# Multicomponent cnoidal waves in cascade quasisynchronous frequency conversion

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It is shown that cascade quasisynchronous frequency conversion due to quadratic nonlinearity can be described in terms of an effective cubic nonlinearity. This enables one to reduce a four-mode interaction problem to solving a system of two coupled nonlinear Schrödinger equations for the amplitudes of the waves participating in both nonlinear processes. Exact analytic solutions of the corresponding system are found in the form of multicomponent cnoidal waves with components expressed through a sum and a difference of two similar fundamental solutions of the Lamé equation with shifted arguments. It is shown that solutions obtained in such a way enable one to optimize the conversion efficiency because of full coverage of the range of possible boundary conditions.

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## I. INTRODUCTION

Cnoidal waves (CWs) are self-consistent periodic solutions of many nonlinear differential equations of second and higher orders—such as the nonlinear Schrödinger equation (NLSE), the Korteweg–de Vries equation (KdV), the sine-Gordon equation, and others [1–7]—and are essentially the modes of corresponding nonlinear problems. When a CW consists of some self-consistent components it is generally referred to as the multicomponent CW (MCW). The MCW notion is widely used in nonlinear hydrodynamics [1,8] and plasma physics [2,9], in description of packets of electronic wave functions (excitons, biexcitons, superconducting pairs, and others), in physics of one-dimensional (1D) chains (conjugate polymers) [10] and two-dimensional (2D) surfaces (ferromagnetics and high-temperature superconductors) [11]. In optics, the MCW concept is also rather universal because taking into account the lowest-order terms in the expansion of nonlinear polarization usually results in equations of such type. Here, MCWs are solutions of 1D problems of nondispersive propagation of pulse trains in optical fibers [3–6,12] and of parametric generation with synchronous pumping [13], of 2D problems of nondiffractive propagation of wave fronts with special periodic transverse structure through photorefractive crystals [7,14], and crystals with quadratic nonlinearity [15].

It was recently shown [16] that MCWs play a crucial role in a classical problem of nonlinear optics—in the description of parametric frequency conversion in media with quadratic nonlinearity [17]. It was proved that an exact analytic solution of a problem of steady-state interaction of three modes with frequencies $\omega_{1-3}$ can be found in a nonstandard way—by increasing the order of the corresponding system of truncated (first-order) nonlinear equations. In doing so, the problem is reduced to three NLSEs coupled to each other only through the boundary conditions. The possibility of such a reduction was interpreted as describing the result of competition of two quadratic nonlinear processes (composition $\omega_1+\omega_2\rightarrow\omega_3$ and decomposition $\omega_3\rightarrow\omega_1+\omega_2$ of quanta) by an effective cascade cubic nonlinearity of Kerr type [18].

Below, through the use of the approach of Ref. [16], we show that, when the wave mismatch is neglected (quasi-phase-matching conditions), four-mode cascade frequency conversion due to quadratic nonlinearity can also be described in terms of such effective cubic nonlinearity. After that, the initial problem is reduced to solving a standard system of two coupled NLSEs for the complex amplitudes of the waves participating in both nonlinear processes [14,19]. We show that the obtained system can be transformed to two identical independent equations, so defining its solutions as the sum and difference of two identical solutions of the same NLSE with shifted arguments. Exact analytic solutions obtained in such a way enable one to optimize the conversion efficiency through full coverage of the range of possible boundary conditions.

We present our paper in the following way. First (Sec. II), we show that, in a four-coupled-mode approximation without wave mismatch, a steady-state cascade frequency conversion problem can be reduced to solving a standard system of two coupled NLSEs governing the complex amplitudes of two waves participating in the two nonlinear processes. Then (Sec. III), we formulate a quasi-phase-matched problem for a periodically poled nonlinear crystal and, after averaging the obtained equations, reduce this problem to a system of two coupled NLSEs. In Sec. IV, we solve the obtained system by separating it into two uncoupled equations and writing down exact analytic solutions for all possible relationships between the coupling constants of the two nonlinear processes. Specific features and nontrivial limits of these solutions are discussed in Sec. V, where we illustrate their peculiarities by changing the coupling constants and boundary conditions (intensities of interacting modes in the input plane). Finally (Sec. VI), we make some conclusions.

## II. CASCADE FREQUENCY CONVERSION AND EFFECTIVE CUBIC NONLINEARITY

Let us consider the propagation of four (subscripts $i=1,\ldots,4$) plane colinear monochromatic waves (modes) at frequencies $\omega_1$, $\omega_2=\omega_1$, $\omega_3=\omega_1+\omega_2=2\omega_1$, and $\omega_4=\omega_1+\omega_3=3\omega_1$ with wave vectors $\mathbf{k}_{1-4}$ and complex amplitudes $A_{1-4}$ in a medium with quadratic nonlinearity. We will suppose that interaction of the waves follows from two nonlinear processes $\omega_1+\omega_{2,3}\rightarrow\omega_{3,4}$ with wave mismatches $\mathbf{\Delta k}_{1,2}=\mathbf{k}_1$

 

$+\mathbf{k}_{2,3}-\mathbf{k}_{3,4}$ correspondingly. Such a model can be used in the case when (i) two modes (subscripts 1 and 2) at the same frequencies $\omega_2=\omega_1$ are of different (orthogonal to each other) polarizations ($\omega_3=\omega_1+\omega_2=2\omega_1$ is a sum-frequency generation of so-called type-II [17]) and (ii) efficiency of generation of all other waves (the modes with $i\neq 1,\ldots,4$) can be neglected. The first suggestion (i) is not of fundamental importance and is used here to simplify all further expressions. The second one (ii) is much more important. This suggestion determines the possibility of highly efficient cascade up- and down-conversion and can be realized by so-called quasi-phase-matching techniques (see next section). Supposing the nonlinearity is of nonresonant character and directing the $z$ axis along $\mathbf{k}_{1-4}$, we write a system of equations, describing the mode interaction, in the form

$$\frac{dA_1}{dz}=-i\beta_1A_2^*A_3\exp(-i\Delta k_1z)-i\beta_2A_3^*A_4\exp(-i\Delta k_2z), \tag{1a}$$

$$\frac{dA_2}{dz}=-i\beta_1A_1^*A_3\exp(-i\Delta k_1z), \tag{1b}$$

$$\frac{dA_3}{dz}=-i2\beta_1A_1A_2\exp(i\Delta k_1z)-i2\beta_2A_1^*A_4\exp(-i\Delta k_2z), \tag{1c}$$

$$\frac{dA_4}{dz}=-i3\beta_2A_1A_3\exp(i\Delta k_2z). \tag{1d}$$

Here $\beta_{1,2}$ are the coupling constants for processes $\omega_1+\omega_{2,3}\rightarrow\omega_{3,4}$.

It is easy to check that there are five second-order integrals $J_{0-4}=$const of Eq. (1), which describe conservation of the energy flux

$$J_0=I_1+I_2+I_3+I_4 \tag{2}$$

and the so-called Manley-Rowe relationships

$$J_1=I_1-2I_2-\frac{1}{2}I_3,\quad J_2=I_1-I_2+\frac{1}{3}I_4,$$

$$J_3=I_1+\frac{1}{2}I_3+\frac{2}{3}I_4,\quad J_4=I_2+\frac{1}{2}I_3+\frac{1}{3}I_4 \tag{3}$$

but only two of these integrals are independent and we can write, for example, that

$$I_2-I_{20}=\frac{1}{2}(I_1-I_{10})-\frac{1}{4}(I_3-I_{30}),$$

$$I_4-I_{40}=-\frac{3}{2}(I_1-I_{10})-\frac{3}{4}(I_3-I_{30}), \tag{4}$$

where $I_i=A_iA_i^*$ are proportional to intensities of the waves and $I_{i0}=A_{i0}A_{i0}^*=A_iA_i^*|_{z=0}$.

