Linear stochastic system with delay: Energy balance and entropy production

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(Received 10 July 2008; revised manuscript received 17 December 2008; published 10 March 2009)

We study the energy balance in a linear stochastic dynamics with delay under the impact of an external periodic force. The linearity of the model, in combination with a response function method, enables us to perform detailed analytic calculations of each term in the energy balance equation. From this, we discuss thermodynamics and entropy production rate σ. With use of the delay time τ and strength of the external force A_0, σ is simply expressed as $\sigma = \sigma_{D,1}(\tau) + A_0^2 \eta(\tau)$, with both $\sigma_{D,1}(\tau)$ and $\eta(\tau)$ positive definite. We thus conclude that even when there is no external force ($A_0=0$), the entropy production rate $\sigma = \sigma_{D,1}(\tau)$ is positive, meaning that the delay force produces work, which is dissipated into a reservoir. Numerical experiments are performed to confirm theoretical results.

DOI: 10.1103/PhysRevE.79.031104 PACS number(s): 05.20.−y, 05.10.Gg, 05.70.−a, 05.40.−a

I. INTRODUCTION

Recently there has been considerable interest in some stochastic systems, whose dynamics are determined by both the present state $x(t)$ and the state $x(t-\tau)$ in the past, with $\tau\,(>0)$ denoting the delay time. Usually this delay is ascribed to a finite speed of information transmission and it is the reason why the effect of delay has been intensively studied in biological systems, i.e., as models to describe postural sway [1], visual feedback [2], and brain activity [3], to mention a few. Delay is also studied in physical systems [4] and to analyze time series [5].

A delay system can show (Hopf) instability for large τ even for a linear system [6]. A strong non-Markovian property may be roughly considered as representing infinitely many degrees of freedom and our tools to study non-Markovian processes are limited, compared with Markovian processes for which many tools, such as the Fokker-Planck equation and master equation, are fully developed and utilized [7,8].

Stochastic systems with delay are thus offering many problems, letting aside applications to biological and physical systems [1–5], interesting from a viewpoint of mathematical physics. Here we consider a stochastic system with delay from the point of view of the energy balance, which seems to have gathered little attention at the moment.

In this paper we show that the delay force or the delayed feedback, which most often is the origin of the delay force, yields dissipation. This is a rather vague statement and in the remainder of this paper we show some detailed calculations to prove this for a simple stochastic system and give its implication. We only note here that the delay force cannot be derived from a potential function and this prevents the system from reaching the equilibrium state even if the system satisfies the fluctuation-dissipation theorem [7,8].

First we introduce the model, whose dynamics is described by a linear Langevin equation with delay [5],

$$dx(t)/dt = -ax(t) - bx(t-\tau) + A_0 \cos(\omega_0 t) + f(t), \quad (1)$$

with the Gaussian random force $f(t)$ satisfying the relation

$$\langle f(t) \rangle = 0, \quad \langle f(t) f(t') \rangle = q\delta(t-t'). \quad (2)$$

Here $q/2$ may be considered as the temperature T of the system. $\tau\,(>0)$ stands for the delay time and the two constants a, b are hereafter assumed to be positive as in the other studies of Eq. (1) with $A_0=0$ [9,10].

At this point let us give several general comments on the model. The system described by Eq. (1) may be considered as the simplest system with feedback control and periodical driving force [5]. In Eq. (1), the effects of inertia are neglected in an overdamping limit under an adiabatic approximation [7,8]. Recently some works have included effects of inertia together with delay [11]. We note in advance that the energy balance developed in Sec. II is applicable also in this case, with a minor modification for internal energy E [see Eq. (6) below], to include a kinetic contribution [12].

If we replace the linear term $-ax(t)$ by a nonlinear one $ax(t)-x(t)^3$, we have a bistable system with delay feedback, which has focused considerable attention in relation to delay effects on barrier crossing [13] or dynamics in a vertical cavity surface emitting laser with optoelectronic feedback [14].

