Direct evidence for conformal invariance of avalanche frontiers in sandpile models

A. A. Saberi,[1] S. Moghimi-Araghi,[2] H. Dashti-Naserabadi,[2] and S. Rouhani[2]
[1]*School of Physics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5531, Tehran, Iran*
[2]*Department of Physics, Sharif University of Technology, P.O. Box 11155-9161, Tehran, Iran*
(Received 5 October 2008; published 26 March 2009)

Appreciation of stochastic Loewner evolution (SLE_κ), as a powerful tool to check for conformal invariant properties of geometrical features of critical systems has been rising. In this paper we use this method to check conformal invariance in sandpile models. Avalanche frontiers in Abelian sandpile model are numerically shown to be conformally invariant and can be described by SLE with diffusivity $\kappa=2$. This value is the same as value obtained for loop-erased random walks. The fractal dimension and Schramm's formula for left passage probability also suggest the same result. We also check the same properties for Zhang's sandpile model.

DOI: 10.1103/PhysRevE.79.031121 PACS number(s): 64.60.av, 45.70.Cc, 11.25.Hf

I. INTRODUCTION

The concept of self-organized criticality (SOC) was first introduced by Bak *et al.* [1] through invention of sandpile models. These models are still the simplest examples of the class of models which show self-organized criticality. A definitive step in analyzing sandpile models was taken in [2], in which Dhar introduced a generalization of Bak, Tang, and Wiesenfeld model. This generalized model was called the Abelian sandpile model (ASM), because of the presence of an Abelian group governing its dynamics. Many different aspects of the model have been considered, for a good review see [3]. It has been shown that the model could be mapped to spanning trees [4] and is related to $c=-2$ conformal field theory [5,6].

There is also another non-Abelian sandpile model introduced by Zhang [7], which is a continuous version of ASM. Although they have different microscopic details but it is expected they fall in a same universality class; this is supported by numerical evidence [8,9].

A relationship between ASM and loop-erased random walk (LERW) has been shown in Ref. [10]. The loop-erased random walk was proposed by Lawler [11]. Such a walk is produced by erasing loops in an ordinary random walk as soon as they are formed. It turns out that the distribution of the LERW is related to the solution of the discrete Laplacian [12] with appropriate boundary conditions. It is also related to the Laplacian random walk [13,14]. The connection between LERW and ASM arises in the following way [10]: starting from a random walk one can produce a tree from it called backward tree. Then one can show that the chemical path on this tree is equivalent to the LERW obtained from the original random walk. Thus statistical properties of chemical paths on spanning trees and LERW's are the same. Using this identification, some analytical and numerical results have been developed. In [15] the upper critical dimension of the ASM was determined and in [16] the above result was confirmed numerically.

Soon afterwards it was realized that LERW belongs to a family of conformally invariant curves called Schramm-Loewner evolution, SLE_κ, with diffusivity constant $\kappa=2$ [17,18]. In this paper we show that LERW appears in certain geometrical aspect of the sandpile dynamics. In contrast with the previous results, we do not consider the chemical path of the spanning trees but consider the curve separating the toppled and untoppled sites, i.e., the avalanche frontier.

This paper is organized as follows. In Sec. II we give some background on the ASM and its properties. Also we introduce Zhang sandpile model very briefly. Section III is devoted to the definition and some references on the SLE. Finally in Sec. IV we present the numerical algorithm its results and discussion.

II. SANDPILE MODEL

We consider the Abelian sandpile model defined on a two-dimensional square lattice $L\times L$. On each site i the height variable h_i is assigned, taking its value from the set $\{1,2,3,4\}$. This variable represents the number of sand grains in the site i. This means that a configuration of the sandpile is given by a set of values $\{h_i\}$.