Notice here that system (1) can be rewritten in Hamiltonian form

$$\frac{dA_i}{dz}=-i\frac{\omega_i}{\omega_1}\frac{\partial H}{\partial A_i^*}, \tag{5}$$

where

$$H=\beta_1A_1A_2A_3^*\exp(i\Delta k_1z)+\beta_2A_1A_3A_4^*\exp(i\Delta k_2z)+\text{c.c.} \tag{6}$$

is a time-averaged density of free energy, which in the case $\Delta k_{1,2}=0$ describes a field-medium interaction [20]. It is easy to see that in the phase-matched case ($\Delta k_{1,2}=0$) $dH/dz\equiv 0$ and $H=H_0=$const is one more integral of system (1). After transition to real variables (intensities $I_i$ and phases $\varphi_i$ of the modes) by the substitution

$$A_i=\sqrt{I_i}\exp(i\varphi_i), \tag{7}$$

expression (6) can be rewritten as

$$H=2\beta_1\sqrt{I_1I_2I_3}\cos\Delta\varphi_1+2\beta_2\sqrt{I_1I_3I_4}\cos\Delta\varphi_2, \tag{8}$$

where $\Delta\varphi_{1,2}=\varphi_1+\varphi_{2,3}-\varphi_{3,4}+\Delta k_{1,2}z$. This gives the evolution equations in the form

$$\frac{dI_i}{dz}=\frac{\omega_i}{\omega_1}\frac{\partial H}{\partial\varphi_i},\quad \frac{d\varphi_i}{dz}=-\frac{\omega_i}{\omega_1}\frac{\partial H}{\partial I_i} \tag{9}$$

known from Ref. [20]. Using Eq. (9) we find $dH/dz=\partial H/\partial z$, which in the most general case gives

$$H(z)=H(z=0)+\Delta k_1[I_2(z)-I_{20}]+\frac{1}{3}\Delta k_2[I_4(z)-I_{40}]. \tag{10}$$

It is very important that $I_{2,4}$ must be used here as explicit functions of $z$.

Following the approach of Ref. [16], we can try to obtain a system of second-order differential equations corresponding to Eq. (1). To do this we change the variables $A_i\rightarrow\widetilde{A}_i$ by the substitution

$$A_i(z)=\widetilde{A}_i(z)\exp(-i\alpha_iz) \tag{11}$$

and choose the constants $\alpha_i$ in such a way that the relations

$$\Delta\alpha_{1,2}=\alpha_1+\alpha_{2,3}-\alpha_{3,4}=\Delta k_{1,2} \tag{12}$$

are fulfilled. After a series of almost the same (see Ref. [16]) simple transformations we obtain

$$\begin{aligned}\frac{d^2\widetilde{A}_1}{dz^2}=&-(\beta_1^2+3\beta_2^2)|\widetilde{A}_1|^2\widetilde{A}_1+\frac{3}{2}(\beta_1^2-3\beta_2^2)|\widetilde{A}_3|^2\widetilde{A}_1\\&+(\beta_1^2J_1+3\beta_2^2J_3-\alpha_1^2)\widetilde{A}_1+(\alpha_1-\alpha_2+\alpha_3)\beta_1\widetilde{A}_2^*\widetilde{A}_3\\&+(\alpha_1-\alpha_3+\alpha_4)\beta_2\widetilde{A}_3^*\widetilde{A}_4,\end{aligned} \tag{13a}$$

$$\begin{aligned}\frac{d^2\widetilde{A}_2}{dz^2}=&-4\beta_1^2|\widetilde{A}_2|^2\widetilde{A}_2-\beta_1^2(2I_{10}-4I_{20}-I_{30})\widetilde{A}_2+\beta_1\beta_2\widetilde{A}_3\widetilde{A}_3\widetilde{A}_4^*\\&-2\beta_1\beta_2\widetilde{A}_1^*\widetilde{A}_1^*\widetilde{A}_4-(\alpha_1-\alpha_2-\alpha_3)\beta_1\widetilde{A}_1^*\widetilde{A}_3-\alpha_2^2\widetilde{A}_2,\end{aligned} \tag{13b}$$

$$\frac{d^2\tilde{A}_3}{dz^2} = -3(\beta_1^2+3\beta_2^2)|\tilde{A}_1|^2\tilde{A}_3 + \frac{1}{2}(\beta_1^2-3\beta_2^2)|\tilde{A}_3|^2\tilde{A}_3 + (\beta_1^2 J_1 + 3\beta_2^2 J_3 - \alpha_3^2)\tilde{A}_3 + 2(\alpha_1+\alpha_2+\alpha_3)\beta_1\tilde{A}_1\tilde{A}_2 - 2(\alpha_1-\alpha_3-\alpha_4)\beta_2\tilde{A}_1^*\tilde{A}_4, \quad (13c)$$

$$\frac{d^2\tilde{A}_4}{dz^2} = 4\beta_2^2|\tilde{A}_4|^2\tilde{A}_4 - \beta_2^2(6I_{10}+3I_{30}+4I_{40})\tilde{A}_4 - 6\beta_1\beta_2\tilde{A}_1\tilde{A}_1\tilde{A}_2 - 3\beta_1\beta_2\tilde{A}_2^*\tilde{A}_3\tilde{A}_3 + 3(\alpha_1+\alpha_3+\alpha_4)\beta_2\tilde{A}_1\tilde{A}_3 - \alpha_4^2\tilde{A}_4. \quad (13d)$$

It is easy to check that only in the case $\Delta k_{1,2}=0$, Eqs. (13a) and (13c) can be reduced to a closed system of two coupled NLSEs which describes interaction of the waves $A_{1,3}$ in terms of an effective cubic nonlinearity. When $\Delta k_{1,2}\neq 0$, the right-hand sides of both equations include the products $\tilde{A}_i\tilde{A}_j$ and $\tilde{A}_i\tilde{A}_j^*$, which makes such a reduced description impossible. Notice here that Eqs. (13b) and (13d) can also come to a similar closed system but in this case in addition to the condition $\Delta k_{1,2}=0$ one must also require real $A_{1,3}$.

### III. QUASI-PHASE MATCHING

Generally, dispersion prevents conditions $\Delta k_{1,2}=0$ from being satisfied [17]. For this reason, a quasi-phase-matching technique is usually used to realize cascade processes [21]. To do this, in a nonlinear medium one can create, for example, a special structure, in which the signs of $\beta_{1,2}$ change periodically along the $z$ axis [22]. This can be described by the replacement $\beta_{1,2}\to\beta_{1,2}g(z)$, where $g(z)$ is an alternating-sign function with spatial period $\Lambda=(2m_{1,2}+1)(2\pi/\Delta k_{1,2})$ given by the coherence lengths $2\pi/\Delta k_{1,2}$ of the two nonlinear processes, $m_{1,2}$ are positive integers. Using a Fourier expansion $g(z)=\sum_{m=-\infty}^{m=+\infty}g_m\exp(i2\pi m\frac{z}{\Lambda})$, averaging Eq. (1), and retaining four synchronous modes in consideration, we obtain the system

$$\frac{dA_1}{dz} = -i\gamma_1 A_2^* A_3 - i\gamma_2 A_3^* A_4, \quad (14a)$$

$$\frac{dA_2}{dz} = -i\gamma_1 A_1^* A_3, \quad (14b)$$

$$\frac{dA_3}{dz} = -i2\gamma_1^* A_1 A_2 - i2\gamma_2 A_1^* A_4, \quad (14c)$$

$$\frac{dA_4}{dz} = -i3\gamma_2^* A_1 A_3. \quad (14d)$$

Here $\gamma_{1,2}=\langle\beta_{1,2}\exp(-i\Delta k_{1,2}z)\rangle_z$ are averaged constants of nonlinear coupling for the processes $\omega_1+\omega_{2,3}\to\omega_{3,4}$, respectively. Further transformation from Eq. (14) to second-order equations gives a sought closed system of two nonlinear equations for $A_{1,3}$ in the form

$$\frac{d^2A_1}{dz^2} = -G_+|A_1|^2A_1 + \frac{3}{2}G_-|A_3|^2A_1 + (|\gamma_1|^2J_1 + 3|\gamma_2|^2J_3)A_1, \quad (15a)$$

$$\frac{d^2A_3}{dz^2} = -3G_+|A_1|^2A_3 + \frac{1}{2}G_-|A_3|^2A_3 + (|\gamma_1|^2J_1 + 3|\gamma_2|^2J_3)A_3 \quad (15b)$$

with the boundary conditions

$$A_1|_{z=0}=A_{10}, \qquad \left.\frac{dA_1}{dz}\right|_{z=0} = -i\gamma_1 A_{20}^* A_{30} - i\gamma_2 A_{30}^* A_{40}, \quad (16a)$$

$$A_3|_{z=0}=A_{30}, \qquad \left.\frac{dA_3}{dz}\right|_{z=0} = -i2\gamma_1^* A_{10}A_{20} - i2\gamma_2 A_{10}^* A_{40}, \quad (16b)$$

where $G_\pm=|\gamma_1|^2\pm 3|\gamma_2|^2$. Notice that while equations for $A_{2,4}$ do not reduce to a similar system [see Eq. (13)], intensities of both these waves can be easily found from Eq. (4). Notice also that analysis of solutions of similar systems is now a subject of intensive research [12,19,23].

