Effective boundary conditions for dense granular flows

Riccardo Artoni,* Andrea Santomaso, and Paolo Canu
Dipartimento di Principi e Impianti di Ingegneria Chimica "I. Sorgato," Università di Padova, Via Marzolo 9, 35100 Padova, Italy
(Received 27 March 2008; revised manuscript received 7 January 2009; published 23 March 2009)

We derive an effective boundary condition for dense granular flow taking into account the effect of the heterogeneity of the force network on sliding friction dynamics. This yields an intermediate boundary condition which lies in the limit between no slip and Coulomb friction; two simple functions relating wall stress, velocity, and velocity variance are found from numerical simulations. Moreover, we show that this effective boundary condition corresponds to Navier slip condition when the model of G. D. R. Midi [Eur. Phys. J. E **14**, 341 (2004)] is assumed to be valid, and that the slip length depends on the length scale that characterizes the system, viz. the particle diameter.

DOI: 10.1103/PhysRevE.79.031304 PACS number(s): 45.70.−n, 47.57.Gc, 47.50.−d

I. INTRODUCTION

Granular media exhibit a wide range of flow regimes [1], as well as a plethora of dynamical instabilities [2]. Focusing on gravity (or shear) driven flows, three regimes have been pointed out: (1) the collisional (*gaslike*) regime, where energy is dissipated by the inelasticity of the collisions, (2) the dense flowing (*liquidlike*) regime, in which particles undergo long-lasting contacts and dissipation occurs through dynamic friction, and (3) the static (*solidlike*) regime, which is capable of maintaining structures due to the threshold nonlinear nature of static friction. These regimes were studied with both experiments and discrete models, with the latter having experienced a great advance in the last years, starting from the work of Cundall and Strack [3].

Reliable continuum models would be of great advantage in simulating granular media, particularly when dealing with complex geometries or flows; in fact a unifying theory is still lacking. In this perspective, regimes (1) and (3) have been worked out with some success in a variety of theoretical studies, respectively, with the kinetic theory of granular gases [4] and with continuum critical state soil mechanics [5]. For the dense regime, various theoretical approaches have been developed (and extensively reviewed in [6]); the last more attractive one is that proposed by the French Research Group on Divided Media (G.D.R. MiDi) based on the inertial number I (see [7–10]), the importance of which was already stated by Goddard [11]. However, despite the great attention toward continuum models and rheologies, and despite the work done to derive boundary conditions (BCs) for rapid collisional flows (both in the case of bumpy [12,13] and flat frictional walls [14]), little effort has been devoted to develop realistic boundary conditions for the velocity field at smooth or rough walls for the case of dense flows even if the crucial role of side walls was recognized, for example, for inclined chute flows [15]. A common experimental approach developed to overcome this issue is the practice of gluing particles to the walls in order to assume a no-slip boundary condition in the interpretation of the results. This intelligent choice is of fundamental importance but has, in our opinion, two major drawbacks: at first, it is known [7] that for high shear rates particles undergo strong slip at the glued particles—bulk particles interface, a slip that adds some difficulty in holding the continuum hypothesis; thus it is not clear whether the glued particles are part of the bulk or of a bumpy wall so that boundary conditions must be expressed on the first moving layer in contact with the glued one. The second drawback of this experimental practice is the partial applicability to real situations: the flow on smooth surfaces such as in hopper discharge usually shows particles slipping at the solid interface. Slip can be promoted or can be an undesired phenomenon, often we are concerned with stick-slip phenomena [16,17], which are common in dry-friction dynamics [18]; in all of these cases, a deeper understanding of the behavior of granular materials flowing near a boundary is needed, and the no-slip boundary condition is not the most valid approach.

