparse networks with $L=O(N)$ provided that $\beta>1$ and $\langle k\rangle>1$. The marginal probability that $L_k=kN_k$ is given by

$$P(L_k=kN_k)=\frac{1}{(kN_k)!}e^{-\beta N_k[\ln(k!)+i\lambda k+i\mu+i\nu k^2]}\frac{\mathcal{Z}_k(L,kN_k,N)}{\mathcal{Z}(L)},\quad(86)$$

with

$$\mathcal{Z}_k(L,\ell,N)=\int dS\int\frac{d\lambda}{2\pi}\int\frac{d\mu}{2\pi}\int\frac{d\nu}{2\pi}\exp[Nf_k(\lambda,\mu,\nu,S,\ell)]\quad(87)$$

and

$$f_k(\lambda,\mu,\nu,\ell)=-i(\langle k\rangle-\ell/N)\lambda-i\mu[1-\ell/(kN)]+\ln\left(\sum_{s\neq k}\frac{1}{(sN_s)!}\exp\left\{sN_s\left(i\lambda+i\mu/s+i\nu s-\frac{\beta}{s}\ln(s!)\right)\right\}\right)-i\nu(S-k\ell/N)-\frac{\beta}{2}\left(\frac{S^2}{\langle k\rangle}\right)^2\frac{1}{N}.\quad(88)$$

If we develop Eq. (86) for $\ell\ll L$ and we use the Stirling approximation for factorials, we get the result that each variable L_k is a Poisson variable with mean $\langle L_k\rangle$ satisfying

$$\frac{\langle L_k\rangle}{k}=\langle N_k\rangle\simeq k^{-\beta-1}e^{i\lambda+\beta+i\mu/k+i\nu k},\quad(89)$$

where we assume that the minimal connectivity of the network is $k>0$. The average $\langle N_k\rangle$ is a power-law distribution with lower and upper effective cutoffs $-i\mu$ and $1/(i\nu)$ fixing the average degree $\langle k\rangle$, with the Lagrangian parameter λ fixing the normalization constant, and finally with β fixing the structural entropy. The distribution of $P(N_k)$ is given by

$$P(N_k)=\frac{k}{(kN_k)!}e^{-\beta N_k\ln(k!)+i\lambda kN_k+i\mu N_k+i\nu(k)^2}.\quad(90)$$

In the limit $\beta\to\infty$ Eq. (90) is extremely peaked around the average degree $k\simeq k^\star=O(\langle k\rangle)$ of the network and the degree distribution N_k decays at large value of N_k as a Poisson distribution, i.e.,

$$P(N_k)\simeq\frac{1}{(kN_k)!}e^{kN_k[-\beta\ln(k^\star!)/k^\star+i\lambda k^\star+i\mu/k^\star+i\nu k^\star]}.\quad(91)$$

Therefore for $\beta\to\infty$ the network is Poisson-like. In the opposite limit of small structural entropy and β small the $P(N_k)$ distribution (90) develops a fat tail decaying like a power law (89) with an exponent $\gamma=\beta+1$. Therefore the natural distribution with a small value of the structural entropies is decaying as a power law and smaller values of the power-law exponent correspond to a smaller value of the structural entropy. When the value of the entropy is minimal, $\beta\to 1$ the degree distribution (91) has a large tail with an exponent $\gamma\to 2$.

VII. CONCLUSIONS

In conclusion we have shown that there is a wide set of network ensembles that can be naturally described by statistical mechanics methods. We have provided the theoretical evaluation of the entropy of these ensembles, quantifying the cardinality of the network ensembles. We believe that the entropy of randomized ensembles constructed from a given real network will have wide applications for inference problems defined on technological social and biological networks. In this paper we have focused on some theoretical problems that can be approached with the use of this quantity. First we have formulated a series of canonical or hidden-variable models that can be used for generating networks with community structure and spatial embedding. Second we have focused on the degree distribution of networks. The degree distributions are not all equivalent. In fact the structural entropy depends strongly on the degree distribution of the network ensemble. In particular, the power-law degree distribution with exponent γ and fixed average degree are associated with a structural entropy that decreases with γ. Nevertheless we have shown that power-law degree distributions are the more likely distributions associated with small structural entropy. This sheds light on the evidence that power-law networks constitute a large universality class in complex networks with a nontrivial level of organization.

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