Following the approach of Ref. [16], let us introduce the magnitudes $X_j$ and the phases $\varphi_j$ of the modes $A_j$ by

$$A_j(z) = X_j(z)\exp[i\varphi_j(z)]. \quad (17)$$

After substitution of Eq. (17) into Eq. (15) and separating the real and imaginary parts, we obtain

$$\frac{d^2X_1}{dz^2} - X_1\left(\frac{d\varphi_1}{dz}\right)^2 = -G_+X_1^3 + \frac{3}{2}G_-X_3^2X_1 + (|\gamma_1|^2J_1 + 3|\gamma_2|^2J_3)X_1, \quad (18a)$$

$$2\frac{dX_1}{dz}\frac{d\varphi_1}{dz} + X_1\frac{d^2\varphi_1}{dz^2} = 0, \quad (18b)$$

$$\frac{d^2X_3}{dz^2} - X_3\left(\frac{d\varphi_3}{dz}\right)^2 = -3G_+X_1^2X_3 + \frac{1}{2}G_-X_3^3 + (|\gamma_1|^2J_1 + 3|\gamma_2|^2J_3)X_3, \quad (18c)$$

$$2\frac{dX_3}{dz}\frac{d\varphi_3}{dz} + X_3\frac{d^2\varphi_3}{dz^2} = 0. \quad (18d)$$

Because the case $X_{1,3}(z)\equiv 0$ is out of our interest, two integrals (see Ref. [16]) for $\varphi_{1,3}$ follow from Eqs. (18b) and (18d). Moreover, it is easy to show that these two integrals are also dependent and that

$$X_1^2\frac{d\varphi_1}{dz} = I_{10}\varphi_{10}' = -\frac{1}{2}H, \quad X_3^2\frac{d\varphi_3}{dz} = I_{30}\varphi_{30}' = -H, \quad (19)$$

where

$$H = \gamma_1^* A_1 A_2 A_3^* + \gamma_1 A_1^* A_2^* A_3 + \gamma_2^* A_1 A_3 A_4^* + \gamma_2 A_1^* A_3^* A_4 = \text{const} \quad (20)$$

and $\varphi_{1,3}|_{z=0}=\varphi_{10,30}$, $d\varphi_{1,3}/dz|_{z=0}=\varphi'_{10,30}$, and $X_i^2|_{z=0}=I_{i0}$.

As in Ref. [16], it follows from Eqs. (18b), (18d), and (19) that if there is a value $z'$, where $X_{1,3}|_{z=z'}=0$ and $dX_{1,3}/dz|_{z=z'}\neq 0$, then for all other values $z$, where $X_{1,3}(z)\neq 0$, the equality $d\varphi_{1,3}/dz\equiv 0$ must be satisfied. This means that $\varphi_{1,3}$ must be stepwise functions of $z$ which can be taken into account ($\varphi_{1,3}$=const) by allowing $X_{1,3}(z)=\pm|A_{1,3}(z)|$ to be negative. If this is not the case, we can find $\varphi_{1,3}(z)$ by integrating Eq. (19) as

$$\varphi_1(z)=\varphi_{10}-\frac{1}{2}H\int_0^z X_1^{-2}(z')dz',$$

$$\varphi_3(z)=\varphi_{30}-H\int_0^z X_3^{-2}(z')dz'. \quad (21)$$

## IV. EXACT ANALYTIC SOLUTIONS

We have shown that the problem under consideration can be reduced to solving a closed system of two ordinary differential equations, which describes interaction of the complex amplitudes $A_{1,3}$ in terms of an effective cubic nonlinearity. After another change of variables

$$z=\tilde{z}/\sqrt{G_+}, \quad (22)$$

the obtained system can be rewritten in the form

$$\frac{d^2X_1}{d\tilde{z}^2}+\frac{1}{4G_+}\frac{H^2}{X_1^3}=-X_1^3+\frac{3}{2}\frac{G_-}{G_+}X_3^2X_1+J_{13}X_1, \quad (23a)$$

$$\frac{d^2X_3}{d\tilde{z}^2}+\frac{1}{G_+}\frac{H^2}{X_3^3}=-3X_1^2X_3+\frac{1}{2}\frac{G_-}{G_+}X_3^3+J_{13}X_3, \quad (23b)$$

where $J_{13}=\frac{|\gamma_1|^2J_1+3|\gamma_2|^2J_3}{|\gamma_1|^2+3|\gamma_2|^2}$. Further we consider situations, when there is a value $z'$ at which $A_1(z')=0$ or $A_3(z')=0$ (that is, one of these two waves is completely depleted or does not present in the plane $z'=z=0$), so that, $H=0$ and $\varphi_{1,3}(z)=\varphi_{10,30}$ (see above).

Notice here, that in the particular case $|\gamma_1|^2=3|\gamma_2|^2$ the obtained system becomes

$$\frac{d^2X_1}{d\tilde{z}^2}=-X_1^3+\frac{1}{2}(J_1+J_3)X_1, \quad (24a)$$

$$\frac{d^2X_3}{d\tilde{z}^2}=-3X_1^2X_3+\frac{1}{2}(J_1+J_3)X_3, \quad (24b)$$

which is a well-known problem of the independent periodic change of amplitude $X_1$ in a medium with Kerr-type nonlinearity [14]. Nevertheless, the period of $X_1(\tilde{z})$ oscillations depends on initial intensities of all other waves (on the sum $J_1+J_3$) and seeking the dependence $X_3(\tilde{z})$ is reduced to solving the second-order Lamé equation [24].

Seeking solutions of Eq. (24a) in forms typical for nonlinearity of this type [14] gives

$$X_1=\sqrt{I_{10}}\mathrm{cn}(\beta\tilde{z},k), \quad (25a)$$

$$X_3=\sqrt{I_{3M}}\mathrm{sn}(\beta\tilde{z},k)\mathrm{dn}(\beta\tilde{z},k) \quad (25b)$$

for $\beta^2=I_{20}-\frac{1}{3}I_{40}$, $k^2=\frac{1}{2}I_{10}(I_{20}-\frac{1}{3}I_{40})^{-1}$, $2(I_{20}-\frac{1}{3}I_{40})\geqslant I_{10}\geqslant 0$ and

$$X_1=\sqrt{I_{10}}\mathrm{dn}(\beta\tilde{z},k), \quad (26a)$$

$$X_3=\sqrt{I_{3M}}\mathrm{sn}(\beta\tilde{z},k)\mathrm{cn}(\beta\tilde{z},k) \quad (26b)$$

for $\beta^2=\frac{1}{2}I_{10}$, $k^2=2I_{10}^{-1}(I_{20}-\frac{1}{3}I_{40})$, $I_{10}\geqslant 2(I_{20}-\frac{1}{3}I_{40})$. Here $1\geqslant k\geqslant 0$ is the modulus of the Jacobi elliptic functions $\mathrm{sn}(z,k)$, $\mathrm{cn}(z,k)$, and $\mathrm{dn}(z,k)$ [25] and is adjustable, the parameter $I_{3M}$ is related to boundary conditions and depends both on the initial intensities $I_{i0}$ and on relations between the initial phases $\varphi_{i0}$ [see Eq. (16b)]. Notice that all other solutions of system (24), including the case $I_{20}-\frac{1}{3}I_{40}\leqslant 0$, are reduced to translation of solutions (25) and (26) along the $\tilde{z}$ axis. The forms written above correspond to the particular case $I_{10}\neq 0$ and $I_{30}=0$ that describes the situation which is analyzed in Sec. V. Here and below expressions for $I_{2,4}(z)$ are not written because they can be easily found from Eq. (4).