Since the above model is linear, some analytic results are known [9,10], e.g., the stationary distribution function $p_{st}(x)$ and the time correlation function $\phi(t) = \langle x(t')x(t'+t)\rangle_{st}$, with $\langle \cdots \rangle_{st}$ denoting an average in the stationary state attained when $A_0=0$ [15]. These are of considerable importance for deepening our understanding of the effects of delay in general and also for studying nonlinear delay systems since analytic results for a linear system are useful for perturbational approaches [16,17].

In view of the many studies on the thermodynamics of Markovian stochastic systems in connection with, e.g., motor protein [18,19] and fluctuation theorem [20,21], it seems important to also investigate the thermodynamics of systems with delay. The purpose of this paper is to study energy balance and entropy production rate σ for the system described by Eqs. (1) and (2).

Since the model (1) is linear, we obtain σ analytically and our result clearly shows that even when there is no external force ($A_0=0$), the entropy production rate $\sigma = \sigma_{D,1}(\tau)$ is posi-

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tive, meaning that the delay force produces work, which is dissipated into the reservoir, and that $p_{st}(x)$ (see above) is a *nonequilibrium* distribution function. This is in sharp contrast with the case $\tau=0$ or $b=0$ in Eq. (1) where $\sigma=0$ and the stationary state is an (Ornstein-Uhlenbeck) equilibrium state. In passing we note that Eq. (1) has been mainly studied in relation to a delay Fokker-Planck equation [17]. Here we use a response function method which is exact for a linear system and convenient for studying a system under an external force.

This paper is organized as follows: In Sec. II energy balance is discussed based on Eqs. (1) and (2). In Sec. III we calculate each contribution to the energy balance equation using the method of response function. We show that we can calculate the entropy production rate σ exactly for the system and that it is positive definite. In Sec. IV the theoretical production rate σ is compared with numerical experiments. The final section is devoted to the summary and conclusion.

II. ENERGY BALANCE

We now consider the energy balance of our system based on Eq. (1), which is expressed in a form more convenient to discuss energy balance and the entropy production rate, as

$$dx(t) = [-ax(t) - bx(t-\tau) + A_0 \cos(\omega_0 t)]dt + \sqrt{q}dw(t), \tag{3}$$

$$\langle dw(t)\rangle = 0, \quad \langle dw(t)dw(t')\rangle = dt\delta(t-t'). \tag{4}$$

In order to consider the energy balance of the system [12], we multiply $dx(t)$ on both sides of Eq. (1), leading to

$$ax(t)\circ dx(t) = [-dx/dt + f(t)]\circ dx(t) - bx(t-\tau)\circ dx(t) + A_0\cos(\omega_0 t)\circ dx(t), \tag{5}$$

where $\circ$ means that the multiplication is in the Stratonovich sense [8].

First we define dE, the increment of the internal energy E, by

$$dE(t) = ax(t)\circ dx(t) = d[ax(t)^2/2]. \tag{6}$$

The work dW done on the system by the external force $A_0\cos(\omega_0 t)$ is given by

$$\begin{aligned} dW(t) &= A_0\cos(\omega_0 t)\cdot dx(t) \\ &= A_0\cos(\omega_0 t)\cdot\{[-ax(t) - bx(t-\tau) \\ &\quad + A_0\cos(\omega_0 t)]dt + \sqrt{q}dw(t)\}. \end{aligned} \tag{7}$$

Since we consider that Eq. (1) is derived under the overdamped approximation [8], the heat from a reservoir dQ is given by [19,21]

$$\begin{aligned} dQ &= [f(t) - dx(t)/dt]\circ dx(t) = [ax(t) + bx(t-\tau) \\ &\quad - A_0\cos(\omega_0 t)]\circ dx(t). \end{aligned} \tag{8}$$

That is, Eq. (8) represents the work done by the random force and the frictional force.