The dynamics of the system is relatively simple. At each time step, a grain of sand is added to a random site, i. Then site is checked for stability, that is, if its height is more than 4, it becomes unstable and topples: it loses four grains of sands, each of them is transferred to one of the four neighbors of the original site. It is common to write $h_j\rightarrow h_j-\Delta_{ij}$ for all j with Δ being discrete Laplacian. As a result of a toppling, the neighboring sites may become unstable and topple and a chain of topplings may happen in the system. If a boundary site topples, one (or two) grains of sand may leave the system, depending on the imposed boundary condition taken. The chain of topplings continue until the system becomes stable, i.e., all the height variables become less than or equal to four. Thus in each time step, the dynamics takes the system from a stable configuration C_m to another stable configuration C_{m+1}. The relaxation process is well defined: it always stops because sand can leave the system at the boundaries and produces the same result independent of the order in which the topplings are performed which is because of the Abelian property.

Under this dynamics the system reaches a well-defined steady state. All the stable configurations fall apart into two subsets: the transient states that do not occur in the steady state and the recurrent states that all occur with the same probability. It has been shown that the total number of recurrent states is det Δ [2]. The criterion that decides whether a configuration is recurrent or not is not a local one. There are some specific clusters, called forbidden subconfigurations (FSCs) that if any of them is found in a stable configuration, it would be a transient configuration. The simplest FSC is a cluster of two adjacent height-one sites. In general an FSC is a height configuration over a subset of sites, such that for any of the sites in this subset, the number of its neighbors within the same subset, is greater than or equal to its height. Such subsets could be as large as the whole system, thus in general you cannot decide easily if a configuration is recurrent or not.

An interesting question would be what is the probability of finding a site with height h or what is the probability of finding a specific cluster of height variables. Even more interesting, is the joint probabilities of such events. These questions have been answered for the case of weakly allowed clusters (WACs) [4]. WACs are the clusters that are not FSC, but if you remove a grain of sand from any of its sites it becomes FSC. The simplest example is one-site height-one cluster.

The correlation functions of all such clusters obey a power law with the same exponent; all the clusters have scaling exponent equal to two. From point of view of critical systems, one expects that in the scaling limit ASM should be expressed via a field theory. There have been found many indications that a specific conformal field theory called the $c=-2$ theory is related to ASM. First of all a connection between ASM and spanning trees has been found [4], therefore it should be related to $q\to 0$ Potts model, which is known to be related to the $c=-2$ theory. Also the exponents of the WAC fit in this theory. In [5] the critical and off-critical two- and three-point correlation functions of 14 simplest WACs were calculated and using these results the scaling fields associated with these WACs were obtained. This result was generalized to arbitrary WAC in [19]. These identifications were done only by comparing the correlation function. In [6] the fields were derived from an action and the way the probabilities are calculated in ASM are translated directly to field theory language to obtain the relevant fields. The $c=-2$ theory is a logarithmic theory [20], and it contains some fields that have logarithmic terms in their correlation functions. Such fields are related to one-site clusters with height more than one [21] although still a direct way to show it is missing. The action of $c=-2$ is $S\sim\int\partial\bar{\theta}\bar{\partial}\theta$, where θ and $\bar{\theta}$ are Grassmannian variables. It is easy to see why the action is related to ASM, just note that the number of recurrent configurations is det Δ and all occur with the same probability. So the partition function of the system is det Δ. This determinant could be written in terms of integrating over Grasmannian variables which leads to the above action in the scaling limit.

Interestingly, it was observed in [12] that the probability distribution of LERW may be written in terms of a Grasmannian path integral, reinforcing the connection between LERW and ASM.

Different properties characterizing an avalanche is the other subject usually investigated in ASM. We call the total number of topplings the size of avalanche and denote it by s. The number of distinct lattice sites toppled is denoted by d which is clearly less than or equal to the size of avalanche. This variable shows the area of the system which is affected by the avalanche. The duration t of an avalanche is the number of update sweeps needed until all sites are stable again. The other characteristic is the linear size of an avalanche which is measured via the radius of gyration of the avalanche cluster and is denoted by R. It was widely assumed In the critical steady state the corresponding probability distributions obey power-law behavior

$$P_\alpha(\alpha) \sim \alpha^{-\tau_\alpha}, \tag{1}$$

where α can be s, d, t, or R.