To analyze the case $|\gamma_1|^2\neq 3|\gamma_2|^2$, let us first introduce normalized variables

$$X_1=\tilde{X}_1, \quad X_3=\sqrt{2|G_+/G_-|}\tilde{X}_3 \quad (27)$$

and turn system (23) into

$$\frac{d^2\tilde{X}_1}{d\tilde{z}^2}=-\tilde{X}_1^3-3\sigma\tilde{X}_3^2\tilde{X}_1+J_{13}\tilde{X}_1, \quad (28a)$$

$$\frac{d^2\tilde{X}_3}{d\tilde{z}^2}=-3\tilde{X}_1^2\tilde{X}_3-\sigma\tilde{X}_3^3+J_{13}\tilde{X}_3. \quad (28b)$$

Here, $\sigma=1$ and $\sigma=-1$ correspond to the cases $|\gamma_1|^2<3|\gamma_2|^2$ and $|\gamma_1|^2>3|\gamma_2|^2$. Notice that the integrability and the character of solutions of systems of such type is determined by the relationship between the coefficients of the nonlinear terms [25].

The case $\sigma=1$ ($|\gamma_1|^2<3|\gamma_2|^2$) is not so difficult because it is known [25] that the substitution

$$\tilde{Y}_\pm=\tilde{X}_1\pm\tilde{X}_3 \quad (29)$$

separates the variables and so reduces system (28) to two independent NLSEs with nonlinearity of focusing type

$$\frac{d^2\tilde{Y}_\pm}{d\tilde{z}^2}=-\tilde{Y}_\pm^3+J_{13}\tilde{Y}_\pm. \quad (30)$$

It is easy to check that the two equations are coupled only through the boundary conditions and have the same proportionality constants in their linear terms. Equality of these constants makes impossible the use of a standard approach in which $Y_\pm$ are supposed to be proportional to two different fundamental solutions $\mathrm{cn}(z,k)$ and $\mathrm{dn}(z,k)$ of the first-order Lamé equation [14], because such solutions are degenerate only for $k=1$ [$\mathrm{cn}(z,k=1)=\mathrm{dn}(z,k=1)=\cosh(z)$], so that $\tilde{X}_1\equiv 0$ or $\tilde{X}_3\equiv 0$ corresponding to parametric bleaching with $I_{1-4}$=const.

However, there are two other possibilities. First, solutions of the two equations in Eq. (30) may be proportional to the same elliptic functions shifted along the $\tilde{z}$ axis, that is,

$$\tilde{Y}_{\pm} = A\mathrm{cn}(\beta\tilde{z} \pm \beta\tilde{z}_0, k) \text{ or } \tilde{Y}_{\pm} = A\mathrm{dn}(\beta\tilde{z} \pm \beta\tilde{z}_0, k). \quad (31)$$

Here, the parameter $\tilde{z}_0$ describes the shift of the functions $\tilde{Y}_{\pm}$ which is taken as symmetrical about $\tilde{z}=0$. This corresponds to the presence of extrema for $I_{1,3}$ on the $\tilde{z}=0$ plane and determines four nontrivial solutions of system (28):

$$\tilde{X}_{1,3} = A\mathrm{cn}(\beta\tilde{z}_0, k)\frac{\mathrm{cn}(\beta\tilde{z}, k)}{1 - k^2\mathrm{sn}^2(\beta\tilde{z}_0, k)\mathrm{sn}^2(\beta\tilde{z}, k)}, \quad (32a)$$

$$\tilde{X}_{3,1} = -A\mathrm{sn}(\beta\tilde{z}_0, k)\mathrm{dn}(\beta\tilde{z}_0, k)\frac{\mathrm{sn}(\beta\tilde{z}, k)\mathrm{dn}(\beta\tilde{z}, k)}{1 - k^2\mathrm{sn}^2(\beta\tilde{z}_0, k)\mathrm{sn}^2(\beta\tilde{z}, k)} \quad (32b)$$

for $\beta^2=A^2-J_{13}$, $k^2=\frac{1}{2}A^2(A^2-J_{13})^{-1}$, $A^2 \geqslant \max(J_{13}, 2J_{13})$,

$$\tilde{X}_{1,3} = A\mathrm{dn}(\beta\tilde{z}_0, k)\frac{\mathrm{dn}(\beta\tilde{z}, k)}{1 - k^2\mathrm{sn}^2(\beta\tilde{z}_0, k)\mathrm{sn}^2(\beta\tilde{z}, k)}, \quad (33a)$$

$$\tilde{X}_{3,1} = -k^2A\mathrm{sn}(\beta\tilde{z}_0, k)\mathrm{cn}(\beta\tilde{z}_0, k)\frac{\mathrm{sn}(\beta\tilde{z}, k)\mathrm{cn}(\beta\tilde{z}, k)}{1 - k^2\mathrm{sn}^2(\beta\tilde{z}_0, k)\mathrm{sn}^2(\beta\tilde{z}, k)} \quad (33b)$$

for $\beta^2=\frac{1}{2}A^2$, $k^2=2(A^2-J_{13})A^{-2}$, and $2J_{13} \geqslant A^2 \geqslant J_{13}$. Here, the values of the real constants $A$ and $\tilde{z}_0$ must be chosen to satisfy the boundary conditions (16) which determine existence regions for the solutions (32) and (33). Notice that returning to the initial variables $X_{1,3}$ breaks the symmetry of expressions (32) and (33) in relation to the index replacement $1 \leftrightarrow 3$.

Second, the solution of one equation in system (30) may be a constant whereas the solution of the other may by proportional to one of the fundamental solutions $\mathrm{cn}(z,k)$ and $\mathrm{dn}(z,k)$ of the first-order Lamé equation, that is,

$$\tilde{Y}_{\pm} = A = \text{const}, \quad (34a)$$

and

$$\tilde{Y}_{\mp} = B\mathrm{cn}(\beta\tilde{z}, k). \quad (34b)$$

This possibility determines four other solutions of system (28):

$$\tilde{X}_{1,3} = \frac{1}{2}[\sqrt{J_{13}} \pm B\mathrm{cn}(\beta\tilde{z}, k)] \quad (35)$$

for $\beta^2=B^2-J_{13}$, $k^2=\frac{1}{2}B^2(B^2-J_{13})^{-1}$, and $B^2 \geqslant 2J_{13} \geqslant 0$;

$$\tilde{X}_{1,3} = \frac{1}{2}[\sqrt{J_{13}} \pm B\mathrm{dn}(\beta\tilde{z}, k)] \quad (36)$$

for $\beta^2=\frac{1}{2}B^2$, $k^2=2(B^2-J_{13})B^{-2}$, and $2J_{13} \geqslant B^2 \geqslant J_{13} \geqslant 0$. Here, the real constant $B$ must be also chosen to satisfy the boundary conditions (16). Notice that in expressions (35) and (36) the indexes can be replaced $1 \leftrightarrow 3$ and, as earlier, due to renormalization of $\tilde{X}_3$, return to the initial variables $X_{1,3}$ breaks the symmetry of Eqs. (35) and (36) regarding this replacement.