The scope of the present work is to determine boundary conditions for the dense flow of granular materials. The analyses developed for collisional flows [12–14] do not apply because in dense systems particles undergo long-lasting contacts, the medium is composed by breaking and forming of force-chain networks, and dissipation is mainly due to friction. In a recent work [19] we used the mixing-length model proposed by G. D. R. MiDi [7] and showed that using a slip boundary condition instead of a no-slip one considerably improved the predictions of the model in the vertical chute configuration; there we used a Coulomb friction condition, which could be a valid alternative to the no-slip condition. In this work we go even further, showing for a simple case that taking into account the effect of the heterogeneity of the force network yields an intermediate boundary condition which lies in the limit between no slip and Coulomb friction; moreover, we show that this effective boundary condition corresponds to Navier slip-length condition if G. D. R. MiDi's model is assumed, and that the slip length depends on the length scale that characterizes the system, viz. the particle diameter.

II. SIMPLE MODEL

We consider a single particle of mass m and diameter d lying on a plane, moving with instantaneous velocity V; the

*riccardo.artoni@unipd.it

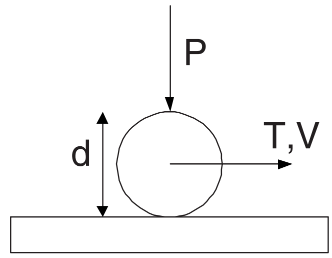

FIG. 1. Schematics of the variables considered in this work.

particle is subjected to a normal force P and to a tangential force T. We will neglect, for simplicity, the effect of couples acting on the particle, considering only translational sliding movements. This choice is based on the consideration that, documented in literature, for many flows of practical interest (e.g., industrial ones), flat surfaces reduce the rotation of particles, and rolling friction is generally considered small compared to sliding friction [20]. Bumpy or geometrically more complex walls are therefore outside the scope of this work. We will assume that due to the heterogeneous nature of the medium the normal force P is a random function of time with a given distribution function. Alternatively, even T could fluctuate but we assume for the sake of simplicity that only the normal force does; qualitative results are not affected by the choice of the fluctuating force. Let F be the friction force; we consider the simplest model of solid friction, e.g., Amontons' law, with only one friction coefficient μ,

$$F=\begin{cases} T & \text{if } V=0 \text{ and } T<\mu P \\ \mu P & \text{else} \end{cases} . \tag{1}$$

The motion of the particle is calculated from Newton's law (Fig. 1):

$$m\frac{dV}{dt}=T-F. \tag{2}$$

If the normal force was constant, only two situations would be possible, corresponding, respectively, to no-slip and Coulomb conditions. But if the force fluctuated, the particle would undergo slip and no-slip events, which globally represent a non-Coulomb slip phenomenon; our aim is to derive an average expression for the slip velocity as a function of the forcings. Let us consider a typical distribution of normal forces of the form:

$$p(f)=a(1-be^{-f^2})e^{-\beta f}, \tag{3}$$

as suggested in [21], where $f=P/P_{\text{ave}}$ and a is a normalization coefficient. This distribution of forces holds for normal forces in uniaxial compression, in a spatial sense; we will make the key assumption that, in dense granular flows, this distribution acts also between successive rearrangements of tangential forces in time. Our choice is supported by the fact that results do not depend on the particular choice of the distribution, apart from one point (the existence of a cutoff value in the force) which will be discussed later, and whose influence is limited. Moreover, Longhi *et al.* [22] showed for a two-dimensional (2D) rapid granular flow that the distribution of forces has the same characteristic exponential tail as in static media. We suppose further that the force is a piecewise linear function whose nodes are extracted from this distribution. Let P_{ave} be the average value of the normal force. We choose the time step between successive force rearrangements to be equal to the relaxation time $\tau=\sqrt{\frac{md}{P_{\text{ave}}}}$; it follows directly that, rescaling t by τ, the time step over which the force rearranges is one. Further rescaling leads to the dimensionless variables: $V'=V\sqrt{\frac{m}{P_{\text{ave}}d}}$ and $T'=T/\mu P_{\text{ave}}$, $P'=P/P_{\text{ave}}$.