Finally, the work $dW_D(t)$ done by the delay force $-bx(t-\tau)$ is introduced as

$$dW_D(t) = -bx(t-\tau)\circ dx(t) = -bx(t-\tau)\cdot dx(t), \tag{9}$$

where the difference between $\circ$ and the Ito integral $\cdot$ vanishes so long as τ is nonzero [8]. From the above, Eq. (5) is rewritten as

$$dE = dQ + dW + dW_D, \tag{10}$$

which may be regarded as the first law of thermodynamics with the extra work dW_D due to the delay force.

Later we will study the entropy production rate and, for convenience of discussion there, let us consider the entropy increment in small time dt. Since $-dQ$ represents the dissipation or heat given to the reservoir, the entropy increment dS_R in the reservoir may be given by

$$dS_R = -dQ/T = (dW + dW_D - dE)/T. \tag{11}$$

The entropy increment dS_S in the system in a stationary state [21,22] is given by

$$dS_S = -\ln p_{st}[x(t) + dx(t)] + \ln p_{st}[x(t)], \tag{12}$$

where $p_{st}[x(t)]$ denotes the distribution function of $x(t)$ in a stationary state. From the above we define the total entropy increment dS by

$$dS \equiv dS_S + dS_R. \tag{13}$$

III. RESPONSE FUNCTION AND ENTROPY PRODUCTION RATE

Stochastic dynamics in delay systems has been mainly studied until now theoretically via the delay Fokker-Planck equation (FPE) [17]. As shown shortly, the method of response function is powerful and we apply it to the energy balance equation (10) in a stationary state, where the average $\langle x(t)\rangle_{st}$ becomes time periodic due to an external force, $\langle x(t)\rangle_{st} = \langle x(t+\tau_0)\rangle_{st}$ with $\tau_0 = 2\pi/\omega_0$.

A. Rresponse function $\chi(t)$

We define the Laplace transform $\tilde{g}(s)$ of $g(t)$ by

$$\tilde{g}(s) \equiv \int_0^\infty dt g(t)\exp(-st). \tag{14}$$

Since we are interested in a stationary state, which may be realized after a long elapse of time starting from an arbitrary initial state, we assume

$$x(t) = 0 \quad (t < 0). \tag{15}$$

Then it is easy to have from Eq. (1) with $F_{ext}(t) = A_0\cos(\omega_0 t)$

$$\tilde{x}(s) = \tilde{\chi}(s)[\tilde{f}(s) + \tilde{F}_{ext}(s)], \tag{16}$$

with the Laplace transform of the response function $\chi(t)$ given by

$$\tilde{\chi}(s) = 1/[s + a + b\exp(-s\tau)]. \tag{17}$$

In a time space we have

$$x(t) = \int_0^t dt' \chi(t-t')[f(t') + F_{ext}(t')] \equiv \delta x(t) + \langle x(t) \rangle. \tag{18}$$

If we define $\tilde{\chi}'(\omega)$ and $\tilde{\chi}''(\omega)$ by

$$\tilde{\chi}(s = i\omega) = \tilde{\chi}'(\omega) - i\tilde{\chi}''(\omega), \tag{19}$$

it is seen that

$$\tilde{\chi}'(\omega) = [a + b\cos(\omega\tau)]/H(\omega), \tag{20}$$

$$\tilde{\chi}''(\omega) = [\omega - b\sin(\omega\tau)]/H(\omega), \tag{21}$$

with $H(\omega) \equiv \{[a+b\cos(\omega\tau)]^2 + [\omega - b\sin(\omega\tau)]^2\}$.

From Eq. (18) we have explicitly in the large time limit

$$\langle x(t) \rangle_{st} \equiv \lim_{t \gg \tau_0} \langle x(t) \rangle = A_0[\tilde{\chi}'(\omega_0)\cos(\omega_0 t) + \tilde{\chi}''(\omega_0)\sin(\omega_0 t)]. \tag{22}$$

Since we are interested in properties in the stationary state [15], we must be careful in interpreting Eq. (18), which is valid for $t > 0$. In the stationary state, Eq. (18) should be rewritten as

$$x(t) = \langle x(t) \rangle_{st} + \delta x(t), \tag{18'}$$

with $\delta x(t)$ denoting the fluctuation in the large time limit.