In the case $|\gamma_1|^2 > 3|\gamma_2|^2$ (i.e., $\sigma=-1$), the approach described above is also applicable. To use it, let us make first a formal replacement $\tilde{z}=i\utilde{z}$ (see Ref. [26]) and then seek solutions in two classes of functions for which

$$\tilde{X}_1(i\tilde{z}) = i\utilde{X}_1(\tilde{z}), \quad \tilde{X}_3(i\utilde{z}) = \utilde{X}_3(\utilde{z}) \quad (37a)$$

or

$$\tilde{X}_1(i\utilde{z}) = \utilde{X}_1(\utilde{z}), \quad \tilde{X}_3(i\utilde{z}) = i\utilde{X}_3(\utilde{z}), \quad (37b)$$

where $\utilde{X}_{1,3}(\utilde{z})$ and $\tilde{X}_{1,3}(\tilde{z})$ are real. Notice that the elliptic functions $\mathrm{sn}(z,k)$, $\mathrm{cn}(z,k)$, and $\mathrm{dn}(z,k)$ satisfy the well-known relations $\mathrm{sn}(iz,k)=i\mathrm{sn}(z,k')\mathrm{cn}^{-1}(z,k')$, $\mathrm{cn}(iz,k)=\mathrm{cn}^{-1}(z,k')$ and $\mathrm{dn}(iz,k)=\mathrm{dn}(z,k')\mathrm{cn}^{-1}(z,k')$, where $k'=\sqrt{1-k^2}$, that is, they satisfy Eq. (37a) and (37b) [24]. After such a replacement, system (28) is rewritten in one of two forms, corresponding to different solution classes

$$\frac{d^2\utilde{X}_1}{d\utilde{z}^2} = -\utilde{X}_1^3 - 3\utilde{X}_3^2\utilde{X}_1 - J_{13}\utilde{X}_1, \quad (38a)$$

$$\frac{d^2\utilde{X}_3}{d\utilde{z}^2} = -3\utilde{X}_1^2\utilde{X}_3 - \utilde{X}_3^3 - J_{13}\utilde{X}_3 \quad (38b)$$

or

$$\frac{d^2\utilde{X}_1}{d\utilde{z}^2} = \utilde{X}_1^3 + 3\utilde{X}_3^2\utilde{X}_1 - J_{13}\utilde{X}_1, \quad (39a)$$

$$\frac{d^2\utilde{X}_3}{d\utilde{z}^2} = 3\utilde{X}_1^2\utilde{X}_3 + \utilde{X}_3^3 - J_{13}\utilde{X}_3. \quad (39b)$$

It is easy to see that now after the substitutions

$$\utilde{Y}_{\pm} = \utilde{X}_3 \pm \utilde{X}_1, \quad (40)$$

we obtain two possible pairs of independent NLSEs

$$\frac{d^2\utilde{Y}_{\pm}}{d\utilde{z}^2} = -\utilde{Y}_{\pm}^3 - J_{13}\utilde{Y}_{\pm} \quad (41a)$$

or

$$\frac{d^2\utilde{Y}_{\pm}}{d\utilde{z}^2} = \utilde{Y}_{\pm}^3 - J_{13}\utilde{Y}_{\pm}. \quad (41b)$$

The two equations of pairs (41a) and (41b) are again coupled only through the boundary conditions and have identical constants of proportionality in the linear terms. However, now these pairs correspond to situations with nonlinearity of either focusing (41a) or defocusing (41b) type. For the same reasons solutions of equations of each pair must be proportional to the same elliptic functions, but by taking into account conditions (37a) and (37b) the shift of their arguments must be imaginary:

$$\utilde{Y}_{\pm} = A\mathrm{cn}(\beta\utilde{z} \pm i\beta\utilde{z}_0, k'), \quad (42a)$$

$$\utilde{Y}_{\pm} = A\mathrm{dn}(\beta\utilde{z} \pm i\beta\utilde{z}_0, k'), \quad (42b)$$

or

$$\underset{\sim}{Y}_{\pm}=A\,\mathrm{sn}(\beta \underset{\sim}{z}\pm i\beta \underset{\sim}{z}_0,k'). \tag{42c}$$

Here, the parameter $\underset{\sim}{z}_0$ describes the shift (so that $i\underset{\sim}{z}_0$ is imaginary) and the modulus of the Jacobi elliptic functions is denoted by $k'$ in order to obtain the modulus $k$ in the final expressions (see below) for all $\tilde{z}$-dependent elliptic functions. The possibilities listed above determine three nontrivial solutions of system (28):

$$\tilde{X}_1=-A\,\mathrm{sn}(\beta\tilde{z}_0,k')\mathrm{dn}(\beta\tilde{z}_0,k')\times\frac{\mathrm{sn}(\beta\tilde{z},k)\mathrm{dn}(\beta\tilde{z},k)}{\mathrm{cn}^2(\beta\tilde{z},k)+(k')^2\mathrm{sn}^2(\beta\tilde{z}_0,k')\mathrm{sn}^2(\beta\tilde{z},k)}, \tag{43a}$$

$$\tilde{X}_3=A\,\mathrm{cn}(\beta\tilde{z}_0,k')\frac{\mathrm{cn}(\beta\tilde{z},k)}{\mathrm{cn}^2(\beta\tilde{z},k)+(k')^2\mathrm{sn}^2(\beta\tilde{z}_0,k')\mathrm{sn}^2(\beta\tilde{z},k)} \tag{43b}$$

for $\beta^2=A^2+J_{13}$, $(k')^2=\frac{1}{2}A^2(A^2+J_{13})^{-1}$, where $A^2\geqslant 2|J_{13}|$ for $J_{13}\leqslant 0$ and any for $J_{13}\geqslant 0$;

$$\tilde{X}_1=-(k')^2A\,\mathrm{sn}(\beta\tilde{z}_0,k')\mathrm{cn}(\beta\tilde{z}_0,k')\times\frac{\mathrm{sn}(\beta\tilde{z},k)}{\mathrm{cn}^2(\beta\tilde{z},k)+(k')^2\mathrm{sn}^2(\beta\tilde{z}_0,k')\mathrm{sn}^2(\beta\tilde{z},k)}, \tag{44a}$$

$$\tilde{X}_3=A\,\mathrm{dn}(\beta\tilde{z}_0,k')\frac{\mathrm{cn}(\beta\tilde{z},k)\mathrm{dn}(\beta\tilde{z},k)}{\mathrm{cn}^2(\beta\tilde{z},k)+(k')^2\mathrm{sn}^2(\beta\tilde{z}_0,k')\mathrm{sn}^2(\beta\tilde{z},k)} \tag{44b}$$

for $\beta^2=\frac{1}{2}A^2$, $(k')^2=2A^{-2}(A^2+J_{13})$ and $2|J_{13}|\geqslant A^2\geqslant|J_{13}|$ for $J_{13}\leqslant 0$;

$$\tilde{X}_1=A\,\mathrm{sn}(\beta\tilde{z}_0,k')\frac{\mathrm{dn}(\beta\tilde{z},k)}{\mathrm{cn}^2(\beta\tilde{z},k)+(k')^2\mathrm{sn}^2(\beta\tilde{z}_0,k')\mathrm{sn}^2(\beta\tilde{z},k)}, \tag{45a}$$

$$\tilde{X}_3=-A\,\mathrm{cn}(\beta\tilde{z}_0,k')\mathrm{dn}(\beta\tilde{z}_0,k')\times\frac{\mathrm{sn}(\beta\tilde{z},k)\mathrm{cn}(\beta\tilde{z},k)}{\mathrm{cn}^2(\beta\tilde{z},k)+(k')^2\mathrm{sn}^2(\beta\tilde{z}_0,k')\mathrm{sn}^2(\beta\tilde{z},k)} \tag{45b}$$

for $\beta^2=J_{13}-\frac{1}{2}A^2$, $(k')^2=A^2(2J_{13}-A^2)^{-1}$, and $A^2\leqslant J_{13}$ for $J_{13}\geqslant 0$.