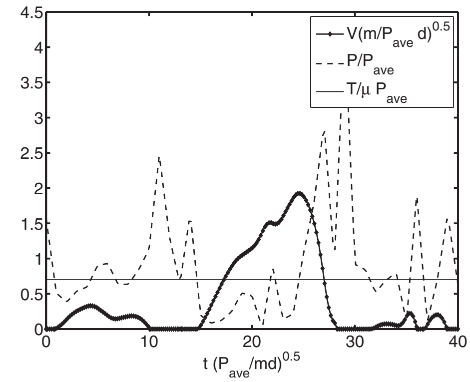

FIG. 2. Example of the local dynamics of the system.

Some comment is needed on the choice of the time step τ. When dealing with granular flows, we consider two fundamental time scales, the time scale related to shear $\tau_\gamma=|\dot{\gamma}|^{-1}$, and the time scale related to pressure, in this case $\tau_p=\sqrt{\frac{md}{P_{\text{ave}}}}$ [7]. Baran and Kondic [23], for the case of rapidly flowing granular materials (with a volume fraction $\nu\sim 0.4$), showed that the autocorrelation function of the stress signal decays strongly to 1 after a time which is $dt\approx 0.01\tau_\gamma$; they also suggested from a comparison between two numerical experiments that dt is an increasing function of τ_γ. Such a dependence is expected in the collisional regime, where the controlling time scale is exactly τ_γ. In the case of dense flows, instead, $\tau_p \ll \tau_\gamma$ is expected so the controlling time scale is provided by pressure rearranging action and it can be safely assumed that $\tau\sim\tau_p$.

If we define $\alpha(t)$ as

$$\alpha=\begin{cases} 0 & \text{if } V=0 \text{ and } T<\mu P \\ 1 & \text{else} \end{cases} , \tag{4}$$

the equation of motion becomes

$$\frac{dV'}{dt'}=\alpha\mu(T'-P'), \tag{5}$$

from which we can compute the average rescaled slip velocity defined as

$$V'_{\text{ave}}=\mu\lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^\tau dt'\left[\int_0^{t'}\alpha(T'-P')dt\right]. \tag{6}$$

We solve numerically the equation of motion; an example of the stick-slip behavior of the system is given in Fig. 2. An initially motionless particle can start to move only if the instantaneous normal force is below the yield threshold. A

