PHYSICAL REVIEW E **79**, 026108 (2009)

# Approximations on the Peano river network: Application of the Horton-Strahler hierarchy to the case of low connections

Samuele De Bartolo,[1,*] Francesco Dell'Accio,[2,†] and Massimo Veltri[1,‡]

[1]*Dipartimento di Difesa del Suolo "V. Marone," Università della Calabria, P. Bucci 42B, Rende I-87030, Italy*

[2]*Dipartimento di Matematica, Università della Calabria, P. Bucci 30A, Rende I-87030, Italy*

(Received 20 October 2008; published 20 February 2009; corrected 3 March 2009)

A network analysis is used to investigate the low connections of natural river channels. At the basin scale, the river networks are analyzed according to the Horton-Strahler hierarchy. We propose a quantitative criterion for the average junction degree as a function of a fixed hierarchical order of the network and independent of the usual scaling laws. The numerical results of this analysis are compared with exact results of the Peano river network, showing differences of the order of $10^{-3}$. This aspect is especially relevant for the characterization of transport and diffusion processes at the basin scale.

DOI: 10.1103/PhysRevE.79.026108 PACS number(s): 05.65.+b, 89.75.Hc, 92.40.Qk, 02.10.Ox

## I. INTRODUCTION

Recently, networks have been studied extensively in many sciences; see the reviews in [1,2] and references therein. In the framework of fluvial geomorphology, river channel networks can be considered as systems with a low degree of connections or junctions. In particular, for river channel systems, the role of these connections can also be analyzed using the criterion of the classical Horton-Strahler hierarchy [3,4]. Besides, in the last decade, a parallel growth of many studies conducted on the mechanisms that control the origin and the dynamics of fluvial structures allowed some authors, also in relation with the above-mentioned hierarchy, to relate some fluvial morphometric magnitudes through scaling laws [5]. In these studies a fundamental role is played by the Peano river basin and the associated network [5–11]. This deterministic network, which is a typical fractal plane-filling structure, can be easily constructed through an iterative procedure, in which all steps are strictly related to the Horton-Strahler criterion, and interior and exterior nodes are calculated directly via geometrical series expansions. Recently the Peano network was also adopted to predict the role of hydrologic controls on invasion processes (of species, populations, propagules, or infective agents, depending on the specifics of reaction and transport) occurring in river basins [12,13]. In this context rivers are considered to be ecological corridors in which the reaction and transport parts can be represented by nodes and links, respectively [13].

In this paper, the average junction degree of the network theory [1,2] is used to provide a descriptor useful for the comparison between fluvial Hortonian structures and the Peano river network. The average junction degree of a natural river network depends on the number of source nodes, but not on the order. In contrast, the average junction degree of the Peano river network depends on both the order and the number of source nodes in force of the regularity of the structure. Therefore, in order to compare substructures of the same order, this descriptor is not useful. To overcome this difficulty we introduce the mean average junction degree as the mean value of the average junction degrees calculated on all Hortonian substructures of the same order $\omega$. This descriptor depends on both the order and the number of source nodes. Using this descriptor, we numerically compare the natural river network provided by the Corace river (Calabria, Italy) with the Peano river network; in line with more recent theories [12,13], we find that the results are in very good agreement, with an error of the order of $10^{-3}$.

## II. HORTON-STRAHLER HIERARCHIES AND NETWORK DESCRIPTORS

In a map that contains a well-developed river network, we can identify distinct fluvial segments, properly called *streams*. The streams are ordered according to a hierarchical magnitude scale by assigning them positive integer numbers [3,4], specified as follows. First of all, going downstream on the river, we identify the source streams, which are also called first-order streams. We assign to each stream of this kind the number 1. The confluence of two first-order streams originates a new stream, which is called a second-order stream. We assign them the number 2. These new streams, as well as all streams of equal order or greater, originate and end at two consecutive confluences, except for the terminal stream, which closes the whole basin. In general, the order $\omega$ of a stream which originates from the junction of two streams of order $i$ and $j$, respectively, is given by

$$\omega = \max\{i,j\} + \delta_{ij}, \tag{1}$$

where $\delta$ is the Kronecker delta. To each stream of order $\omega$, we assign the number $\omega$. A sequence of two or more streams of equal order is called a channel of the same order. A substructure of order $\omega$ is any maximal subset of connected streams which does not contain any stream of order greater than $\omega$.