From the linearity of the system (1) and the Gaussian property of the noise (2), the stationary distribution function for $x(t)$ is also Gaussian with the average $\langle x(t) \rangle_{st}$, Eq. (22), and the variance $K \equiv \langle [\delta x(t)]^2 \rangle$, which is independent of the external force and is already given in Ref. [9]. Thus it reads as

$$p_{st}(\delta x) = \exp[-\delta x^2/(2K)]/\sqrt{(2\pi K)}. \tag{23}$$

The time correlation function is defined by

$$\phi(t) \equiv \lim_{t' \to \infty} \langle \delta x(t') \delta x(t' + t) \rangle. \tag{24}$$

From the definition of $\delta x(t)$ in Eq. (18′) and Eq. (2), we have

$$\phi(t) = q \int_0^\infty dt' \chi(t') \chi(t' + t). \tag{25}$$

It is straightforward to calculate $\phi(t)$ from Eq. (25) (see the Appendix).

We know there has been no explicit calculation of the time correlation function $\phi(t)$ so far and hence we pursue $\phi(t)$ a little further. We show $\phi(t)$ obtained by inverting the Fourier transform Eq. (A7) in Fig. 1 for the case $a=0.1$ and $b=0.2$ and in Fig. 2 for the case $a=0.2$ and $b=0.1$. It is noted that when τ becomes large, $\phi(t)$ behaves more oscillatory and $\phi(0)=K$ becomes larger. This is more eminent for the case $a=0.1$, $b=0.2$ than the case $a=0.2$, $b=0.1$, reflecting the Hopf instability of the linear delayed system, Eq. (1) [6,10]. For the case $b > a$ we know that the system becomes unstable when

$$\tau > \tau_c \equiv \cos^{-1}(-a/b)/\sqrt{b^2 - a^2}. \tag{26}$$

For $a=0.1$ and $b=0.2$, $\tau_c \simeq 12.1$. It is noted that $\phi(t)$ obtained from a delay FPE [10] coincides numerically with the $\phi(t)$ above within the width of the curves in Figs. 1 and 2.

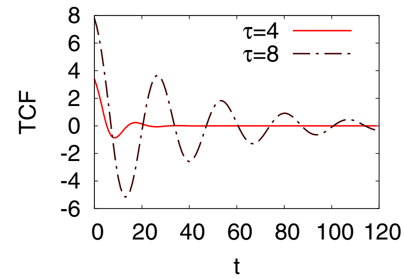

FIG. 1. (Color online) Time correlation function (TCF) $\phi(t)$ for the system Eq. (1) with $a=0.1$, $b=0.2$, and $q=1.0$. $\tau=4$ (full curve) and $\tau=8$ (dashed curve) calculated by Fourier inverting Eq. (A7).

B. Calculation of $\oint \langle dZ \rangle_{st}/\tau_0$ $(Z=E, W, W_D)$

We now calculate $\langle dZ \rangle/dt \equiv \oint \langle dZ \rangle_{st}/\tau_0$ $(Z=E, W, W_D)$ with $\oint$ standing for the integral over one period τ_0 in a stationary state. From Eq. (6) we have

$$\langle dE \rangle/dt = (a/2)[\langle x(\tau_0) \rangle_{st}^2 + \langle \delta x(\tau_0)^2 \rangle_{st} - \langle x(0) \rangle_{st}^2 - \langle \delta x(0)^2 \rangle_{st}] = 0, \tag{27}$$

where we use $x(t) = \langle x(t) \rangle_{st} + \delta x(t)$ [see Eq. (18′) and the fact that $\langle x(\tau_0) \rangle_{st} = \langle x(0) \rangle_{st}$ and $\langle \delta x(\tau_0)^2 \rangle_{st} = \langle \delta x(0)^2 \rangle_{st}$.

