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Fluctuation properties of an effective nonlinear system subject to Poisson noise

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We study the work fluctuations of a particle, confined to a moving harmonic potential, under the influence of friction and external asymmetric Poissonian shot noise. This type of noise generalizes the usual Gaussian noise and induces an effective interaction between the noise and the potential, leading to an effectively nonlinear system with singular features. On the basis of an analytic solution we investigate the roles of time scales, symmetries, and singularities in the context of nonequilibrium fluctuations. Our results highlight the nonuniversality of the steady-state fluctuation theorem in stochastic systems.

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Poissonian shot noise (PSN) is a natural description of fluctuations in nature [1,2] and is ubiquitous in physics and, e.g., electric engineering. Importantly, PSN generalizes the usual Gaussian noise, to which it converges in a certain limit, and is the paradigm of *external noise*: the dissipation and the noise originate from physically independent mechanisms [2]. This implies that the fluctuation-dissipation theorem does not have to be satisfied for PSN, contrary to thermal Gaussian noise. In this Rapid Communication, we present a theory of the effect of PSN on the work fluctuations in a paradigmatic nonequilibrium model, namely, a particle with friction confined to a moving harmonic potential. This system allows us to investigate a number of important physical concepts in the context of nonequilibrium fluctuations. (i) *Time scales*: PSN introduces characteristic times in the system that crucially influence its fluctuation behavior. (ii) *Symmetry* of the noise: PSN is usually asymmetric (one-sided). (iii) *Singularities*: the asymmetry of the noise gives rise to an effective interaction between the noise and the potential, leading to an effectively nonlinear system with singular features. Nevertheless, the work distribution can be calculated in analytical form in our model and is completely characterized by its time scales. The interplay of these time scales leads to certain striking transitions in the behavior of the work fluctuations.

Furthermore, we discuss the validity of the steady-state fluctuation theorem (SSFT) for the work fluctuations in our model. In a nonequilibrium steady state (NESS), the SSFT states that the probability distribution $\Pi_\tau(p)$ of finding a particular value of the dimensionless work p over time τ satisfies a certain symmetry relation of the form (cf. [3–6])

$$\frac{\Pi_\tau(p)}{\Pi_\tau(-p)} \cong e^{c\tau p}, \tag{1}$$

where $\cong$ indicates the asymptotic behavior for large τ, and c is a constant. For deterministic systems in a compact phase space, the SSFT for the work fluctuations is expected to hold universally under the chaotic hypothesis [4]. However, stochastic systems do not exhibit the same generality in their fluctuation behavior. In the nonequilibrium particle model mentioned above, the SSFT for the work fluctuations has been shown to hold when the noise has Gaussian statistics [7]. When the noise is given by Lévy noise [8] or, as we will show here, by PSN, the SSFT is violated [9].

We consider a system consisting of a particle under the influence of a time-dependent harmonic force as well as friction and external noise. In the overdamped regime [10], the equation of motion for the position $x(t)$ of the particle in the laboratory frame reads

$$\alpha\dot{x}(t) = -\kappa[x(t) - vt] + \xi(t). \tag{2}$$

Here, the force $-\kappa[x(t)-vt]$ stems from a parabolic potential $U(x,t)=\kappa(x-vt)^2/2$ which moves with constant velocity v, where either $v>0$ or $v<0$. The parameter κ denotes the strength of the potential, α the friction coefficient, and $\xi(t)$ stochastic noise from the environment. For the noise, we consider

$$\xi(t) = z(t) - \lambda\Gamma_0, \tag{3}$$

where $z(t)$ is white PSN of the form [1]

$$z(t) = \sum_{k=1}^{n_t} \Gamma_k \delta(t - t_k), \tag{4}$$

which has the mean value $\langle z(t)\rangle=\lambda\Gamma_0$, so that $\xi(t)$ of Eq. (3) has zero mean for convenience. In Eq. (4), n_t denotes the number of pulses in time t, which is determined by the Poisson counting process $P(n_t)=(\lambda t)^n e^{-\lambda t}/n!$. The parameter λ is the average number of kicks per unit time (rate of kicks) so that there are λt kicks occurring in the time interval $[0,t]$. When a kick occurs, its amplitude Γ_k is sampled randomly from a distribution $p(\Gamma)$. In the following, we focus on the usual case of *one-sided* shot noise so that $\Gamma_k>0$ for all k, and we assume an exponential distribution of amplitudes $p(\Gamma_k)=\Gamma_0^{-1}e^{-\Gamma_k/\Gamma_0}$. The PSN Eq. (4) with exponentially distributed Γ_k has the characteristic functional [1]