However, the above solutions do not exhaust all situations determined by the boundary conditions. Really, variants of $I_{10-40}$ can arise for $J_{13}<0$ in Eq. (41a) and (41b). For nonlinearity of focusing type this case is described by solutions (43) and (44), whereas in the case of the nonlinearity of defocusing type, solution (45) does not exist. However, despite the existence of singular points, the function $\frac{\mathrm{sn}(iz,k)\mathrm{dn}(iz,k)}{\mathrm{cn}(iz,k)}=i\frac{\mathrm{sn}(z,k')\mathrm{dn}(z,k')}{\mathrm{cn}(z,k')}=i\sqrt{\frac{1-\mathrm{cn}(2z,k')}{1+\mathrm{cn}(2z,k')}}=\mathrm{if}(z,k')$, which is not a fundamental solution of the Lamé equation, satisfies each equation of the pair (41b). Moreover, by a shift of arguments, existence of such points does not prevent looking for solutions of Eq. (41b) in the form

$$\underset{\sim}{Y}_{\pm}=Af(\beta\underset{\sim}{z}\pm i\beta\underset{\sim}{z}_0,k'), \tag{46}$$

that gives the expressions

$$\tilde{X}_1(\tilde{z})=A\frac{\mathrm{sn}(2\beta\tilde{z}_0,k')\mathrm{dn}(2\beta\tilde{z},k)}{1+\mathrm{cn}(2\beta\tilde{z}_0,k')\mathrm{cn}(2\beta\tilde{z},k')}, \tag{47a}$$

$$\tilde{X}_3(\tilde{z})=A\frac{\mathrm{dn}(2\beta\tilde{z}_0,k')\mathrm{sn}(2\beta\tilde{z},k)}{1+\mathrm{cn}(2\beta\tilde{z}_0,k')\mathrm{cn}(2\beta\tilde{z},k')} \tag{47b}$$

for $\beta^2=\frac{1}{2}A^2$, $(k')^2=\frac{1}{2}(A^2+J_{13})A^{-2}$, and $A^2\geqslant|J_{13}|$. As previously, the values of the real constants $A$ and $\tilde{z}_0$ here must be matched to the boundary conditions (16) which determine domains of existence of solutions in such of the forms (43)–(45) and (47). Notice that in the case $|\gamma_1|^2>3|\gamma_2|^2$ the symmetry of expressions for $\tilde{X}_{1,3}$ in relation to replacement $1\leftrightarrow 3$ of the indexes is initially broken by requirements (37a) and (37b).

## V. SPECIFIC FEATURES OF SOLUTIONS

To illustrate the character and peculiarities of the solutions obtained we consider here the case when $I_{10,20}\neq 0$ and $I_{30}=I_{40}=0$ (that is, $H=0$). For such boundary conditions, two low-frequency modes ($A_{1,2}$) play the role of two-component pumping which is used to generate two high-frequency modes ($A_{3,4}$). Recall that the position of the plane $z=0$ is conventional and the argument of any solution listed above can be arbitrary shifted. Thus, to satisfy the mentioned boundary conditions, we will use solutions (43) and (44) shifted along the $z$ axis by a quarter of their period.

Let us introduce the plane $(\varepsilon,N)$ defined by the two parameters $\varepsilon=3|\gamma_2|^2/|\gamma_1|^2-1\geqslant -1$ and $N=I_{10}/I_{20}\geqslant 0$ which describe the relationship between nonlinear coupling constants and the role of boundary conditions for solutions (25), (26), (32), (33), (43)–(45), and (47). Regions of existence of these solutions are limited on this plane by separatrixes (see Fig. 1)

$$\varepsilon_0(N)=-1/N, \tag{48a}$$

$$\varepsilon_{\pm}(N)=\frac{2}{N}[2-N\pm\sqrt{2(2-N)}]. \tag{48b}$$

Separatrixes $\varepsilon_{\pm}(N)$ are substantially two branches of a double-valued solution of the equation $N(\varepsilon)=8(\varepsilon+1)/[\varepsilon^2+4(\varepsilon+1)]$ and analytically continue each other at the point ($\varepsilon=0$, $N=2$). Separatrixes $\varepsilon_0(N)$ and $\varepsilon_-(N)$ are tangent to each other at the point ($\varepsilon=-2/3$, $N=3/2$) and are also analytically joined at this point. This joining results in formation of two intersecting curves with $k=0$ and $k=1$, respectively (Fig. 1). Their intersection point ($\varepsilon=-2/3$, $N=3/2$) is singular (see below). The regions $|\gamma_1|^2<3|\gamma_2|^2$ [$\varepsilon>0$, solutions (32) and (33)] and $|\gamma_1|^2>3|\gamma_2|^2$ [$\varepsilon<0$, solutions (43)–(45) and (47)] are located above and below the line $\varepsilon=0$ which corresponds to solutions (25) and (26) (Fig. 1).

Expressions (25), (32), and (43) are proved to be responsible for the region on the left side of the separatrixes $\varepsilon_{\pm}(N)$, where all three solutions can be rewritten in unified form as

$$\frac{X_1}{|A_{10}|}=\frac{\mathrm{cn}(\alpha z)}{1-\frac{\sqrt{1+\varepsilon N}-1}{2\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{49a}$$

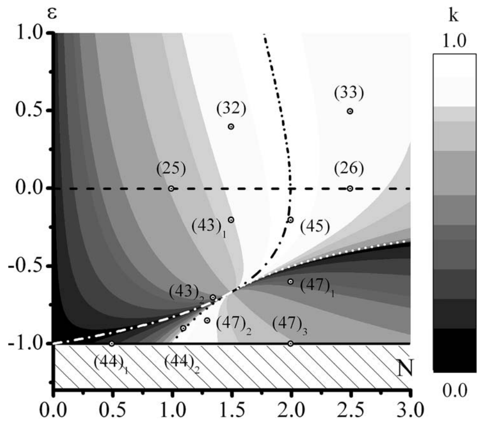


FIG. 1. Grayscale map of the modulus $k$ of the Jacobi elliptic functions on the plane $(\varepsilon, N)$ for $I_{10,20} \neq 0$ and $I_{30}=I_{40}=0$. The line $\varepsilon=0$ and the boundary $\varepsilon=-1$ are shown by dash and solid lines, whereas the separatrixes $\varepsilon_0(N)=-1/N$ and $\varepsilon_{\pm}(N)=\frac{2}{N}[2-N\pm\sqrt{2(2-N)}]$ are identified by short dash, short dash-dot, and dash-dot lines. The segments corresponding to $k=1$ and $k=0$ values are shown by black and white lines. Numerical parameters $\varepsilon=3|\gamma_2|^2/|\gamma_1|^2-1$ and $N=I_{10}/I_{20}$ describe the relationship between nonlinear coupling constants $|\gamma_{1,2}|^2$ and the role of boundary conditions $(I_{10,20})$.

$$\frac{X_2}{|A_{10}|} = N^{-1/2}\frac{1-\dfrac{\sqrt{1+\varepsilon N}-1+N}{2\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}{1-\dfrac{\sqrt{1+\varepsilon N}-1}{2\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{49b}$$

$$\frac{X_3}{|A_{10}|} = \sqrt{2}(1+\varepsilon N)^{-1/4}\frac{\mathrm{sn}(\alpha z)\mathrm{dn}(\alpha z)}{1-\dfrac{\sqrt{1+\varepsilon N}-1}{2\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{49c}$$

$$\frac{X_4}{|A_{10}|} = \frac{1}{2}\sqrt{\frac{3N(1+\varepsilon)}{1+\varepsilon N}}\frac{\mathrm{sn}^2(\alpha z)}{1-\dfrac{\sqrt{1+\varepsilon N}-1}{2\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{49d}$$

$$k = \frac{\sqrt{2(\sqrt{1+\varepsilon N}-1)+N(2+\varepsilon)}}{2(1+\varepsilon N)^{1/4}},$$
$$\alpha = \sqrt{\frac{2}{N}}(1+\varepsilon N)^{1/4}|\gamma_1||A_{10}|. \tag{49e}$$

Expressions (26), (33), and (45) correspond to the region on the right side of the separatrixes $\varepsilon_{\pm}(N)$ above the separatrix $\varepsilon_0(N)$ for $\varepsilon > -2/3$, where all these solutions can be also rewritten in unified form as

$$\frac{X_1}{|A_{10}|} = \frac{\mathrm{dn}(\alpha z)}{1-2\dfrac{\varepsilon}{2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{50a}$$

$$\frac{X_2}{|A_{10}|} = N^{-1/2}\frac{1-2\dfrac{1+\varepsilon+\sqrt{1+\varepsilon N}}{2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}{1-2\dfrac{\varepsilon}{2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{50b}$$

$$\frac{X_3}{|A_{10}|} = 2N^{-1/2}\sqrt{\frac{2(1+\sqrt{1+\varepsilon N})}{2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}}}$$
$$\times\frac{\mathrm{sn}(\alpha z)\mathrm{cn}(\alpha z)}{1-2\dfrac{\varepsilon}{2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{50c}$$

$$\frac{X_4}{|A_{10}|} = 2N^{-1/2}\frac{\sqrt{3(1+\varepsilon)}(1+\sqrt{1+\varepsilon N})}{2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}}$$
$$\times\frac{\mathrm{sn}^2(\alpha z)}{1-2\dfrac{\varepsilon}{2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{50d}$$

$$k = 2\sqrt{\frac{\varepsilon\sqrt{1+\varepsilon N}}{[2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}](\sqrt{1+\varepsilon N}-1)}},$$
$$\alpha = \sqrt{\frac{2}{N}}\frac{(1+\varepsilon N)^{1/4}}{k}|\gamma_1||A_{10}|. \tag{50e}$$