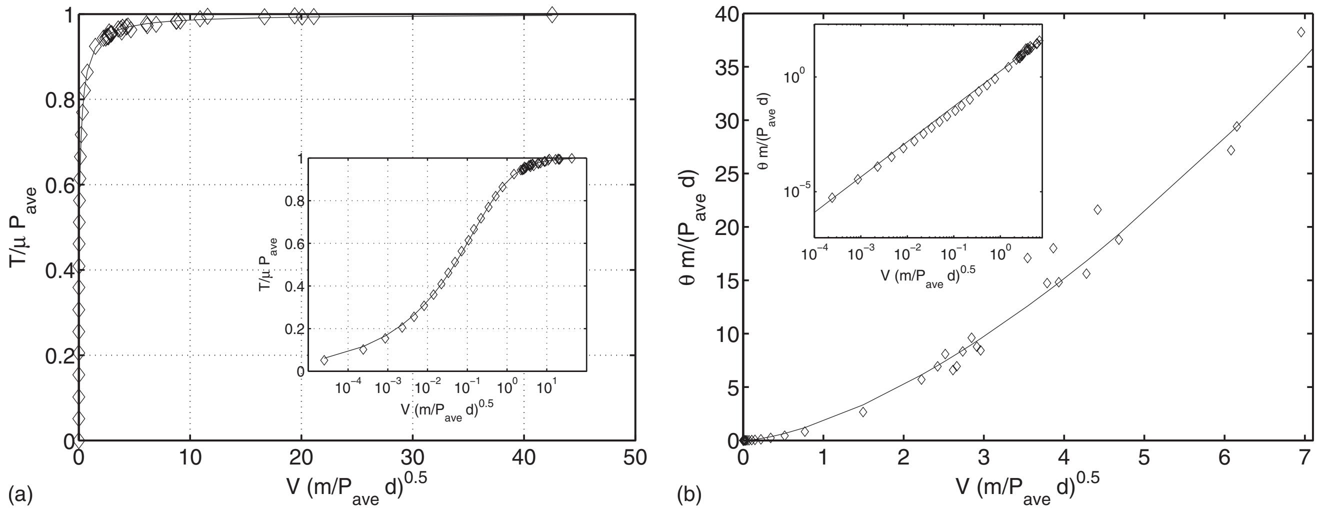

FIG. 3. Dependence of statistics of particle velocity on statistics of force. (a) Rescaled average pulling force vs average slip velocity. (b) Rescaled velocity variance vs average slip velocity. Best fits from Eqs. (7) and (8) are also included.

moving particle can decelerate only if the normal force is higher than the threshold. Moreover, it is clear from Fig. 2 that the area in which normal forces oppose motion is larger than the area in which they promote motion; it is the dynamical nature of the system that causes the body to have a nonnull average velocity. It would be desirable to find a relationship between the average slip velocity computed by means of Eq. (6) and the rescaled average tangential force (which corresponds to a rescaled effective friction coefficient, being $\mu_{\mathrm{eff}}/\mu = T/\mu P_{\mathrm{ave}}$). After solving Eq. (5) it is possible to look at the dependence of the statistics of the particle motion on the average value of the force in Fig. 3.

III. DISCUSSION

The curves evidence a no-slip limit at low values of the rescaled force T', and a Coulomb limit for $T' \to 1$. The way V'_{ave} approaches zero depends on the nature of the distribution $p(f)$: if the distribution had an upper cutoff value, it would be easy to conclude that the system had a sort of yield-stress behavior at the wall, with a finite range of T' giving $V'_{\mathrm{ave}}=0$; in the other case, without cutoff, the average velocity would be zero only for $T'=0$. This is the only point in which the choice of the distribution function qualitatively changes something in the results; however, the fast decrease in the tail in the distribution, if not giving a "plastic" behavior, would give some sort of pseudoplastic behavior because of the need to impose a certain stress to obtain an appreciable slip. So, with a certain loss of exactness, it is possible to assume also a yield-stress formulation for the boundary condition (BC).

It is interesting to note that also the variance of the distribution of the instantaneous particle velocities, corresponding to the concept of granular temperature, which we express as $\theta = \langle [V(t) - V_{\mathrm{ave}}]^2 \rangle$, where brackets denote time averaging, grows when T' increases, which is similar to the behavior of the slip velocity. Due to its definition, θ is made dimensionless with the position $\theta' = \theta \frac{m}{Pd}$. In Fig. 3 correlation between granular temperature and *average* velocity is shown to follow a power-law behavior. From a general standpoint, the boundary conditions can be expressed with the help of the following fitting functions (in the following, subscript ave will be eliminated for the sake of simplicity):

$$T' = \left(\frac{V'}{V' + c_1} \right)^{\gamma}, \tag{7}$$

$$\theta' = c_2 V'^{\beta}, \tag{8}$$

where $\gamma > 0$, $\beta < 0.5$, and $c_1, c_2 > 0$ are fitting parameters. A very good fit is obtained for $\gamma \approx 0.28$, $\beta \approx 1.5$, $c_1 \approx 0.51$, and $c_2 \approx 1.8$. In the figures the fit is represented as a solid line. Equations (7) and (8) are the simplest expressions for the effective boundary conditions that can be applied at the wall characterized by a particle-wall friction coefficient μ. These functions are an important result of this work: we have obtained two boundary conditions which are characterized by simplicity and direct applicability to continuum simulations of granular flows.

IV. DEPENDENCE ON THE PARAMETERS OF THE MODEL

In the preceding section we showed that the average behavior of stick-slip events can be interpreted by means of simple relations between dimensionless stress and velocity. Our aim is to propose a simple tool allowing treatment of the regime between no slip and continuous sliding in an average sense.