However it is shown that while the distribution of avalanche areas obeys finite-size scaling that of total number of topplings does not, rather it is characterized by a multifractal spectrum [22]. In this paper we concentrate on the area distribution and hence will assume that finite-size scaling holds. If one is interested in other properties of avalanches such as total number of topplings, either he or she has to consider a multifractal spectrum or to consider waves of toppling [16].

The above exponents are calculated numerically [23,24], also using specific assumptions some (different) analytic results have been obtained [25]. The exponents are not independent, as an example because the region that the sites topple is a compact one and does not have holes in it, the area s of the region should be proportional to R^2 statistically. This induces the relation $\tau_r=2\tau_s+1$ between the exponents.

Other versions of sandpile models have been considered [7,26]. In [7], Zhang introduced a model in which the height variables were continuous and are called energy. At any time step a random amount of energy is added to a random site. If the energy of the site becomes more than a specific amount, called threshold, it becomes active and topples: it loses all its energy, which is equally distributed among its nearest neighbors. In his original paper, Zhang observed, based on results of numerical simulation, that for large lattices, in the stationary state the energy variables tend to concentrate around discrete values of energy; he called this the emergence of energy "quasiunits." Then, he argues that in the thermodynamic limit, the stationary dynamics should behave as in the discrete ASM. Zhang model does not have the Abelian property, therefore little analytic results is at hand. However the numerical simulations show that it exhibits finite-size scaling property Eq. (1) [9,27].

These scaling relations imply that there should be some related geometric structures in the avalanches. We consider avalanche clusters in the steady state in which all sites have experienced toppling at least once. Then, in the following sections using theory of SLE, we investigate the statistics of the avalanche boundaries (see Fig. 1).

III. STOCHASTIC LOEWNER EVOLUTION

Critical behavior of various systems can be coded in the behavior of their geometrical features. In two dimensions, the criticality shows itself in the statistics of interfaces, e.g., domain walls. The domain walls are some nonintersecting curves which directly reflect the status of the system in ques-

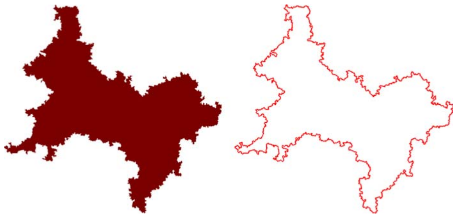

FIG. 1. (Color online) An avalanche cluster (left) consisting all sites that have toppled at least once and its frontier (right).

tion. For example, consider one of the prototype lattice models which can be interpreted in terms of random nonintersecting paths, Ising model, which we consider it in the physical domain, i.e., upper half plane $\mathbb{H}$. To impose an interface growing from zero on the real line to infinity, a fixed boundary condition can be considered in which all spins in the right and left sides of the origin are up and down, respectively. At zero temperature the interface is a straight line and increasing the temperature leads the interface to a random nonintersecting curve. In the 1920s, it has been shown by Loewner [28] that any such curve in the plane which does not cross itself can be created, in the continuum limit, by a dynamical process called Loewner evolution with a suitable continuous driving function ξ_t as

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi_t}, \tag{2}$$

where, if we consider the hull K_t, the union of the curve and the set of points which cannot be reached from infinity without intersecting the curve, then $g_t(z)$ is an analytic function which maps $\mathbb{H}\backslash K_t$ into the $\mathbb{H}$ itself.

For the mentioned Ising model, at zero temperature the interface can be described in the continuum limit by Loewner evolution with a specific constant driving function. At higher temperatures less than critical temperature T_c, the driving function might be a complicated random function. At $T = T_c$, the system and the interfaces as well are conformally invariant (in an appropriate sense), i.e., they are invariant under local scale transformations. Schramm showed [17] that the consequences of conformal invariance for a set of random curves are such that the driving function in the Loewner evolution should be proportional to a standard Brownian motion B_t (which is known as stochastic-Schramm Loewner evolution or SLE_κ). Therefore $\xi_t = \sqrt{\kappa} B_t$ so that $\langle \xi_t \rangle = 0$ and $\langle (\xi_t - \xi_s)^2 \rangle = \kappa |t-s|$ (for more precise mathematical definitions and theorems see the review paper [29] and references therein).

