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Anisotropic magnetohydrodynamic spectral transfer in the diffusion approximation

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A theoretical model of spectral transfer for anisotropic magnetohydrodynamic (MHD) turbulence is introduced, approximating energy transport in wave vector ($\boldsymbol{k}$) space as a nonlinear diffusion process, extending previous isotropic k-space diffusion theories for hydrodynamics and MHD. This formal closure at the spectral equation level may be useful in space and astrophysical applications.

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Turbulence applications often require time evolution of the energy spectrum and therefore confront the classical closure problem. Standard closures [1–5] provide substantial insight, but become rather intricate for anisotropic turbulence unless one builds in phenomenological constraints [6–8]. Lacking a more general theory, one-dimensional or isotropic [9] models are often used, e.g., in astrophysical stochastic particle acceleration by wave particle interactions [10,11], pickup ion theories [12], or reacceleration in galaxy clusters [13]. The need for more realistic spectral models has also been emphasized in coronal [14] and solar wind [15] contexts. Here we extend the diffusive spectral transport closure to magnetohydrodynamics (MHD) in which anisotropy is induced by a large-scale magnetic field.

Leith [16] introduced a k-space diffusion model for evolution of the energy spectrum in isotropic three-dimensional incompressible hydrodynamic turbulence—namely, $\frac{\partial E(k)}{\partial t} = \frac{1}{k^2}\frac{\partial}{\partial k}[k^2 D^{\rm iso}(k)\frac{\partial E(k)}{\partial k}]$—ignoring forcing and dissipative effects. On physical grounds the diffusion coefficient $D^{\rm iso}$ should depend on the spectral transfer time scale $\tau^{\rm sp}(k)$, and dimensional analysis gives $D^{\rm iso}(k)=\frac{k^2}{\tau^{\rm sp}(k)}$. In the isotropic case the only time scale available is the nonlinear time $\tau^{\rm nl}(k)=1/(ku_k)\equiv\tau^{\rm sp}(k)$, where $u_k=\sqrt{k\mathcal{E}(k)}$ is the characteristic speed at wave number k and $\mathcal{E}(k)$ is the omnidirectional spectrum. [The total energy per mass is $\int\mathcal{E}(k)dk=\int E(\boldsymbol{k})d^3k$.] Requiring that the energy flux $\boldsymbol{F}=-\hat{\boldsymbol{k}}k^2D^{\rm iso}\partial E(k)/\partial k$ be independent of k yields the Kolmogorov spectral form $\mathcal{E}(k)=C_{\rm Kol}\epsilon^{2/3}k^{-5/3}$, with $C_{\rm Kol}\doteq 0.42$. The experimental value is ≈ 1.6 [17]; this discrepancy is fixed by introducing a constant in the definition of $D^{\rm iso}$. An extension of Leith's model to isotropic MHD [9] adheres to the approximation that spectral transfer is *local* in wave number k [18,19]. Models including parametrized mixtures of diffusion and advection in k space have been proposed for MHD [14,20].

In MHD, a large-scale magnetic field $\boldsymbol{B}_0$ induces a distinctive anisotropy, in which spectral transfer in the $\boldsymbol{B}_0$ direction is suppressed [21–26] due to interference between counterpropagating Alfvén fluctuations [27,28]. This anisotropy occurs relative to both global (dc) [21–26] and *local* [3,29,30] magnetic field directions.

Here we describe spectral anisotropy, relative to $\boldsymbol{B}_0$, in a diffusion approximation. We postulate that the diffusion of the modal (kinetic plus magnetic) energy spectrum occurs according to

$$\frac{\partial E(\boldsymbol{k})}{\partial t} = \frac{\partial}{\partial k_i}\left[D_{ij}(\boldsymbol{k})\frac{\partial}{\partial k_j}E(\boldsymbol{k})\right], \tag{1}$$

where D_{ij} is the diffusion tensor, which can depend on $\boldsymbol{B}_0$ and (in principle nonlocally) the energy spectrum at all wave vectors. Summation on repeated indices is implied.