Expressions (44), shifted by a quarter of their period along the z axis, are responsible for the region on the left side of the separatrix $\varepsilon_0(N)$ below the separatrix $\varepsilon_-(N)$ for $\varepsilon < -2/3$, where the corresponding solution can be rewritten as

$$\frac{X_1}{|A_{10}|} = \frac{\mathrm{cn}(\alpha z)\mathrm{dn}(\alpha z)}{1+(2+\varepsilon)\dfrac{\sqrt{1+\varepsilon N}-1}{2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{51a}$$

$$\frac{X_2}{|A_{10}|} = N^{-1/2}\frac{1+\varepsilon\dfrac{\sqrt{1+\varepsilon N}-1}{2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}{1+(2+\varepsilon)\dfrac{\sqrt{1+\varepsilon N}-1}{2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{51b}$$

$$\frac{X_3}{|A_{10}|} = 2N^{-1/2}\sqrt{\frac{2(\sqrt{1+\varepsilon N}-1)}{2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}}}$$
$$\times\frac{\mathrm{sn}(\alpha z)}{1+(2+\varepsilon)\dfrac{\sqrt{1+\varepsilon N}-1}{2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{51c}$$

$$\frac{X_4}{|A_{10}|} = 2N^{-1/2}\frac{\sqrt{3(1+\varepsilon)}(\sqrt{1+\varepsilon N}-1)}{2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}} \times \frac{\mathrm{sn}^2(\alpha z)}{1+(2+\varepsilon)\dfrac{\sqrt{1+\varepsilon N}-1}{2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}}\mathrm{sn}^2(\alpha z)}, \tag{51d}$$

$$k = \sqrt{\frac{(1-\sqrt{1+\varepsilon N})[2+3\varepsilon+(2+\varepsilon)\sqrt{1+\varepsilon N}]}{(1+\sqrt{1+\varepsilon N})[2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}]}},$$

$$\alpha = \sqrt{\frac{(1+\sqrt{1+\varepsilon N})[2+3\varepsilon-(2+\varepsilon)\sqrt{1+\varepsilon N}]}{2\varepsilon N}}|\gamma_1||A_{10}|. \tag{51e}$$

Expressions (47) correspond to the last region, located below the separatrix $\varepsilon_0(N)$, where this solution can be rewritten in the form

$$\frac{X_1}{|A_{10}|} = \frac{2\eta}{\eta+1}\frac{\mathrm{dn}(\alpha z)}{1+\dfrac{\eta-1}{\eta+1}\mathrm{cn}(\alpha z)}, \tag{52a}$$

$$\frac{X_2}{|A_{10}|} = N^{-1/2}\frac{(2+\varepsilon)(\eta+1)-2}{(2+\varepsilon)(\eta+1)}\frac{1+\dfrac{(2+\varepsilon)(\eta-1)+2}{(2+\varepsilon)(\eta+1)-2}\mathrm{cn}(\alpha z)}{1+\dfrac{\eta-1}{\eta+1}\mathrm{cn}(\alpha z)}, \tag{52b}$$

$$\frac{X_3}{|A_{10}|} = N^{-1/2}\frac{2}{\eta+1}\sqrt{\frac{2\eta}{2+\varepsilon}}\frac{\mathrm{sn}(\alpha z)}{1+\dfrac{\eta-1}{\eta+1}\mathrm{cn}(\alpha z)}, \tag{52c}$$

$$\frac{X_4}{|A_{10}|} = 2N^{-1/2}\frac{\sqrt{3(1+\varepsilon)}}{(2+\varepsilon)(\eta+1)}\frac{1-\mathrm{cn}(\alpha z)}{1+\dfrac{\eta-1}{\eta+1}\mathrm{cn}(\alpha z)}, \tag{52d}$$

$$k = \sqrt{\frac{2+(2+\varepsilon)(\eta-1)N}{2(2+\varepsilon)\eta N}}, \quad \alpha = \sqrt{2(2+\varepsilon)\eta}|\gamma_1||A_{10}|, \tag{52e}$$

$$\eta = \sqrt{1-8\frac{1+\varepsilon}{N(2+\varepsilon)^2}}.$$

Finally, the solution for the singular point ($\varepsilon=-2/3$, $N=3/2$) can be easily obtained as a limit of the above-written expressions what gives an algebraic solution of solitary type

$$\frac{X_1}{|A_{10}|} = \frac{1}{1+\lambda z^2}, \tag{53a}$$

$$\frac{X_2}{|A_{10}|} = \frac{1}{\sqrt{6}}\frac{2-\lambda z^2}{1+\lambda z^2}, \tag{53b}$$

$$\frac{X_3}{|A_{10}|} = \frac{2\sqrt{\lambda}z}{1+\lambda z^2}, \tag{53c}$$

$$\frac{X_4}{|A_{10}|} = \sqrt{\frac{3}{2}}\frac{\lambda z^2}{1+\lambda z^2}, \tag{53d}$$

$$\lambda = \frac{2}{3}|\gamma_1|^2 I_{10}. \tag{53e}$$

Notice, to determine $X_{2,4}(z)$ in Eqs. (49)–(53), we used expressions (4) and took into account the sign of $dA_{2,4}/dz$ near the points $X_{2,4}=0$ [see Eqs. (14b) and (14d)].

Peculiarities of these solutions are illustrated by Fig. 2 which show the dependence of $X_{1-4}$ (normalized by $|A_{10}|$) on $z$ (normalized by $\alpha$) as the values of $\varepsilon$ and $N$ are changed. The given dependencies correspond to expressions (25a) and (25b) [(49) for $\varepsilon=0$, Fig. 2(a)], (26) [(50) for $\varepsilon=0$, Fig. 2(b)], (32) [(49) for $\varepsilon>0$, Fig. 2(c)], (33) [(50) for $\varepsilon>0$, Fig. 2(d)], (45) [(50) for $\varepsilon<0$, Fig. 2(e)], and (47) [(52), Figs. 2(f)–2(h)], as well as, to shifted solutions (43) [(49) for $\varepsilon<0$, Figs. 2(i) and 2(j)] and (44) [(51a)–(51e), Figs. 2(k) and 2(l)]. The values of $\varepsilon$ and $N$ used for all the cases are shown on the plane $(\varepsilon,N)$ by the points marked with the number of the corresponding expressions (Fig. 1).

It is easy to see that the sharpest changes in the character of $X_{1-4}(z)$ are observed near to the separatrixes that determine our choice of points for calculations. On the border $\varepsilon=-1$, the amplitude $X_4(z)$ vanishes and expressions (51a)–(51e) and (52) become the classical analytic formulas describing the generation of the wave $A_3$ from the waves $A_{1,2}$ [17]. One can see also that while all solutions shown in Fig. 2 are built by using the fundamental solutions of the first-order Lamé equation (see Ref. [16]), in addition to the period-doubling $2K\rightarrow 4K$, which takes place on passing from the fixed-sign function $\mathrm{dn}(z,k)$ to functions $\mathrm{sn}(z,k)$ and $\mathrm{cn}(z,k)$ having alternating signs, such a doubling is observed here for all components with alternating signs. Here $K=K(k)$ is the complete elliptic integral of the first kind, which determines both the period of the fundamental solutions of the first-order Lamé equation (i.e., the Jacobi elliptic functions [24]) and the period of the above-written analytic solutions.