The diffusivity κ classifies different universality classes and is related to the fractal dimension of the curves D_f as

$$D_f = 1 + \kappa/8. \tag{3}$$

After invention of SLE, many of its properties and applications have been appeared by both mathematicians and physicists. Its connection with conformal field theory has also been made explicit in a series of papers by Bauer and Bernard [30]. It has been also appeared in various physical subjects such as two-dimensional turbulence [31,32], spin glasses [33], nodal lines of random wave Functions [34], experimental deposited WO_3 surface [35], and also in two-dimensional Kardar-Parisi-Zhang surface [36]. The connection between SLE and some lattice models in the scaling limit is also proven or conjectured today. For example, two-dimensional LERW is a random curve, whose continuum limit is proven to be an SLE_2 [17]. Self-avoiding random walk (SAW) [37] and cluster boundaries in the Ising model [38] are also conjectured to be $\mathrm{SLE}_{8/3}$ and SLE_3, in the scaling limit, respectively.

One of the calculations, which has been made by SLE which will be referred later, is the probability that the trace of SLE in domain $\mathbb{H}$ passes to the left of a given point at polar coordinates (R, ϕ). It was studied by Schramm using the theory of SLE in [39]. Because of scale invariance, this probability depends only on ϕ and has been shown that

$$P_\kappa(\phi) = \frac{1}{2} + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{8-\kappa}{2\kappa}\right)}\, {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}; \frac{3}{2}; -\cot^2(\phi)\right)\cot(\phi). \tag{4}$$

In the following, we will use these statements to show that the avalanche frontiers in the both ASM and Zhang's model can be described by SLE_2.

IV. NUMERICAL DETAILS: TEST FOR CONFORMAL INVARIANCE

In this section, using the scaling relations and theory of stochastic Loewner evolution introduced in previous sections, we show that the conformal field theory which describes the sandpile models algebraically, can be derived from a quite different approach, i.e., investigation of the statistics and symmetries of some well-defined geometric features during the sandpile dynamics. To this end, we consider the avalanche clusters in the steady-state regime during the dynamics: including all sites which topple at least once at each time step when adding a grain to a random site of the system makes it unstable (see Sec. II). Then we get an ensemble of the boundary of these clusters as suitable candidates to study their statistics and possible conformal invariance. We compare our results with similar ones for known models which their relations with sandpile models is made explicit, i.e., LERW.

To investigate the statistical behavior of the avalanche boundaries (loops) in the ASM model, we first calculate their fractal dimension by using the scaling relation between their radius of gyration R and their perimeter length l, i.e., $l \sim R^{D_f}$. As shown in Fig. 2, the fractal dimension is very close to the one for LERW which is proven to be 5/4, (the best fit to the data yields $D_f = 1.24 \pm 0.02$).

It is also discussed in [40] that the mean area of the loops scales with their perimeter length as $A \sim l^{2/D_f}$. The inset of Fig. 2 shows the comparison of this relation with one calculated for avalanche boundaries in ASM model. The same results can be obtained for the avalanche boundaries in Zhang's model which have not been shown here.

This fractal dimension D_f is consistent with the fractal dimension of SLE_2 curves in the scaling limit [see Eq. (3)].

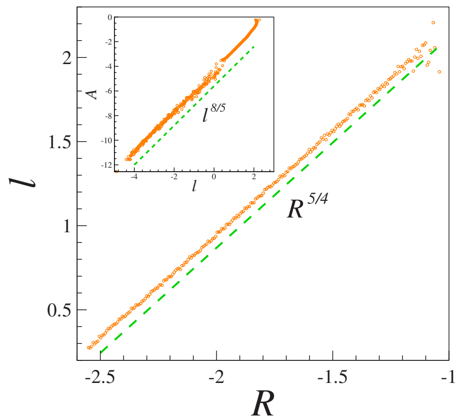

FIG. 2. (Color online) Main frame: log-log plot of the perimeter of avalanche frontiers (loops) l versus the radius of gyration R for ASM model simulated on squared lattice with size of 1024^2. Inset: log-log plot of the average area of loops A vs the length l. The dashed lines show the results for LERW.