The parameters of the proposed functions should be obtained experimentally; however some issues must be considered, which are related to the rather simple model we assumed, to see how the obtained relations depend on the assumptions made. First of all, we used a particular choice of the force distribution $p(f)$ [21], taken from statistics of static granular packings. In Fig. 4 we show how the qualitative behavior of the obtained functions is not affected by the par-

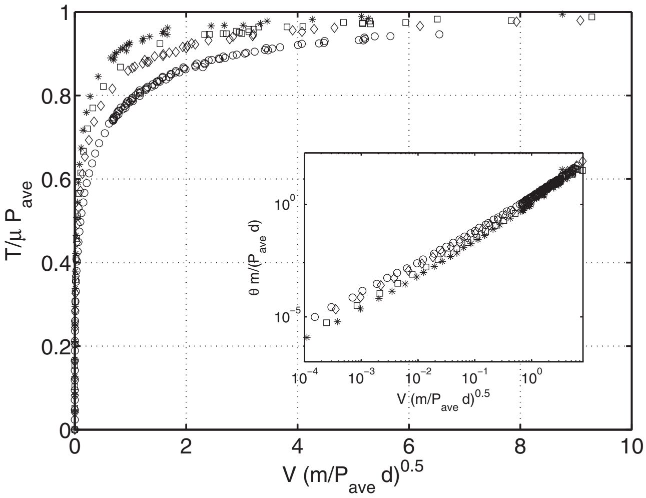

FIG. 4. Curves obtained varying the force distribution: Eq. (3) (squares), half-Gaussian-like (diamonds), same as Eq. (3) but with β=0.15 (circles), and β=3 (stars). (outer panel) Rescaled average pulling force vs rescaled average slip velocity (inner panel). Rescaled velocity variance vs rescaled average slip velocity.

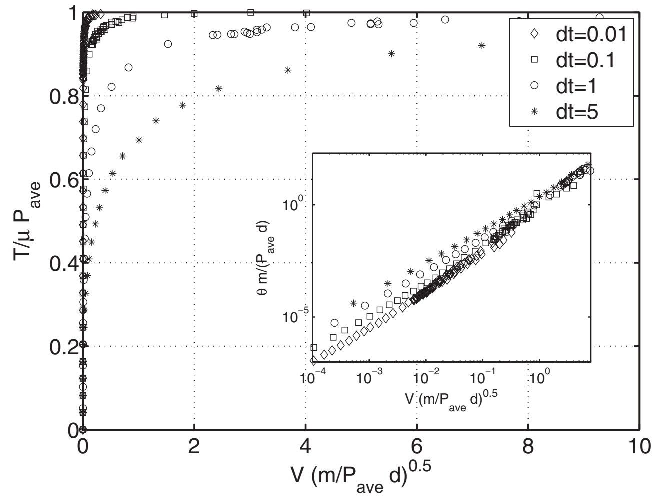

FIG. 6. Data obtained varying the time step of force changing. (outer panel) Rescaled average pulling force vs rescaled average slip velocity (inner panel). Rescaled velocity variance vs rescaled average slip velocity.

ticular choice of the force distribution but seem to come only from the stochastic behavior of the force. Other distributions were also considered, giving the same qualitative behavior. From a more quantitative standpoint, the amount of slip increases as the variance of the distribution increases (increasing β corresponds to increasing the variance of the force distribution). Another issue that shall be considered is the friction model assumed, which is quite simple. Assuming a different model, with two friction coefficients (a static friction coefficient higher than the sliding friction coefficient), does not affect the results, also quantitatively (see Fig. 5). Again, the main feature causing the intermediate stick-slip behavior is the randomness of the force, related to the presence (and the mechanisms of breaking and forming) of force chains. Regarding the variations in the force in time, we