Anisotropy immediately engenders complications with regard to standard ideas of wave *number* locality (typically defined as the three wave vectors in a triad having relative magnitudes within a factor of 2). The Alfvén wave propagation effect itself is nonlocal, involving the magnetic field at the longest wavelengths ($k\to 0$). Within a single k shell, the relative influence of propagation varies from modes with very low frequencies (i.e., $\boldsymbol{k}\cdot\boldsymbol{B}_0\approx 0$), where nonlinear effects are dominant [21,31–33], to high-frequency modes ($\boldsymbol{k}\cdot\boldsymbol{B}_0 \approx kB_0$), for which the dynamics may be mainly wave like. Thus the strength of diffusion should vary around a k shell. On the other hand, a model cannot be completely local in vector $\boldsymbol{k}$, because it would then lack resonant transfer [22–25]. Here we construct a model that maintains locality in the following sense: for diffusion at wave vector $\boldsymbol{k}$ we consider triads $\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{r}$ such that $|\boldsymbol{p}|\sim k$ (usually $k/2<|\boldsymbol{p}|<2k$ is considered local). Thus, two of the three wave vectors lie near the k shell of interest. We call this *modified locality*.

A time scale relevant to fluctuations near wave vector $\boldsymbol{k}$ is the nonlinear time scale $\tau_k^{\rm nl}=1/(kZ_k)$, where the characteristic speed for modes with $|\boldsymbol{k}|\sim k$ is $Z_k=\sqrt{k\mathcal{E}^{MHD}(k)}$. Also dynamically relevant is the Alfvén time $\tau^{\rm A}(\boldsymbol{k})=1/|\boldsymbol{k}\cdot\boldsymbol{B}_0| \equiv 1/|k_\parallel B_0|$, with $k_\parallel=k\cos\theta$. Alfvén speed units $\boldsymbol{b}\to\boldsymbol{b}/\sqrt{4\pi\rho}$ are employed. Sometimes, a direction-averaged Alfvén time is used, $\tau_k^{\rm A}=1/(kB_0)$ [4,28,34]; however, this is an oversimplification. Fluctuations near $\boldsymbol{k}$ are wave like if $\tau^{\rm A}(\boldsymbol{k})<\tau_k^{\rm nl}$. Conversely, when $\tau^{\rm A}(\boldsymbol{k})>\tau_k^{\rm nl}$, nonlinear effects will dominate over wave effects.

Another time scale related to energy transfer is the spectral transfer time for the k shell: $\tau_k^{\rm sp}$. For isotropic turbulence this is related to the rate of energy transfer, $\epsilon_k=Z_k^2/\tau_k^{\rm sp}$, from modes $<k$ to modes $>k$. When both nonlinear and wave propagation effects are important, $\tau_k^{\rm sp}$ can be estimated using the "golden rule" $\tau_k^{\rm sp}\tau_k^{(3)}=(\tau_k^{\rm nl})^2$ [3,4]. The additional time scale $\tau_k^{(3)}$ is the lifetime, or decorrelation time scale, of the triple correlations that are responsible for driving turbulence. For isotropic hydrodynamics, $\tau_k^{(3)}=\tau^{\rm nl}$. In MHD the triple

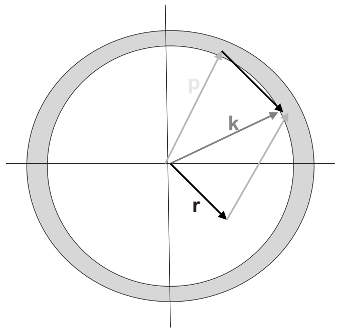

FIG. 1. Wave vector triads $\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{r}$. Here $\boldsymbol{k}$ and $\boldsymbol{p}$ are restricted to lie on the same spherical shell.

time may be estimated [3,34] by summing wave and nonlinear rates: $1/\tau_k^{(3)}=1/\tau_k^{\mathrm{nl}}+1/\tau_k^{\mathrm{A}}$ (cf. [3,4,28,35]). Combining this with the golden rule yields

$$\frac{1}{\tau_k^{\mathrm{sp}}}=\frac{kZ_k}{1+|B_0|/Z_k}. \tag{2}$$

For anisotropic situations, we modify the above ideas to accommodate nonuniform distribution of energy on k shells. For example, as $k_\parallel$ varies over a shell, so too does the Alfvén time. Since energy density varies on a shell, we introduce a modal nonlinear time scale $\tau^{\mathrm{nl}}(\boldsymbol{k})$. Let us define $Z^2(\boldsymbol{k})=k^3E(\boldsymbol{k})$ and, accordingly, the nonlinear rate per unit solid angle on the spherical shell, $1/\tau^{\mathrm{nl}}(\boldsymbol{k})=kZ^2(\boldsymbol{k})/Z_k$. This weights contributions to the transfer rate from various parts of a k shell in a way that is additive in the same sense as the energy. Consequently, for any symmetry of turbulence, integration of this rate over angles gives the standard nonlinear rate for the shell, kZ_k.