This suggests that the scaling limit of the avalanche frontiers may be conformally invariant in the same universality class of LERW.

A simple way to check this proposition can be done using Eq. (4). Since in this equation it is supposed that the curves are in domain $\mathbb{H}$, so we have to be careful about reference domain. We assume that any avalanche frontier is in the plane, and then we can consider any arbitrary straight line which crosses the loop at two points $x_0=0$ and x_∞ as real line. Then we cut the portion of the curve which is above the real line. To have a curve starting from origin and tending to infinity, we use the map $\varphi(z)=x_\infty z/(x_\infty-z)$ for all points of the curve [41]. Doing so for all frontiers, we would have an ensemble of such curves and we can check Schramm's formula [Eq. (4)] for them.

Figure 3 shows the result for avalanche frontiers of both ASM and Zhang's model. The result is most consistent with the prediction for SLE_2 curves.

Now we are in a position to extract the Loewner driving function ξ_t, in Eq. (2), for these avalanche boundaries and

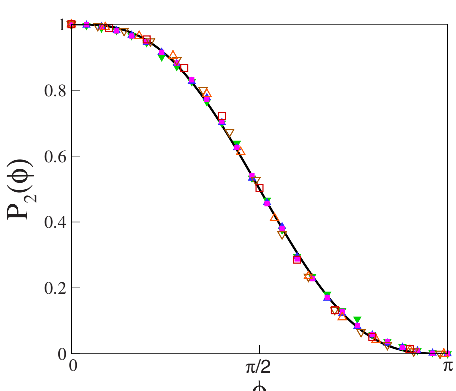

FIG. 3. (Color online) The probability that an avalanche frontier of ASM model (filled symbols) and Zhang model (open symbols) in domain $\mathbb{H}$, passes to the left of a point at polar coordinates (R,ϕ) for $R=0.05$, 0.1, and 0.2. The solid line shows the prediction of SLE for $\kappa=2$ [$P_2(\phi)$ in Eq. (4)].

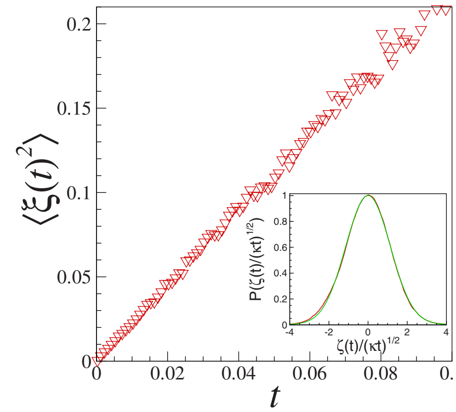

FIG. 4. (Color online) Statistics of the driving function $\xi(t)$ for the avalanche boundaries of ASM model. Main frame: the linear behavior of $\langle\xi(t)^2\rangle$ with the slope $\kappa=2.1\pm0.1$. Inset: the probability distribution function of the noise $\xi(t)/\sqrt{\kappa t}$ for $0\le t\le 0.05$.

examine whether they are Brownian motion. This is another direct check which shows the behavior of the curves under local scale transformations. We use successive conformal maps according to the algorithm introduced by Bernard *et al.* [32] based on the approximation that driving function is a piecewise constant function.