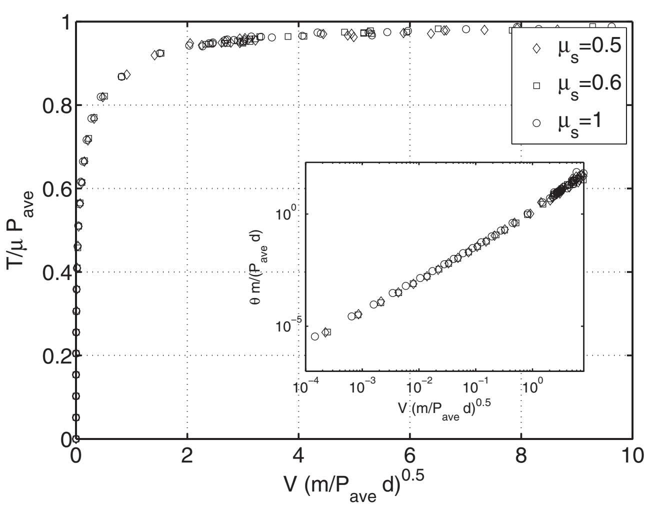

FIG. 5. (Outer panel) Average dimensionless stress vs average dimensionless velocity and (inner panel) dimensionless granular temperature vs average dimensionless velocity, varying the coefficient of static friction.

identified as a typical time scale the rearrangement time $\tau = \sqrt{\frac{md}{P_{\text{ave}}}}$; this choice might be questionable, particularly in high shear situations (when the inertial number $I \gg 1$), where the characteristic time related to shear is small compared to the rearrangement time. Therefore the validity of our approach should be rigorously valid in the limit of small I while experimental work is needed to verify and extend its validity in other regimes. In this perspective it can be useful to verify whether the time step over which the force changes has an impact on the final curves although $\Delta t \sim \tau$ seems a reasonable assumption. From results reported in Fig. 6 we can see that the larger the time step, the larger the amount of slip predicted by the model for the same average pulling force. However, it is important to underline that the qualitative behavior of the effective boundary conditions does not depend on particular choices for the distribution, the time step, or the friction model, and that it can be well represented by means of the proposed Eqs. (7) and (8). However, the parameters in these equations should be determined experimentally, from local measurements of slip velocity and wall stresses; in this perspective, we suggest that, to develop boundary conditions more suitable for gravity (i.e., stress) driven flows, slip measurements should be also done in gravity driven situations, for example, in the vertical chute configuration.

V. INTERPRETATION OF THE RESULTS BY MEANS OF A NAVIER BOUNDARY CONDITION

Navier boundary condition, relating the slip velocity and the gradient of the velocity normal to the boundary via a slip length λ, is a common way to characterize slip in fluid flows in microchannels and nanochannels; however, there is not a single plot of this condition in the V' vs T' diagram because such a plot needs information on the relationship between stresses and deformation rates in form of constitutive relations. For a Newtonian fluid, considering the force acting on a surface $S \sim d^2$,

$$V' = \lambda \frac{\mu}{\eta} \sqrt{\frac{mP}{d^5}} T', \tag{9}$$

which is linear and parametric in $\frac{\mu}{\eta}\sqrt{\frac{mP}{d^5}}$. A Bingham yield-stress fluid will have an explicit relation of the form:

$$V' = \lambda \frac{\mu}{\eta'} \sqrt{\frac{mP}{d^5}} (T' - T'_Y) \quad (\text{for } T' > T'_Y), \tag{10}$$

where T'_Y is the rescaled yield stress and η' is the viscosity coefficient in Bingham's model. So a Navier condition for Bingham's model in the V' vs T' plot is a line shifted by T'_Y and again parametric in $\frac{\mu}{\eta}\sqrt{\frac{mP}{d^5}}$.

Both of these relationship obviously do not conform to the behavior obtained from the model developed in this work; assuming a mixing-length model as G. D. R. MiDi's [7], where $\frac{T}{P} = \mu(I)$, with $I = \frac{\dot{\gamma}}{\sqrt{P/md}}$ (the difference in the expression of I of the previous literature is due to the fact that here P is a normal *force*, not a pressure), the assumption of $\mu(I) = \mu_s + \frac{\mu_2 - \mu_s}{I_0/I + 1}$ (taken from Jop *et al.* [8,15]) yields for a Navier BC:

$$V' = \frac{\lambda}{d} \frac{T' - \mu_s/\mu}{\mu_2/\mu - T'} \quad (\text{for } T' > \mu_s/\mu), \tag{11}$$