$$G_z[g(t)] = \exp\left\{\lambda\int_0^\infty \left(\frac{1}{1 - i\Gamma_0 g(t)} - 1\right)\mathrm{d}t\right\}, \tag{5}$$

leading to δ-correlated cumulants with mean $\langle z(t)\rangle=\lambda\Gamma_0$ [2].

Changing to coordinates in a comoving frame, $y(t)\equiv x(t)-vt$, the equation of motion (2) reads

$$\dot{y}(t) = -\frac{1}{\tau_r}y(t) - v_e + \frac{1}{\alpha}z(t), \tag{6}$$

where we have defined the effective drift velocity $v_e\equiv v+\lambda\Gamma_0/\alpha$. If we move the potential for a time period τ, a certain amount of work is done on the particle, namely,

$$W_\tau = -\kappa v \int_0^\tau y(t)\mathrm{d}t. \tag{7}$$

In an experimental setup, where the harmonic potential can be induced, e.g., via lasers [11], Eq. (7) is the mechanical work needed in order to overcome the particle friction. Importantly, if the potential is stationary (i.e., $v=0$), no work is performed. This means that we ignore the work due to the purely stochastic motion of the particle, which contributes to the heat in the case of thermal noise [7,12].

In Eq. (6), $\tau_r \equiv \alpha/\kappa$ denotes the characteristic relaxation time of the particle in the harmonic potential. It is crucial that PSN gives rise to two additional time scales in our model, namely, the characteristic time scale of the fluctuations $\tau_\lambda \equiv \lambda^{-1}$, which is the mean waiting time between two successive pulses, and the time scale

$$\tau_p \equiv \frac{\Gamma_0}{\alpha|v|}, \tag{8}$$

which relates the mean amplitude of fluctuations and the dissipation due to the driving. The three time scales τ_r, τ_λ, and τ_p completely determine the properties of the work fluctuations.

From the Langevin equation (6) we can infer two important properties of our model. First, upon averaging of Eq. (6) we obtain the mean position in the NESS $\langle y(t)\rangle = -v\tau_r$, and consequently from Eq. (7) the mean work $\langle W_\tau\rangle = \alpha v^2\tau$, which is always positive, in agreement with the second law. Second, we find that there exists a *minimal value* y^* of the position coordinate. This can be seen if we solve Eq. (6) without the stochastic term $z(t)$, which yields $y^* \equiv -v_e\tau_r$ in the NESS. The important observation is that $z(t)$ only provides kicks in the positive direction [cf. $\Gamma_k > 0$ in Eq. (4)] so that y^* is a singular *cutoff*, i.e., no positions $y(t) < y^*$ can be reached.

Then Eq. (7) yields from the position cutoff y^* a work cutoff W_τ^*, i.e., an *extremal value of the work*,

$$W_\tau^* = -\kappa v y^* \tau = \langle W_\tau\rangle[1+\sigma(v)\tau_p/\tau_\lambda], \tag{9}$$

where $\sigma(v)$ denotes the sign function. It is important to note that the spatial asymmetry of the noise induces a qualitatively different behavior of the work fluctuations depending on the sign of v. In the case $v>0$, the value of W_τ^* is always positive and corresponds to the maximum work done on the system in time τ. For $v<0$, on the other hand, W_τ^* is the minimum work value. In that case, W_τ^* can be either positive or negative depending on the ratio of the two time scales τ_λ and τ_p [cf. Eq. (9)]. In the case $\tau_\lambda > \tau_p$, W_τ^* is positive and no negative work fluctuations can occur. Furthermore, $W_\tau^*=0$ if $\tau_p = \tau_\lambda$ and $W_\tau^* < 0$ if $\tau_\lambda < \tau_p$.