The procedure is based on applying the map $G_{t,\xi} = x_\infty\{\eta x_\infty(x_\infty-z)+[x_\infty^4(z-\eta)^2+4t(x_\infty-z)^2(x_\infty-\eta)^2]^{1/2}\}/\{x_\infty^2(x_\infty-z)+[x_\infty^4(z-\eta)^2+4t(x_\infty-z)^2(x_\infty-\eta)^2]^{1/2}\}$ on all the points z of the curve approximated by a sequence of $\{z_0=0,z_1,\ldots,z_N=x_\infty\}$ in the complex plane, where $\eta=\varphi^{-1}(\xi)$ and again $\varphi(z)=x_\infty z/(x_\infty-z)$, in which the dimensionless parameter t is used for parametrization of each curve. At each step, by using the parameters $\eta_0=\varphi^{-1}(\xi_0)=[\mathrm{Re}\, z_1 x_\infty-(\mathrm{Re}\, z_1)^2-(\mathrm{Im}\, z_1)^2]/(x_\infty-\mathrm{Re}\, z_1)$ and $t_1=(\mathrm{Im}\, z_1)^2x_\infty^4/\{4[(\mathrm{Re}\, z_1-x_\infty)^2+(\mathrm{Im}\, z_1)^2]^2\}$, one point of the curve z_0 is swallowed and the resulting curve is rearranged by one element shorter. This operation yields a set containing N numbers of ξ_{t_k} for each curve.

Figure 4 shows analysis of statistics of the ensemble of the driving functions. Within the statistical errors, it converges to a Gaussian process with the linear behavior of $\langle\xi(t)^2\rangle$ and the slope $\kappa=2.1\pm0.1$.

The predicted universality class for avalanche frontiers of sandpile models with diffusivity $\kappa=2$ is consistent with the central charge of conformal field theory with $c=-2$, given by the relation $c=(8-3\kappa)(\kappa-6)/2\kappa$, which is supposed to define the ASM model [5,6].

All these evidences show another example that the theory of SLE can define (or predict) the conformal field theory which describes the system.

V. CONCLUSION

In this paper, we analyzed the statistics of avalanche frontiers that appear in the geometrical features of sandpile dynamics. Using the theory of SLE, we found numerically that the curves are conformally invariant with the same properties as LERW, with diffusivity of $\kappa=2$. This relation with LERW which has been obtained in a quite different way, with respect to the previous studies, suggests that logarithmic conformal field theory with central charge $c=-2$, defining the system is in agreement with that obtained from algebraic approach.