In Eq. (7), replacing $x(t)$ by its average plus the fluctuation, Eq. (18′), and noting the relation (2) we have

$$\langle dW \rangle/dt = [A_0^2/2H(\omega_0)]\omega_0[\omega_0 - b\sin(\omega_0\tau)] \equiv T\sigma_W. \tag{28}$$

Similarly using the relations $\langle \delta x(t) \delta x(t-\tau) \rangle_{st} = \phi(\tau)$ and $\langle \delta x(t) \delta x(t) \rangle_{st} = \phi(0)$ we obtain

$$\langle dW_D \rangle/dt = \langle dW_D \rangle_1/dt + \langle dW_D \rangle_2/dt, \tag{29}$$

where

$$\langle dW_D \rangle_1/dt = ab\phi(\tau) + b^2\phi(0) \equiv T\sigma_{D,1}(\tau), \tag{30}$$

$$\begin{aligned} \langle dW_D \rangle_2/dt = (bA_0^2/2)\{&[b + a\cos(\omega_0\tau)] \\ &\times [(\tilde{\chi}'(\omega_0))^2 + \tilde{\chi}''(\omega_0))^2] \\ &- [\tilde{\chi}'(\omega_0)\cos(\omega_0\tau) - \tilde{\chi}''(\omega_0)\sin(\omega_0\tau)]\} \\ \equiv T\sigma_{D,2}. \end{aligned} \tag{31}$$

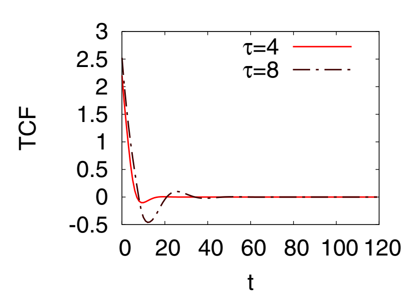

FIG. 2. (Color online) Time correlation function (TCF) $\phi(t)$ for the system Eq. (1) with $a=0.2$, $b=0.1$, and $q=1.0$. $\tau=4$ (full curve) and $\tau=8$ (dashed curve) calculated by Fourier inverting Eq. (A7).

It is noted that $T\sigma_{D,1}$ is independent of the external force and turns out to be positive (see Sec. III C). Also it is readily noticed that $T(\sigma_W+\sigma_{D,2})=[A_0^2/2H(\omega_0)]\omega_0^2$, which is positive. These quantities are related to the entropy production rate in a delay system in the next subsection.

C. Entropy production rate σ

From Eq. (11) and the above, we have

$$\langle dS_R\rangle/dt = \sigma_{D,1}(\tau) + [A_0^2/2TH(\omega_0)]\omega_0^2 \equiv \sigma_{D,1}(\tau) + A_0^2\eta(\tau). \tag{32}$$

The entropy increment of the system, Eq. (12), is easily calculated from Eq. (23) and we have

$$\begin{aligned}\langle dS_S\rangle/dt &= (1/2K\tau_0)[\langle \delta x(\tau_0)^2\rangle_{st} - \langle \delta x(0)^2\rangle_{st}] \\ &= (1/2K\tau_0)(K-K) = 0.\end{aligned} \tag{33}$$

Introducing the entropy production rate σ by $\sigma=\langle dS_R\rangle/dt+\langle dS_S\rangle/dt$, we finally arrive at our main result:

$$\sigma = \sigma_{D,1}(\tau) + A_0^2\eta(\tau). \tag{34}$$

Equation (34) shows that σ is decomposed into two parts: $\sigma_{D,1}(\tau)$, the entropy production rate of a system without the external force, and $A_0^2\eta(\tau)$ (>0), representing the contribution to σ from the external force. As we prove below, $\sigma_{D,1}(\tau)>0$ and Eq. (34) may be considered as the second law of thermodynamics.