To compose the spectral transfer rate associated with $\boldsymbol{k}$, we employ the golden rule in the form $\tau^{\mathrm{sp}}\tau^{(3)}=\tau_k^{\mathrm{nl}}\tau^{\mathrm{nl}}(\boldsymbol{k})$, with $1/\tau^{(3)}(\boldsymbol{k})=1/\tau_k^{\mathrm{nl}}+1/\tau^{\mathrm{A}}(\boldsymbol{k})$, obtaining

$$\frac{1}{\tau^{\mathrm{sp}}(\boldsymbol{k})}=\frac{kZ^2(\boldsymbol{k})/Z_k}{1+|B_0\cos\theta|/Z_k}. \tag{3}$$

From this, one can form a diffusion coefficient as $D_\Omega(\boldsymbol{k},\boldsymbol{p})=\frac{k^2}{\tau^{\mathrm{sp}}(\boldsymbol{p})}$, interpreted as the diffusion per solid angle about $\boldsymbol{k}$ due to excitations at $\boldsymbol{p}$. This scalar diffusion coefficient is anisotropic on a k shell, but cannot on its own account for preferential *directions* of spectral transfer; in particular, it lacks suppression of transfer parallel to $\boldsymbol{B}_0$ [22–26,36–39].

To allow for anisotropic spectral transfer, we consider the tensor character of the diffusion. All spectral transfer in incompressible MHD is such that two modes interact to drive a third mode only if their wave vectors satisfy the triad condition $\boldsymbol{p}+\boldsymbol{r}=\boldsymbol{k}$. For a given $\boldsymbol{k}$ we consider the effect of two classes of triad interactions, with $\boldsymbol{p}$ restricted to lie on the same wave vector shell as $\boldsymbol{k}$—i.e., $k=p$ (Fig. 1). Except for equilateral triads, $\boldsymbol{r}$ will lie either inside the shell or outside it. Thus, $\boldsymbol{p}$ and $\boldsymbol{k}$ are local in magnitude, but some $\boldsymbol{r}$ are not.

First (class I), suppose energy is transferred between modes $\boldsymbol{k}$ and $\boldsymbol{r}$ under influence of mode $\boldsymbol{p}$. We term $\boldsymbol{p}$ the *spectator* mode. Evidently the sense of transfer at wave vector $\boldsymbol{k}$ is in the direction of $\boldsymbol{p}$. For a fixed $\boldsymbol{k}$, summing over all $\boldsymbol{p}$ on its k shell accounts for transfer of energy between $\boldsymbol{k}$ and the modes $\boldsymbol{r}$ whose wave vector tips lie on the sphere of radius k with center at $\boldsymbol{k}$.

Second (class II), consider triads in which energy is transferred between modes $\boldsymbol{k}$ and $\boldsymbol{p}$ under the influence of a spectator mode $\boldsymbol{r}$. For these interactions, the direction of transfer is clearly that of $\boldsymbol{r}$.

These two classes are a physically reasonable extension of the classical turbulence theoretic notion of local triad interactions to the case of anisotropy. Both classes fit neatly into our notion of modified locality. For the total diffusion tensor, we have $D_{ij}(\boldsymbol{k})=D_{ij}^{\mathrm{I}}+D_{ij}^{\mathrm{II}}$.

The class-I contribution to the diffusion tensor is

$$D_{ij}^{\mathrm{I}}(|\boldsymbol{k}|)=\int d\Omega_p\hat{p}_i\hat{p}_jD_\Omega(\boldsymbol{k},\boldsymbol{p})=k^2\int d\Omega_p\frac{\hat{p}_i\hat{p}_j}{\tau^{\mathrm{sp}}(\boldsymbol{p})}, \tag{4}$$

where $\mathrm{d}\Omega_p$ is the differential solid angle with respect to $\boldsymbol{p}$, $\hat{p}_i$ is the ith Cartesian component of $\boldsymbol{p}/p$, and $\boldsymbol{r}=\boldsymbol{k}+\boldsymbol{p}$ is understood [40]. Note that D^{I} is independent of the direction of $\boldsymbol{k}$. We employ the nonlinear rate of the spectator mode $\boldsymbol{p}$, associated with terms like $\boldsymbol{v}(\boldsymbol{p})\cdot\nabla\boldsymbol{v}(\boldsymbol{k})$—i.e., $kZ^2(\boldsymbol{p})/Z_p$.