The avalanche front is expected to be an SLE_2 from circumstantial evidence. The ASM model has been argued to be related to $c=-2$ logarithmic conformal field theory which is turn is related to SLE with κ equal to either 2 or 8. However as $\kappa=8$ is a space filling curve, not a good candidate for the avalanche front leaving us with $\kappa=2$. A more definite reasoning, we note that the way an avalanche is formed one can define a burning algorithm: at each step, the site i topples if its height h_i is larger than the number of those of its nearest neighbors which have not toppled in the previous step. This burning algorithm leads to a tree that spans the whole area of the avalanche. Hence the avalanche front is expected to be the dual of the spanning tree thus must have $\kappa=2$. It is worth to mention that actually the waves of toppling are described by spanning trees and hence an avalanche is superposition of a few spanning trees. However if there are not many waves one expects that the frontiers have the same statistics as the waves. An improvement of this results may be obtained using wave-frontier statistics. Work in this direction is in progress.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[2] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
[3] D. Dhar, Physica A **369**, 29 (2006).
[4] S. N. Majumdar and D. Dhar, Physica A **185**, 129 (1992).
[5] S. Mahieu and P. Ruelle, Phys. Rev. E **64**, 066130 (2001).
[6] S. Moghimi-Araghi, M. A. Rajabpour, and S. Rouhani, Nucl. Phys. B **718**, 362 (2005).
[7] Y.-C. Zhang, Phys. Rev. Lett. **63**, 470 (1989).
[8] A. Diaz-Guilera, Europhys. Lett. **26**, 177 (1994).
[9] S. Lubeck, Phys. Rev. E **56**, 1590 (1997).
[10] S. Majumdar, Phys. Rev. Lett. **68**, 2329 (1992).
[11] G. F. Lawler, Duke Math. J. **53**, 249 (1986).
[12] M. Bauer, *School on Stochastic Geometry, the Stochastic Loewner Evolution, and Nonequilibrium Growth Processes*, Lecture Notes in Theoretical Physics (ICTP, Trieste, Italy, 2008).
[13] J. W. Lyklema and C. Evertsz, J. Phys. A **19**, L895 (1986).
[14] G. F. Lawler, J. Phys. A **20**, 4565 (1987).
[15] V. B. Priezzhev, J. Stat. Phys. **98**, 667 (2000).
[16] D. V. Ktitarev, S. Lubeck, P. Grassberger, and V. B. Priezzhev, Phys. Rev. E **61**, 81 (2000).
[17] O. Schramm, Isr. J. Math. **118**, 221 (2000).
[18] G. F. Lawler, O. Schramm, and W. Werner, Ann. Probab. **32**, 939 (2004).
[19] M. Jeng, Phys. Rev. E **71**, 016140 (2005).
[20] V. Gurarie, Nucl. Phys. B **410**, 535 (1993); M. A. Flohr, Int. J. Mod. Phys. A **18**, 4497 (2003); M. A. Rajabpour, S. Rouhani, and A. A. Saberi, Fortschr. Phys. **55**, 1289 (2007).
[21] M. Jeng, G. Piroux, and P. Ruelle, J. Stat. Mech. (2006) P10015.
[22] C. Tebaldi, M. De Menech, and A. L. Stella, Phys. Rev. Lett. **83**, 3952 (1999).
[23] S. S. Manna, Physica A **179**, 249 (1991); S. Lubeck and K. D. Usadel, Phys. Rev. E **55**, 4095 (1997).
[24] K. Christensen and Z. Olami, Phys. Rev. E **48**, 3361 (1993).
[25] L. Pietronero, P. Tartaglia, and Y.-C. Zhang, Physica A **173**, 22 (1991); S. N. Majumdar and D. Dhar, *ibid.* **185**, 129 (1992).
[26] S. S. Manna, J. Phys. A **24**, L363 (1991).
[27] A. Giacometti and A. Diaz-Guilera, Phys. Rev. E **58**, 247 (1998); R. Pastor-Satorras and A. Vespignani, Eur. Phys. J. B **18**, 197 (2000).
[28] K. Löwner, Math. Ann. **89**, 103 (1923).
[29] W. Kager and B. Nienhuis, J. Stat. Phys. **115**, 1149 (2004); J. Cardy, Ann. Phys. **318**, 81 (2005); M. Bauer and D. Bernard, Phys. Rep. **432**, 115 (2006).
[30] M. Bauer and D. Bernard, Commun. Math. Phys. **239**, 493 (2003); Phys. Lett. B **543**, 135 (2002); **557**, 309 (2003).
[31] D. Bernard, G. Boffetta, A. Celani, and G. Falkovich, Nat. Phys. **2**, 124 (2006).
[32] D. Bernard, G. Boffetta, A. Celani, and G. Falkovich, Phys. Rev. Lett. **98**, 024501 (2007).
[33] C. Amoruso, A. K. Hartmann, M. B. Hastings, and M. A. Moore, Phys. Rev. Lett. **97**, 267202 (2006); D. Bernard, P. Le Doussal, and A. A. Middleton, Phys. Rev. B **76**, 020403(R) (2007).
[34] J. P. Keating, J. Marklof, and I. G. Williams, Phys. Rev. Lett. **97**, 034101 (2006); E. Bogomolny, R. Dubertrand, and C. Schmit, J. Phys. A **40**, 381 (2007).
[35] A. A. Saberi, M. A. Rajabpour, and S. Rouhani, Phys. Rev. Lett. **100**, 044504 (2008).
[36] A. A. Saberi, M. D. Niry, S. M. Fazeli, M. R. Rahimi Tabar, and S. Rouhani, Phys. Rev. E **77**, 051607 (2008).
[37] T. Kennedy, Phys. Rev. Lett. **88**, 130601 (2002).
[38] S. Smirnov, European Mathematical Society **2**, 1421 (2006); e-print arXiv:0708.0039.
[39] O. Schramm, Electron. Commun. Probab. **6**, 115 (2001).
[40] J. Cardy, e-print arXiv:cond-mat/9409094.
[41] To preserve the reflection symmetry, we take the sign of x_∞ for each curve, plus or minus with probability $\frac{1}{2}$.