which reaches an asymptote for $T' \to \mu_2/\mu$, and is zero in the range $0 - \mu_s/\mu$. Thus, to unify the curves and represent the results obtained from the simple model of wall friction presented in this work, λ must be a function of the form:

$$\lambda = kd\zeta(T'), \tag{12}$$

where $\zeta(T')$ accounts for the change in the position of the asymptote and can be expressed simply as

$$\zeta = \frac{\mu_2/\mu - T'}{1 - T'}. \tag{13}$$

An important result is given in Eq. (12); to unify the curves as obtained in the "experiments," λ must be a multiple of d: this is actually an important result, being d the only internal length scale of the system, and so the best choice as a basis for estimating the slip length. This applies also to the other BCs, where λ should be a multiple of d, as well, but also a function of P and ρ (assuming $m = \rho d^3$ yields $\lambda \sim \frac{\eta}{\mu}\frac{d}{\sqrt{P\rho}}$). The typical form of V' vs T' curves for the various models is given in Fig. 7.

To resume, the intermediate efficient boundary condition we are looking for can be qualitatively expressed as Navier slip condition in a mixing-length framework, the slip length corresponding to a multiple of the particle diameter. A step further can be made in the direction of determining a value for λ. Let us admit the yield-stress behavior of Pouliquen's form for $\mu(I)$, and suppose that $\mu_2 \approx \mu$ (remember that μ is the particle-wall friction coefficient). In this perspective the slip length is simply proportional to the particle diameter and the best fit gives $\lambda/d \approx 0.2$. This value gives a sort of minimum slip length; in the case with $\mu_2 > \mu$ the slip length diverges for $T' \to 1$.

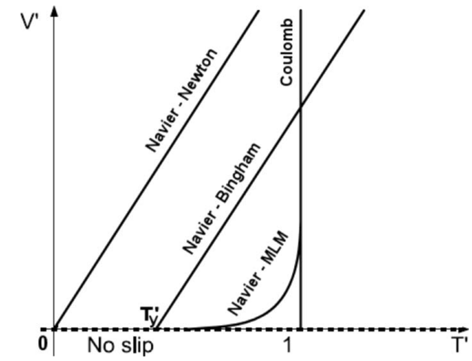

FIG. 7. Rescaled average velocity vs average pulling force for different BCs/constitutive laws. The slope of Newton and Bingham lines is $\lambda \frac{\mu}{\eta'}\sqrt{\frac{mP}{d^5}}$, where λ is assumed to be almost constant with respect to T'.

It is important to note that V' is an analog of the inertial number I for near wall flows, and T' is an effective wall friction coefficient as $\mu(I)$ is for the bulk; thus it is interesting to note that the shape of the curve $T'(V')$ is very close to that of $\mu(I)$; this can lead to some ideas on the origin of the effective friction coefficient in the bulk remembering that the effective wall friction coefficient derives from the assumption of heterogeneous forces.

In this work we do not aim to define the correct functional form for these BCs (even if a very good fit was obtained for this simple case) but we want to underline that real boundary conditions (even in simplified setups) are not no-slip-like or Coulomb-like, and assumption of one of these limiting BCs can introduce errors in the physical validity of granular flow models; this slip behavior can be captured by a modified Navier condition, where the slip length is proportional to the particle diameter.