The position cutoff y^* can be interpreted as an infinite barrier in the potential so that the noise induces an *effective nonlinearity* in the potential (see Fig. 1) [13]. This means that our model exhibits an effective interaction between the noise and the potential. Furthermore, due to these cutoffs, the distributions of position and work are generally non-

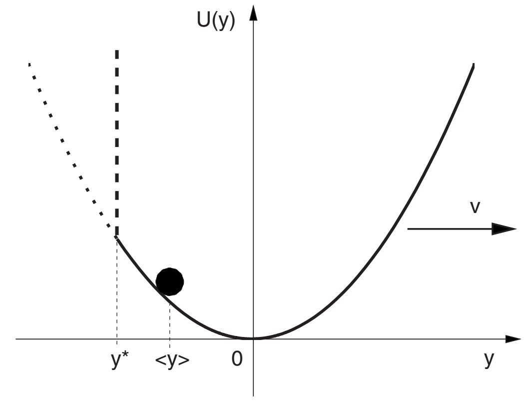

FIG. 1. The harmonic potential (full and dotted lines) changes due to the PSN into an effective potential (full and dashed lines). The dashed line, starting at the cutoff point y^*, represents physically an effective infinite barrier for the particle. The black dot indicates the particle at its mean position $\langle y\rangle$ for $v>0$.

Gaussian unless one considers the Gaussian limit of the PSN [1],

$$\lambda \to \infty, \quad \Gamma_0 \to 0, \tag{10}$$

with $\lambda\Gamma_0^2$=const. In addition, if this noise is thermal, i.e., it originates from an equilibrium heat bath, one requires $\lambda\Gamma_0^2 = \alpha/\beta$ to satisfy the fluctuation-dissipation theorem, where β is interpreted as inverse temperature.

In order to determine the work distribution for arbitrary λ and Γ_0 values, we first derive an exact expression for the characteristic function of the work. To this end, we note that if the characteristic functional of $y(t)$, defined as

$$G_y[h(t)] \equiv \left\langle \exp\left\{ i\int_0^\infty h(t)y(t)\mathrm{d}t \right\}\right\rangle, \tag{11}$$

is known, we obtain the characteristic function of work $G_{W_\tau}(q)$ by choosing the particular test function $\bar{h}(t) = -q\kappa v\Theta(\tau - t)$ in Eq. (11) [8]. This follows immediately from the definition of work Eq. (7),

$$G_{W_\tau}(q) \equiv \langle e^{iqW_\tau}\rangle = \left\langle \exp\left\{ -iq\kappa v\int_0^\tau y(t)\mathrm{d}t \right\}\right\rangle = G_y[\bar{h}(t)]. \tag{12}$$

In turn, the characteristic functional G_y is related to the noise functional G_z, Eq. (5), via the equation of motion (6) and is formally given by [8,14]

$$G_y[h(t)] = e^{iy_0k_0 - iv_e\int_0^\infty k(t)\mathrm{d}t} G_z[k(t)/\alpha]. \tag{13}$$

Here, the function $k(t)$ is defined as [14]

$$k(t) \equiv \int_t^\infty h(s)e^{(t-s)/\tau_r}\mathrm{d}s, \tag{14}$$

and $k_0 \equiv k(t=0)$ as well as $y_0 \equiv y(t=0)$. Since we are interested in the properties in the NESS, we have to sample the initial condition y_0 from the NESS distribution of positions. This distribution can be found by solving the Master equation associated with the Langevin equation (6) (cf. [2]) and reads [15]

$$p(y) = \frac{1}{\Gamma(\lambda\tau_r)}\frac{\alpha}{\Gamma_0}\left(\frac{\alpha}{\Gamma_0}(y-y^*)\right)^{\lambda\tau_r-1} e^{-(y-y^*)\alpha/\Gamma_0}. \quad (15)$$

Using Eqs. (13)–(15), we obtain the characteristic function of work by substituting the functions $k(t)$ and k_0, calculated with $\bar{h}(t)$, into Eq. (13) and performing the average over y_0 with respect to the distribution Eq. (15). After some algebra, one obtains the characteristic work function

$$G_{W_\tau}(q) = [1 + iq\Gamma_0 v(1-e^{-\tau/\tau_r})]^{\lambda\tau_r\{[1/(1+iq\Gamma_0 v)]-1\}} \times \exp\left\{iqW_\tau^* + \lambda\tau\left(\frac{1}{1+iq\Gamma_0 v}-1\right)\right\}, \quad (16)$$

with a pole at $q=i/(\Gamma_0 v)$.