In the case under consideration, two modes $A_{1,2}$ play the role of two-component pumping which is used to generate two other modes $A_{3,4}$. This means that the possibility of pumping depletion is of great importance. It is easy to see that the intensity of at least one of the pumping components almost always can vanish. The only exclusion is solution (50) for $\varepsilon>0$ [Fig. 2(d)] where the minimal [superscript (min)] pumping intensities are determined by expressions

$$I_1^{(\mathrm{min})} = I_{10}\left[1-\frac{8(\varepsilon+1)}{N(\varepsilon+2)^2}\right], \quad I_2^{(\mathrm{min})} = I_{20}\left(\frac{\varepsilon}{2+\varepsilon}\right)^2. \tag{54}$$

In solutions (50) for $\varepsilon\leqslant 0$ and (52) for $-2/3<\varepsilon\leqslant 0$ the minimal intensity $I_1^{(\mathrm{min})}$ is determined by the same expression (54) and corresponds to points where $I_2=0$ [Figs. 2(e) and 2(f)]. However, for $\varepsilon<-2/3$, solution (52) exhibits two different minima $I_1^{(\mathrm{min})}=I_{10}(\varepsilon N+1)/\varepsilon N$ arising at points where

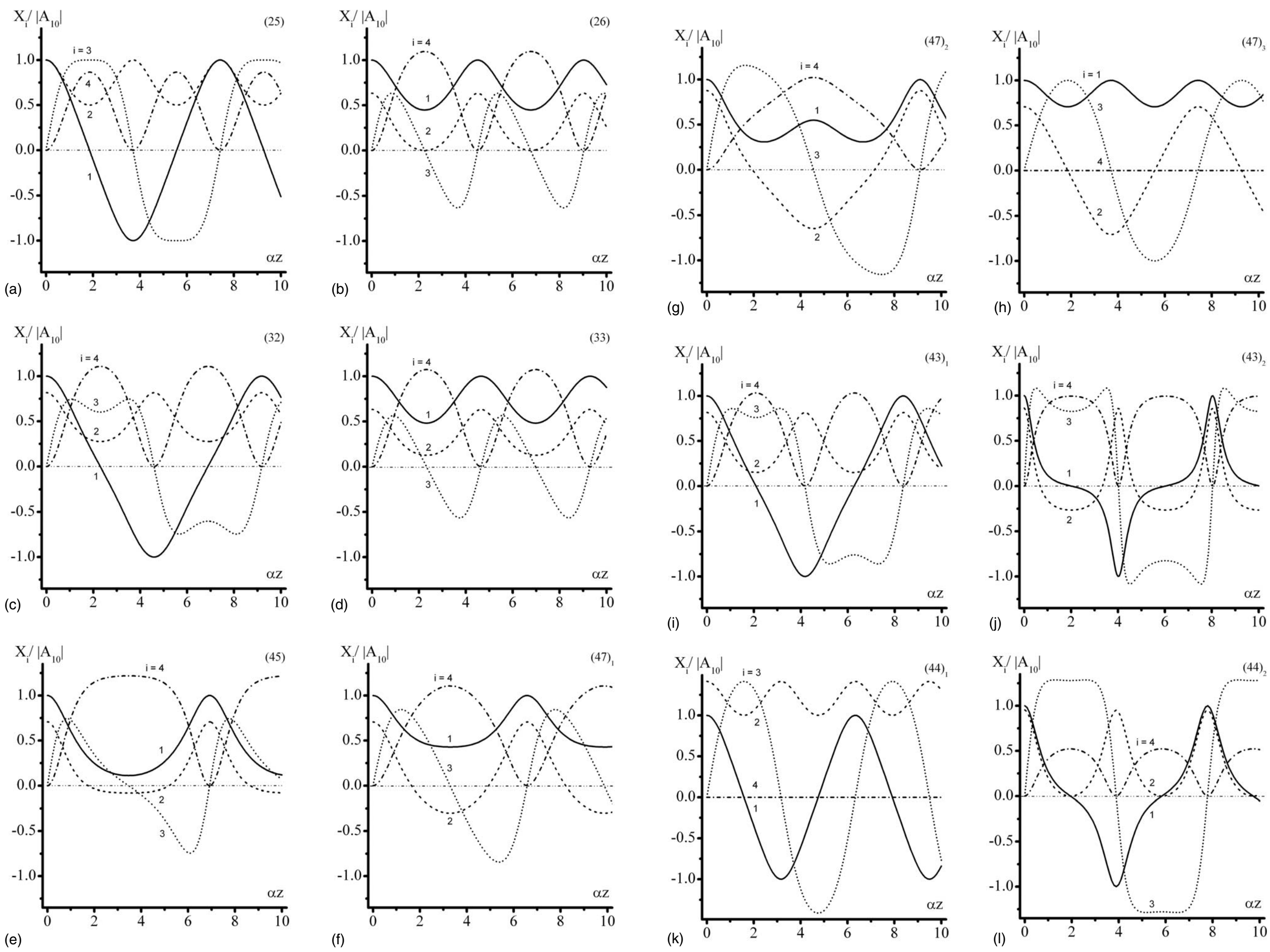


FIG. 2. Evolution of $X_{1–4}(z)$ as the values of parameters $\varepsilon$ and $N$ are changed. $X$ and $Y$ axes are normalized by $\alpha$ and $|A_{10}|$. Shown plots correspond to expressions (25) (a), (26) (b), (32) (c), (33) (d), (45) (e), and (47) [(f)–(h)] as well as to shifted solutions (43) and (44) [(i)–(l)]. The $\varepsilon$ and $N$ values used are shown on the plane $(\varepsilon, N)$ by points marked by the solution number (see Fig. 1).

the intensities $I_{2–4}$ of all other modes are neither minimal nor maximal [Fig. 2(g)]. Notice that the dependence $I_3(z)$ demonstrates extrema of the latter type in almost all the solutions obtained [Figs. 2(b)–2(g), 2(i), and 2(j)]. In solutions (49) and (51a)–(51e), the intensity $I_1^{(\min)}$ is zero, because $X_1(z)$ is an alternating-sign function [Figs. 2(a), 2(c), and 2(i)–2(l)]. Here, the minimal intensity of the second pumping component is determined by the expression

$$I_2^{(\min)} = I_{20}\left(\frac{1+\varepsilon-\sqrt{1+\varepsilon N}}{\varepsilon}\right)^2. \tag{55}$$

It is evident that, because of full covering of the range of possible boundary conditions, the obtained analytic solutions provide one with a possibility to optimize the conversion efficiency in any concrete situation. For example, the choice of $\varepsilon$ and $N$ values corresponding to solution (45) near separatrix $\varepsilon_-(N)$ results in up-conversion to frequency $\omega_4$ with maximal efficiency [Fig. 2(e)].

## VI. CONCLUSIONS

With the use of an approach similar to Ref. [16], we show that in cases when one can neglect the wave mismatch (quasi-phase-matching conditions) the process of parametric interaction of four modes in cascade frequency conversion with quadratic nonlinearity can be also described in terms of an effective cubic nonlinearity. After that, the initial problem is reduced to solving a standard system of two coupled NLSEs for the complex amplitudes of the waves participating in both nonlinear processes [14,19]. This system is fully integrable and can be transformed to two identical independent NLSEs by a simple change of variables. This defines its solutions in an unusual MCW form: as a sum and a difference of two solutions of the same NLSE, identical but for a shift in arguments. Exact analytic solutions obtained in such a way enable one to optimize the conversion efficiency because the range of possible boundary conditions is fully covered.

It is natural, that all the above-written analytic solutions can be obtained in other ways. For example, in the particular case $I_{30}=0$ and $\varphi_{i0}=\text{const}$, a full set of solutions similar to Eq. (25), (26), (32), (33), (43)–(45), and (47) has been obtained by using a traditional unwieldy approach [17]—with successive solution of a classical system of truncated first-order differential equations (14).

Notice here that the technique used above for constructing particular solutions of a system of two NLSEs in the form of a sum and a difference of two identical fundamental solutions with shifted arguments can be rather universal and, as far as we know, has never been used. In our opinion, this technique can be useful in all cases when the variables of a problem under consideration can be separated [25,27]. We emphasize here that exact analytic solutions of a system of two NLSEs in the form (28) have also not been obtained before.


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