To resume, a simple model of a particle sliding with the simplest frictional law on a plane has been developed in this paper to determine effective boundary conditions for dense granular flows. To account for the heterogeneity of the medium, the particle is subjected to a random normal force while a constant tangential force is assumed for simplicity. The dynamics consists of stick-slip events, which are related to the heterogeneity in the stress field; we reported the resulting dependence of the average tangential force on the average slip velocity and on the variance of the velocity of the particle (i.e., granular temperature), thus providing two possible effective boundary conditions for the velocity and granular temperature fields. The results are well fitted by simple laws and represent for the velocity field an intermediate behavior between Coulomb's law (at high velocities) and the no-slip boundary condition. Granular temperature is related to the velocity by a simple power-law behavior. The functional form we propose can be adopted as a general tool to quantify this intermediate behavior, as it does not depend on the particular choice of the force distribution or the friction model adopted. The approach is developed for the situation, of wide applicability, of flat walls. The case of bumpy walls, which is not treated in the present contribution, requires more complex models, including rotations of the particles and higher dimensionality, and for these reasons it should be addressed by means of numerical simulations taking into account also the bulk behavior; however, the functions developed in this work, which describe the intermediate stick-slip behavior, could be tested in that case and be a basis to develop *a posteriori* correlations. In addition, we demonstrated that the curve obtained by numerical simulation satisfies a modified slip-length Navier boundary condition within a mixing-length model of granular flow, with the slip length being proportional to the characteristic length of the system, the particle diameter. Further experimental work is needed to estimate the parameters and test the validity of the boundary conditions developed in real situations. In this perspective, we propose that experimental work should be done in assessing slip velocity—wall stress relations in gravity driven flows, and suggest for such an estimation the vertical chute geometry.

[1] H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys. **68**, 1259 (1996).

[2] P. King, P. Lopez-Alcaraz, H. Pacheco-Martinez, C. Clement, A. Smith, and M. Swift, Eur. Phys. J. E **22**, 219 (2007).

[3] P. A. Cundall and O. D. L. Strack, Geotechnique **29**, 47 (1979).

[4] J. T. Jenkins and S. B. Savage, J. Fluid Mech. **130**, 187 (1983).

[5] A. N. Schofield and C. P. Wroth, *Critical State Soil Mechanics* (McGraw-Hill, New York, 1968).

[6] O. Pouliquen and F. Chevoir, C. R. Phys. **3**, 163 (2002).

[7] G. D. R. Midi, Eur. Phys. J. E **14**, 341 (2004).

[8] P. Jop, Y. Forterre, and O. Pouliquen, Nature (London) **441**, 727 (2006).

[9] O. Pouliquen, C. Cassar, Y. Forterre, P. Jop, and M. Nicolas, *Proceedings of Powders & Grains 2005*, edited by R. Garcia-Rojo, H. J. Herrmann, and S. McNamara (Balkema, Rotterdam, 2005), p. 850.

[10] F. da Cruz, S. Emam, M. Prochnow, J.-N. Roux, and F. Chevoir, Phys. Rev. E **72**, 021309 (2005).

[11] J. D. Goddard, Acta Mech. **63**, 3 (1986).

[12] M. W. Richman, Acta Mech. **75**, 227 (1988).

[13] M. W. Richman and C. S. Chou, ZAMP **39**, 885 (1988).

[14] J. T. Jenkins and M. Y. Louge, Phys. Fluids **9**, 2835 (1997).

[15] P. Jop, Y. Forterre, and O. Pouliquen, J. Fluid Mech. **541**, 167 (2005).

[16] A. Baldassarri, F. Dalton, A. Petri, S. Zapperi, G. Pontuale, and L. Pietronero, Phys. Rev. Lett. **96**, 118002 (2006).

[17] S. Nasuno, A. Kudrolli, A. Bak, and J. P. Gollub, Phys. Rev. E **58**, 2161 (1998).

[18] F. Heslot, T. Baumberger, B. Perrin, B. Caroli, and C. Caroli, Phys. Rev. E **49**, 4973 (1994).

[19] R. Artoni, A. Santomaso, and P. Canu, Europhys. Lett. **80**, 34004 (2007).

[20] U. Tüzün, M. J. Adams, and B. J. Briscoe, Chem. Eng. Sci. **43**, 1083 (1988).

[21] D. M. Mueth, H. M. Jaeger, and S. R. Nagel, Phys. Rev. E **57**, 3164 (1998).

[22] E. Longhi, N. Easwar, and N. Menon, Phys. Rev. Lett. **89**, 045501 (2002).

[23] O. Baran and L. Kondic, Phys. Fluids **17**, 073304 (2005).