Similarly, for class-II interactions we have

$$D_{ij}^{\mathrm{II}}(\boldsymbol{k})=k^2\int d\Omega_p\frac{\hat{r}_i\hat{r}_j}{\tau^{\mathrm{sp}}(\boldsymbol{r})}. \tag{5}$$

As $\boldsymbol{p}=\boldsymbol{k}-\boldsymbol{r}$, it follows that D^{II} will depend on the direction of $\boldsymbol{k}$, as well as its magnitude. In general, $\boldsymbol{r}$ does not lie on the k shell, and we consider separately the local and nonlocal contributions: Class IIa comprises triads with $k/2\leq|\boldsymbol{r}|\leq 2k$, corresponding to values of $\boldsymbol{p}$ on the k shell and lying outside the cone $\theta=\theta_0$ centered on $\boldsymbol{k}$ ($\theta_0\approx 1/2$ rad). In $\tau^{\mathrm{sp}}(\boldsymbol{r})$ we employ $B_0/Z_r\approx B_0/Z_k$, giving, for k in the inertial range,

$$D_{ij}^{\mathrm{IIa}}(\boldsymbol{k})=k^2\int_0^{2\pi}d\phi\int_{-1}^{7/8}d\mu\frac{\hat{r}_i\hat{r}_jrZ^2(\boldsymbol{r})/Z_r}{1+\frac{B_0}{Z_k}|\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{B}}_0|}, \tag{6}$$

where $\boldsymbol{k}=k\hat{\boldsymbol{e}}_3$ defines the polar axis for ϕ and $\mu=\cos\theta$. Note that the integration is over the angles for $\boldsymbol{p}$.

Class IIb considers the (nonlocal) wave vectors $\boldsymbol{r}$ such that $|\boldsymbol{r}|<k/2$, so that $\boldsymbol{p}$ lies within the cone $\theta\leq\theta_0$ centered on $\boldsymbol{k}$. These couplings act mainly to isotropize the energy distribution on the k shell and contribute little to transfer in the $\hat{\boldsymbol{k}}$ direction. Here we emphasize local couplings and neglect the D^{IIb} contributions, although they can be important for strongly anisotropic spectra. This completes our development of the model.

We now present several illustrative examples. We begin with the *isotropic* case—i.e., $B_0=0$, and $Z^2(\boldsymbol{p})=Z_k^2/4\pi$—independent of the angles (θ,ϕ) on the sphere. Using $rZ^2(\boldsymbol{r})/Z_r\approx kZ_k/4\pi$ in (6) gives

$$D_{ij}^{\mathrm{I}}=\frac{k^3Z_k}{3}\delta_{ij},\quad D_{ij}^{\mathrm{IIa}}=0.22k^3Z_k\left[\delta_{ij}+\frac{19}{15}\hat{k}_i\hat{k}_j\right]. \tag{7}$$

Recall that the ith component of the diffusive flux is $F_i=-(D_{ij}^{\mathrm{I}}+D_{ij}^{\mathrm{II}})\partial E(\boldsymbol{k})/\partial k_j$. For isotropy, $\partial E(\boldsymbol{k})/\partial\boldsymbol{k}$ is in the $\boldsymbol{k}$ direction, and the flux can be rewritten in terms of a scalar diffusion coefficient $D\approx(5/6)k^3Z_k$, corresponding to a $C_{\mathrm{Kol}}\approx 0.48$, in near agreement with Leith [16].

However, isotropy cannot persist when B_0 is nonzero, but nonetheless, as an initial condition, it provides useful insights. Equations (4) and (6) can be evaluated exactly for this case, after approximating $rZ^2(\boldsymbol{r})/Z_r$ as $kZ_k/4\pi$. The expressions are cumbersome; however, the leading-order terms as $a=B_0/Z_k\to\infty$ are approximately

$$\frac{D^{\mathrm{I}}_{ij}}{k^3Z_k}\approx\frac{0.5}{a}\left[\delta^{\perp}_{ij}\ln\left(\frac{a}{2}\right)+\hat{B}_i\hat{B}_j\right],$$