From Eq. (30) and the relation $a\phi(0)+b\phi(\tau)=q/2$ [10], we have

$$\sigma_{D,1}(\tau) = [ab\phi(\tau) + b^2\phi(0)]/T = [(b^2-a^2)K + aT]/T. \tag{35}$$

If $b>a>0$, then $\sigma_{D,1}(\tau)>0$. On the other hand, if $a>b>0$, we know that [10]

$$K = T[1 + (b/\omega_1)\sinh(\omega_1\tau)]/[a + b\cosh(\omega_1\tau)], \tag{36}$$

where

$$\omega_1 \equiv \sqrt{a^2-b^2}. \tag{37}$$

With the use of these relations we see that

$$\begin{aligned}\sigma_{D,1}(\tau) = b[&b + a\cosh(\omega_1\tau) \\ &- \omega_1\sinh(\omega_1\tau)]/[a + b\cosh(\omega_1\tau)].\end{aligned} \tag{38}$$

The denominator of Eq. (38) is positive and it is readily seen that

$$\begin{aligned}[b + a\cosh(\omega_1\tau)]^2 = &\omega_1^2\sinh^2(\omega_1\tau) \\ &+ b^2\cosh^2(\omega_1\tau) + 2ab\cosh(\omega_1\tau) \\ &+ a^2 > \omega_1^2\sinh^2(\omega_1\tau).\end{aligned} \tag{39}$$

Thus we can conclude that the entropy production rate $\sigma_{D,1}(\tau)$ is positive for $a>0$ and $b>0$.

A final comment on Eq. (34) is on its relation with the linear response theory. When $b=0$, we have

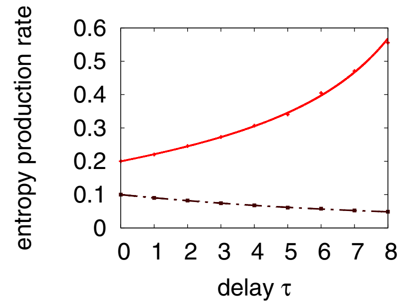

FIG. 3. (Color online) Entropy production rate $\sigma_{D,1}(\tau)$ for the system Eq. (1) with $a=0.1$, $b=0.2$ (full curve) and $a=0.2$, $b=0.1$ (dashed curve) as a function of τ.

$$\sigma = (A_0^2/2T)\omega_0^2/(a^2 + \omega_0^2), \tag{40}$$

which is expressed in a form $\sigma=(\omega_0/2)A_0^2\chi''(\omega_0)$ for a system without delay usually obtained under a linear response approximation [23,24].

IV. NUMERICAL RESULTS AND SIMULATIONS

In Fig. 3, we plot $\sigma_{D,1}(\tau)$, Eq. (30), as a function of τ for $a=0.1$, $b=0.2$ (full curve) and for $a=0.2$, $b=0.1$ (dotted curve). Reflecting the instability, Eq. (26), we observe large $\sigma_{D,1}(\tau)$ for large τ in the case $a=0.1$, $b=0.2$. When $\tau=0$ we should have $\sigma=0$ because the system is in the thermodynamic (canonical) equilibrium state. The apparent discrepancy present in Fig. 3 stems from the fact that we should take the stochastic integral in Eq. (9) to be the Storatonovich type [8] when τ is exactly zero. In the case $\tau=0$ we have

$$\begin{aligned}\sigma_{D,1}(\tau=0) &= \lim_{z\to\infty} - (b/T)\int_0^z d[x(t)^2/2]/z \\ &= \lim_{z\to\infty}(-b/2z)[x(z)^2 - x(0)^2] = 0,\end{aligned} \tag{41}$$

as it should be. In our numerical calculation in Fig. 3 we employed the Ito interpretation which is valid so long as $\tau>0$. In this case simple calculations show in the limit of $\tau\to 0$ that

$$\begin{aligned}\sigma(\tau=0) &= \lim_{z\to\infty} - b\int_0^z (d[x(t)^2] - \{dx(t)\}^2)/(2zT) \\ &= \lim_{z\to\infty}(-b/2zT)[x(z)^2 - x(0)^2 - 2Tz] \to b,\end{aligned} \tag{42}$$

which is in accord with the results in Fig. 3.