For a discussion of the properties of work fluctuations, we introduce the scaled dimensionless work value p, defined by $p \equiv W_\tau/\langle W_\tau\rangle$. The distribution of p follows from the inverse Fourier transform of G_{W_τ},

$$\Pi_\tau(p) = \frac{\alpha v^2\tau}{2\pi}\int_{-\infty}^{\infty} G_{W_\tau}(q)e^{-iqp\alpha v^2\tau}\mathrm{d}q. \quad (17)$$

To the best of our knowledge, there is no exact result for the inverse Fourier transform of Eq. (17). However, for large τ the integral will be dominated by its saddle point and can be analytically obtained using the method of steepest descent [16]. Neglecting terms of order $\tau^{-1/2}$ yields then straightforwardly the analytic form of the distribution $\Pi_\tau(p)$,

$$\Pi_\tau(p) \cong \frac{1}{\sqrt{4\pi}}\frac{\sqrt{\tau/\tau_\lambda}}{|p^*-1|} \times\left(\sqrt{\frac{p^*-p}{p^*-1}}\right)^{-(\tau_r/\tau_\lambda)\{\sqrt{[(p^*-p)/(p^*-1)]}-1\}-(3/2)} \times\exp\left\{-\frac{\tau}{\tau_\lambda}\left(\sqrt{\frac{p^*-p}{p^*-1}}-1\right)^2\right\}, \quad (18)$$

where p^* denotes the rescaled extremal value of work [cf. Eq. (9)],

$$p^* \equiv \frac{W_\tau^*}{\langle W_\tau\rangle} = 1 + \sigma(v)\frac{\tau_p}{\tau_\lambda}. \quad (19)$$

Equation (18) shows that the distribution $\Pi_\tau(p)$ is completely specified by the times τ_r, τ_λ, τ_p, and τ. Furthermore, $\Pi_\tau(p)$ is always real, since both p^*-p and p^*-1 are either positive ($v>0$) or negative ($v<0$). For very large τ, $\Pi_\tau(p)$ exhibits the large deviation form

$$\Pi_\tau(p) \cong e^{-\tau I(p)} \quad (20)$$

with rate function

$$I(p) \equiv \frac{1}{\tau_\lambda}\left(\sqrt{\frac{p^*-p}{p^*-1}}-1\right)^2. \quad (21)$$

One notices that two different singularities appear in Eq. (18). First, the derivative of $\Pi_\tau(p)$ diverges for $p\to p^*$ as $\Pi'(p)\propto|p^*-p|^{-1}$ in leading order. This means that the approach of $\Pi_\tau(p)$ to the cutoff has a vertical slope. Second, one notices that $\Pi_\tau(p)$ itself diverges for $p\to p^*$ if $\tau_r/\tau_\lambda<3/2$. This divergence for small τ_r/τ_λ at $p=p^*$ can be understood physically by considering the nature of the work and position cutoffs in our model. If τ_r/τ_λ is sufficiently small, the system relaxes "too quickly" in between the stochastic kicks and thus spends most of its time at the position that it would assume without noise, i.e., at y^*. Consequently, the particle will predominantly acquire work W_τ^* over time τ leading to a divergence in the work distribution at $p=p^*$.

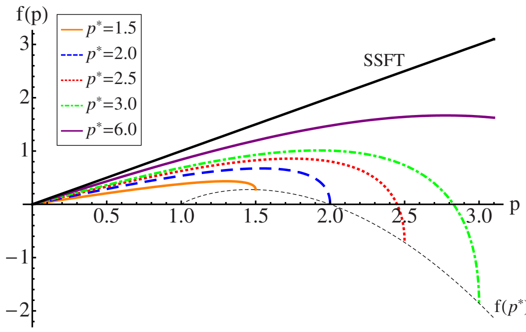

FIG. 2. (Color online) The fluctuation function $f(p)$ of Eq. (23) for various p^* values and $v>0$. The approach to $f(p^*)$ is with a vertical slope, while $f(p^*)$ itself remains finite (thin dashed line). For $p^*>2$, $f(p)$ becomes negative in the interval $p\in[p_+,p^*]$. We observe a pronounced linear regime for small p values, which is a general consequence of the large deviation form of $\Pi_\tau(p)$ [18]. In the Gaussian limit, where $p^*\to\infty$, the SSFT is recovered.