$$\frac{D^{\mathrm{IIa}}_{ij}}{k^3Z_k}\approx\frac{0.4}{a}\begin{cases}\left[\delta^{\perp}_{ij}+\dfrac{14}{9}\hat{B}_i\hat{B}_j\right], & \boldsymbol{k}\parallel\boldsymbol{B}_0,\\ \left[\left(\delta^{\perp}_{ij}+\dfrac{6}{5}\hat{k}_i\hat{k}_j\right)\ln\dfrac{a}{2}+\hat{B}_i\hat{B}_j\right], & \boldsymbol{k}\perp\boldsymbol{B}_0,\end{cases} \tag{8}$$

where $\delta^{\perp}_{ij}=\delta_{ij}-\hat{B}_i\hat{B}_j$ and $\hat{\boldsymbol{B}}=\boldsymbol{B}_0/B_0$.

Finally, for the case of moderate anisotropy, we evaluate the same exact expressions (not shown) with $a=B_0/Z_k=1$:

$$\frac{D^{\mathrm{I}}_{ij}}{k^3Z_k}=\frac{1}{4}[\delta^{\perp}_{ij}+0.77\hat{B}_i\hat{B}_j],$$

$$\frac{D^{\mathrm{IIa}}_{ij}}{k^3Z_k}=\begin{cases}0.14[\delta^{\perp}_{ij}+2\hat{B}_i\hat{B}_j], & \boldsymbol{k}\parallel\boldsymbol{B}_0,\\ 0.17[(\delta^{\perp}_{ij}+1.2\hat{k}_i\hat{k}_j)+0.82\hat{B}_i\hat{B}_j], & \boldsymbol{k}\perp\boldsymbol{B}_0.\end{cases} \tag{9}$$

Although this instantaneously isotropic case is somewhat artificial, the results (8) and (9) reveal important features of the **k**-space diffusion model. Comparing the large-a results (8) with the $B_0=0$ ones (7) shows that asymptotically the diffusion coefficients are all reduced by a factor $\sim Z_k/B_0$. Moreover, the tensor structure has altered, with the piece parallel to the magnetic field (indicated by $\hat{\boldsymbol{B}}\hat{\boldsymbol{B}}$) falling off faster with Z_k/B_0 than the perpendicular pieces do. Clearly, all transfer is greatly weakened when $\boldsymbol{B}_0$ is strong—parallel transfer most of all—even if the spectrum is isotropic. The $a=1$ case, Eqs. (9), has a tensor structure similar to the large-a case and already shows suppression of parallel transfer.

A simple parametrization that captures the essential elements of both limits is, for the D^I contributions,

$$D^{\mathrm{I}}_{ij}\approx\frac{k^3Z_k}{2a}\begin{pmatrix}\frac{1}{2}+\ln a & 0 & 0\\ 0 & \frac{1}{2}+\ln a & 0\\ 0 & 0 & 1-\frac{5}{8a}\end{pmatrix}, \tag{10}$$

which is approximately valid for $a\geqslant 1$; here, $\hat{\boldsymbol{B}}_0=\hat{\boldsymbol{e}}_3$.

For anisotropic MHD, it is sometimes useful to view the spectrum as consisting of low- and high-frequency parts, the distinction being which side of the "equal-time-scale" boundary, $\tau^{\mathrm{A}}(\boldsymbol{k})\approx\tau^{\mathrm{nl}}_k$, the **k** mode lies on [6,21,32,41–45]. For B_0 large, low-frequency turbulence concentrates in a narrow region near $k_{\parallel}=0$, where $\tau^{\mathrm{nl}}_k<\tau^{\mathrm{A}}(\boldsymbol{k})$ is satisfied. In reduced MHD (RMHD) [41,42] and "critical balance" [44] models, one assumes that the higher-frequency part of the spectrum is not highly populated. RMHD should be a reasonable approximation when $1/a\sim\delta B/B_0\ll 1$, initial conditions and driving are at low frequency, and the flow remains nearly incompressible. As an example, we examine **k** diffusion due to a low-frequency component, with an amplitude $Z^2(\boldsymbol{k})\sim B_0Z_k$ on a narrow part of each k shell, centered around the $k_{\parallel}=0$ plane. This scaling with B_0 will support an order-1 diffusive energy flux due to activity within a narrow band. A second (weak) component is assumed to lie on the other side of the equal-time-scale boundary, $\tau^{\mathrm{A}}(\boldsymbol{k})<\tau^{\mathrm{nl}}_k$. Thus, with $\mu_k=\cos\theta_k=\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}}_0$, a model spectrum is $Z^2_{\mathrm{RMHD}}(\boldsymbol{k})=\gamma\frac{B_0Z_k}{8\pi}$, where $\gamma=1$ for $|\mu_k|\leqslant\frac{1}{a}$ and $\gamma=e^{1-a|\mu_k|}$ when $|\mu_k|>\frac{1}{a}$. This is consistent with RMHD models, critical balance spectra, and fits to simulation data [39,44]. Substituting into (4) and (6) leads to the following large-a results:

$$D^{\mathrm{I}}_{ij}\approx 0.17k^3Z_k\left[\delta^{\perp}_{ij}+\frac{0.56}{a^2}\hat{B}_i\hat{B}_j\right], \tag{11}$$

$$\frac{D^{\mathrm{IIa}}_{ij}}{k^3Z_k}\approx 0.14\left[\delta^{\perp}_{ij}+0.14\hat{k}_i\hat{k}_j+\frac{0.33}{a^2}\hat{B}_i\hat{B}_j\right],\quad \boldsymbol{k}\perp\boldsymbol{B}_0. \tag{12}$$

Therefore we conclude that the asymptotic diffusion coefficient for a model RMHD (plus weak high-frequency waves) spectrum is

$$D_{ij}\approx 0.33k^3Z_k\begin{pmatrix}0.94 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0.43/a^2\end{pmatrix}, \tag{13}$$

where $\hat{\boldsymbol{k}}=\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{B}}_0=\hat{\boldsymbol{e}}_3$. Note that the total perpendicular flux $\epsilon_{\perp}\sim\int d\mu F_{\perp}$ through the k shell is order unity, since the spectral density $E(\boldsymbol{k})=O(B_0)$ and the angular width on the shell surface $k_{\parallel}/k=O(1/B_0)$. The parallel flux near $\boldsymbol{k}\perp\boldsymbol{B}_0$, on the other hand, is smaller by a factor of $O(a^{-2})$. This is because nonlocal effects (D^{IIb}) are needed to cause a thin shell to spread in the parallel direction. For **k** and $\boldsymbol{B}_0$ approximately parallel and **k** deep in the "wave region," D^{IIa}_{ij} is $O(\exp(-a/4)/a)$ and vanishes rapidly. Thus, $D_{ij}(\boldsymbol{k}\parallel\boldsymbol{B}_0)$ is well approximated by (11) and leading-order transfer is again perpendicular, with parallel flux down by a factor a^{-2}. This is completely in accordance with ideas of dominant perpendicular spectral transfer in MHD with a strong B_0 [21–26]. The simple form of the model exemplified by (13) suggests possible utility in practical calculations.

In summary, we have developed a model of anisotropic MHD spectral transfer based on **k**-space diffusion. It supports analytic investigation for special cases, returns to isotropy when appropriate, and is numerically tractable in other circumstances. Due to its numerical simplicity, this model may prove to be useful in space physics and astrophysics studies where full simulation of MHD turbulence is not feasible, but spectral information is needed.

Related models [14,20] treat spectral transfer as a mixture of "advection" and diffusion in $\boldsymbol{k}$ space. The advective part, introduced in [46], is problematic as it produces pulses in wave number (unpublished numerical tests). Both models diffuse in $k_{\parallel}$ and in the $k_{\perp}$ direction either advect [20] or advect and diffuse [14]. The former case is specialized to strong B_0 and employs particular spectra such as those found in RMHD [6,32,41–45], although the relationship to weak turbulence is also examined. Neither of the above models examines how transport and diffusion behave as the mean magnetic field B_0 varies from strong (highly anisotropic) to zero (isotropic).

The purely diffusive phenomenology developed herein accounts for anisotropy and also goes over smoothly to the isotropic hydrodynamic case. (Note that substantial progress has been made in understanding closure for the case of extremely large B_0 and its effect on weakly nonlinear spectral transfer [37,47].) Naturally, numerical factors such as those in (10) and (13) may require tuning by comparisons with simulations and numerical studies of the model are planned. It might also be advantageous to extend the model to include the effects of cross helicity and to introduce polarization (vector component) effects. It seems clear, for example, that the diffusion coefficient for Z^{+} will depend upon Z^{-} and vice versa. Further study of nonlocal effects on $\boldsymbol{k}$-space diffusion may be important too, especially for highly anisotropic spectra.

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