In our numerical experiments we solved the Langevin equation (1) ($q=2T=1$) for long time and calculated σ by time averaging, e.g., $\sigma_W \simeq (1/T)\int_0^L dW/L (L \gg \tau_0)$. Whenever the external force is introduced, we take $A_0=1$, and $\omega_0=0.05$. Experimental results are shown by squares in Fig. 3 and these agree well with the theoretical results, Eq. (30).

In Figs. 4 and 5 we plot σ_W, $A_0^2\eta(\tau)=\sigma_W+\sigma_{D,2}$, and σ for $a=0.1$, $b=0.2$, and $a=0.2$, $b=0.1$, respectively, as a function of τ. Our theory is observed to well reproduce the experimental results. As shown here, entropy production σ_W due to the external force can take negative values. However, as

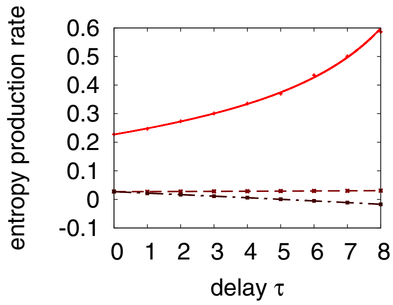

FIG. 4. (Color online) Entropy production rate σ (full curve), $A_0^2\eta(\tau)=\sigma_W+\sigma_{D,2}$ (dotted curve), and σ_W (dashed curve) as a function of τ for $a=0.1$, $b=0.2$. Results from numerical experiments are shown only by squares.

shown in Eq. (32) the sum $A_0^2\eta(\tau)=\sigma_W+\sigma_{D,2}$ becomes positive.

For the system with $a=0.1$ and $b=0.2$ we plot in Fig. 4 σ only in the small delay region $\tau<8$ because of the instability, Eq. (26), at $\tau=\tau_c=12.1$. For the system with $a=0.2$ and $b=0.1$ we plot σ up to $\tau=200$ and we observe approximately periodic behavior of σ in Fig. 5, which results from the periodic (in τ) structure of $A_0^2\eta(\tau)$ with the period $\tau_0=125.6$ [see Eq. (32) and $H(\omega_0)$ given just below Eq. (21)].

V. SUMMARY AND CONCLUSION

In this paper we studied the stochastic linear delay system (1), focusing on its energy balance and thermodynamic properties. The first law was expressed as Eq. (10) with an extra term dW_D, Eq. (9), which denotes the work done by the delay force. In contrast to Markovian systems, the entropy production due to the external force σ_W, Eq. (28), itself is not necessarily positive. However, by including part of the contributions of the delay force, $\sigma_{D,2}$, which depends on an external force, the sum $A_0^2\eta(\tau)=\sigma_W+\sigma_{D,2}$, see Eq. (32), becomes positive.

The other part $\sigma_{D,1}(\tau)$, Eq. (30) or Eq. (35), which gives the entropy production rate for a system without the external force $A_0=0$, turned out to be positive. From this we interpret that the stationary state, attained for the system with $A_0=0$, is not an equilibrium state. This is in sharp contrast with the Markovian case, $b=0$ in Eq. (1). For this Ornstein-Uhlenbeck system, the entropy production rate is zero and the stationary distribution, Eq. (23) with $K=T/a$, actually represents an equilibrium (or canonical) distribution function. Thus we may consider that the delay force $-bx(t-\tau)$, although it may be regarded as an internal force, does work to be dissipated to a reservoir. It is noted that this dissipation is expressed in terms of the time correlation function $\phi(t)$ for the nonequilibrium stationary state.