In the asymptotic regime $\tau\to\infty$, the work distribution Eq. (18) is dominated by the large deviation form Eq. (20). In order to further discuss the fluctuation properties of the work, we consider the fluctuation function

$$f_\tau(p) \equiv \frac{1}{a\langle W_\tau\rangle}\ln\frac{\Pi_\tau(p)}{\Pi_\tau(-p)}. \quad (22)$$

Here, the constant a is defined as $a\equiv\alpha/(\lambda\Gamma_0^2)$, which becomes $a=\beta$ in the limit where PSN converges to thermal Gaussian noise [cf. Eq. (10)]. From Eq. (20), we obtain

$$f(p) = 2(p^*-1)p + 2(p^*-1)^2\left(\sqrt{\frac{p^*-p}{p^*-1}}-\sqrt{\frac{p^*+p}{p^*-1}}\right), \quad (23)$$

defined on the interval $[-p^*,p^*]$ [17]. In the Gaussian limit, Eq. (10), $p^*\to\infty$ [cf. Eq. (19)] and $f(p)$ indeed yields the SSFT, as can be seen by expanding the square roots in Eq. (23) in powers of p. Therefore, for finite p^* the SSFT Eq. (1) is violated. We now characterize the behavior of $f(p)$ for $v>0$ and $v<0$ separately.

(i) For $v>0$, we find that Eq. (23) has the zeros $p_0=0$ and $p_\pm=\pm 2\sqrt{p^*-1}$ [due to the antisymmetry of $f_\tau(p)$ with respect to p, we neglect the negative root p_-]. The crucial observation is now that $f(p)$ from Eq. (23) becomes negative in $[p_+,p^*]$ for the critical value $p^*>2$ (see Fig. 2). This means that in this regime, negative fluctuations have a higher probability than positive ones. From Eq. (19), we find that $p^* > 2$ if $\tau_p > \tau_\lambda$.

(ii) For $v < 0$, we have to distinguish two regimes. First, for $\tau_\lambda > \tau_p$ no negative work fluctuations can occur [cf. Eq. (9)] and therefore $f(p)$ is not defined in this parameter regime. Second, for $\tau_\lambda < \tau_p$ we have $p^* < 0$. In this case, $f(p)$ is only zero at $p=0$ and always positive for $p > 0$.

The fluctuation function $f(p)$, therefore, exhibits striking properties in two different parameter regimes. First, if $v < 0$ and $\tau_\lambda > \tau_p$, there is no fluctuation relation *at all*. Second, if $v > 0$ and $\tau_p > \tau_\lambda$, negative fluctuations of a certain magnitude are *more likely to occur* than corresponding positive ones. This negative regime of $f(p)$ originates from the strongly asymmetric tails of the work distribution $\Pi_\tau(p)$: the negative tail decays exponentially while the positive tail is bounded by the cutoff at p^*. Despite the existence of a negative regime of the fluctuation function $f(p)$, the second law is never violated: the mean value of the work is always positive, $\langle W_\tau \rangle = \alpha v^2 \tau$.

PSN occurs quite naturally in electric circuits, where the discreteness of the electron charge causes time-dependent fluctuations of the electric current. Our theory could thus be tested in an experiment similar to the resistance-capacitor dipole of [19], if the Brownian Nyquist noise can be sufficiently suppressed. We note that our theory could also be adapted to an experiment similar to that of Mahadevan *et al.* [20], where a lubricated rod of a hydrogel sliding on a soft vibrating substrate is considered as a model for biomimetic ratcheting motion. Instead of the purely oscillatory vibrations of [20], one could induce asymmetric PSN, which could lead to work fluctuations with features similar to those presented in this Rapid Communication.

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