For the system described by Eq. (1), $\phi(t)$ plays two important roles. One is a relationship usually observed for equilibrium statistical mechanics that the variance K of the stationary distribution function (23) is determined by the $t=0$ value of $\phi(t)$. The other one is rather special to a delay system, where the entropy production rate $\sigma_{D,1}(\tau)$ depends on $\phi(\tau)$ also. This dependence is reasonable in view of the energy balance equation (5), in which we have a term of the form $bx(t-\tau)\circ dx(t)$. The average of this term might naturally depend on $\phi(\tau)$. For nonlinear systems also there are cases where $\phi(\tau)$ plays an important role to characterize system properties. For example, in noise induced dynamics in some bistable systems with delay [13], the sign of $\phi(\tau)$ reflects the sign of strength of delay feedback with the delay time τ.

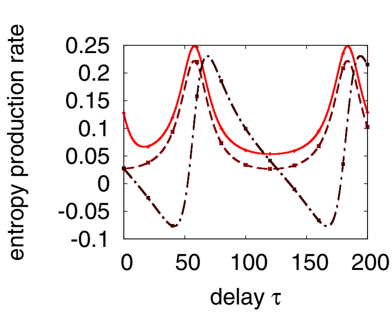

FIG. 5. (Color online) Entropy production rate σ (full curve), $\sigma_W+\sigma_{D,2}$ (dotted curve), and σ_W (dashed curve) as a function of τ for $a=0.2$, $b=0.1$. Results from numerical experiments are shown by squares.

APPENDIX

With use of inverse Laplace transformation, we have

$$\chi(t)=(1/2\pi)\int_{-\infty}^{\infty}d\omega\tilde{\chi}(s=i\omega)\exp(i\omega t). \tag{A1}$$

From Eq. (24) we have

$$\phi(t)=iq/(2\pi)^2\int d\omega_1\tilde{\chi}(s=i\omega_1)\int d\omega_2\tilde{\chi}(s=i\omega_2)\times\exp(i\omega_2 t)/[\omega_1+\omega_2+i\epsilon] \tag{A2}$$

with $\epsilon>0$. From the well-known formula $\lim_{\epsilon\to 0+}1/(x+i\epsilon)=P(1/x)-i\pi\delta(x)$, with P denoting the Cauchy pripinal value, we have

$$\phi(t)=\phi_1(t)+\phi_2(t), \tag{A3}$$

with $\phi_1(t)$ given by

$$\phi_1(t)=iq/(2\pi)^2P\int d\omega_1\tilde{\chi}(s=i\omega_1)\times\int d\omega_2\tilde{\chi}(s=i\omega_2)\exp(i\omega_2 t)/[\omega_1+\omega_2], \tag{A4}$$

and $\phi_2(t)$ given by

$$\phi_2(t)=q\pi/(2\pi)^2\int d\omega\{[\tilde{\chi}'(\omega)]^2+[\tilde{\chi}''(\omega)]^2\}\exp(i\omega t). \tag{A5}$$

As for $\phi_1(t)$ we apply the causality principle [23] $\tilde{\chi}(s=i\omega)=(i/\pi)P\int d\omega'\tilde{\chi}(s=i\omega')/(\omega'-\omega)$ to find that $\phi_1(t)=\phi_2(t)$. Thus we finally have

$$\phi(t) = (q/2\pi) \int d\omega\{[\tilde{\chi}'(\omega)]^2 + [\tilde{\chi}''(\omega)]^2\}\exp(i\omega t). \tag{A6}$$

If we define the Fourier transform (spectral function) of $\phi(t)$ by $\tilde{\phi}(\omega) \equiv (2/\pi)\int_0^\infty \exp(i\omega t)\phi(t)$, we readily obtain from Eqs. (20) and (21) that

$$\tilde{\phi}(\omega) = (q/\pi)/\{[a + b\cos(\omega\tau)]^2 + [\omega - b\sin(\omega\tau)]^2\}, \tag{A7}$$

which is identical to the one given by a delay Fokker-Planck equation for $A_0=0$ [10].

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