From the mathematical point of view, a river network is a graph that contains no cycles—i.e., a set of connected arcs whose basic elements are the nodes (which enclose the junctions) and the edges (the streams). We distinguish between

*samuele.debartolo@unical.it

†fdellacc@unical.it

‡veltrim@dds.unical.it

 

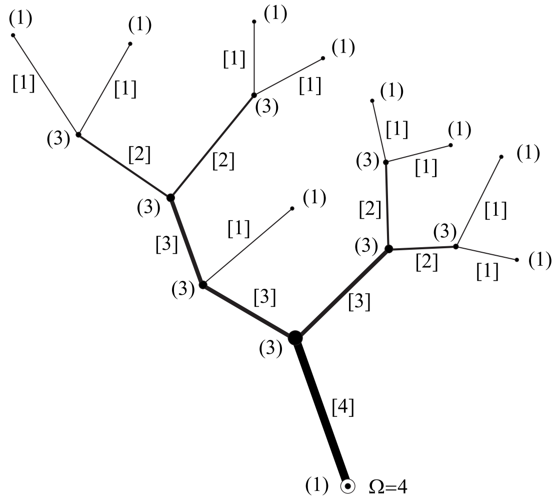


FIG. 1. An example of a Horton-Strahler river network tree. In the figure, the junction degrees (parentheses), the hierarchical order $\omega$ (brackets), and the outlet order $\Omega$ (maximum) are specified.

source nodes (the river heads), internal nodes (the junctions), and the radix or outlet. In general, the number of tributary streams which connect to a single node is equal to 2 and, in this case, we are in the presence of a directed acyclic digraph (DAG). In some cases, however, fractures or junction faults can increase the number of tributary streams to a same junction. The number $n_n$ of nodes in a river network is related to the number $n_s$ of source nodes and to the number $n_j$ of junction nodes by the equation

$$n_n = n_s + n_j + 1. \tag{2}$$

The number of streams, $n_e$, as well as the numbers $n_n$ and $n_j$ can be easily obtained from the number of source nodes as follows:

$$n_j = n_s - 1, \tag{3}$$

$$n_n = 2n_s, \tag{4}$$

$$n_e = n_n - 1 = 2n_s - 1. \tag{5}$$

When the number of tributary streams that join into a single node is greater than 2, as in random graph theory [14] or in modern network theory, more descriptors are needed. A graph which has $k$ nodes, $k-1$ arcs, and no cycles is a tree of order $k$. In the link-based random model by Shreve [15], two river networks are called topologically identical if their schematic map projections can be continuously deformed and rotated in the plane of projection so as to become congruent. In the analysis of river networks by means of graph theory, topologically identical networks are identified.

A local descriptor is the junction degree of a node, denoted by $k_n$, that is the number of its inflow and outflow tributaries. In Hortonian networks, this number is equal to 1 for the source and outlet nodes and it is on average 3 for the interior nodes (Fig. 1). The total junction degree of a network, denoted by $\mathcal{K}$, is the number of the junction degrees over all $n$ nodes:

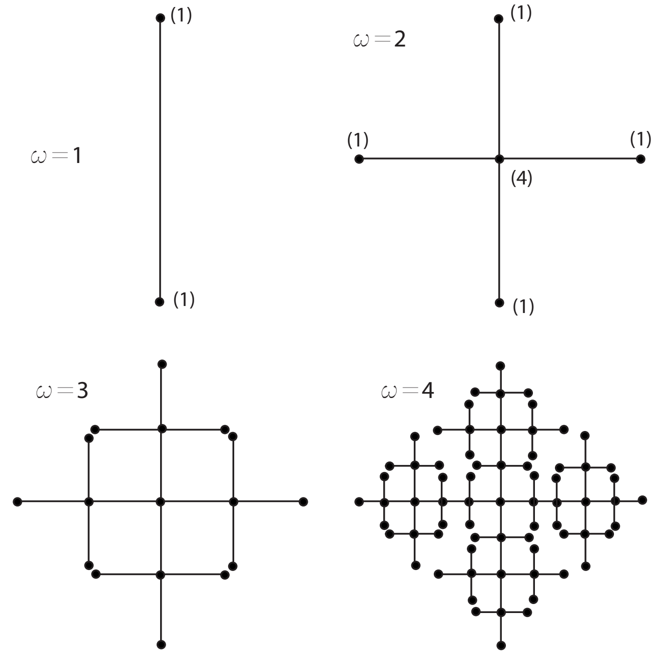


FIG. 2. Construction of the Peano river network for a fourth Horton-Strahler hierarchical order. In the figure is also shown the number of connections for interior and exterior nodes at the first- and second-order steps, respectively.

$$\mathcal{K} = \sum_n k_n. \tag{6}$$

For a tree of order $k$, the following relation holds:

$$\mathcal{K} = 2(k-1). \tag{7}$$

The average junction degree is, by definition [1],

$$\langle k_n \rangle = \frac{\mathcal{K}}{k}, \tag{8}$$

and using Eq. (7) for a tree of order $k$, we find [1]

$$\langle k_n \rangle = 2 - 2/k. \tag{9}$$

In the case of large trees, which are characterized by a very high number of nodes, $\langle k_n \rangle$ approaches 2. Moreover, reading at the right-hand side of Eq. (7) the number of arcs doubled, the average degree can be easily related, through Eqs. (3)–(5) and (8), to the number of junctions, source nodes, or streams, respectively.

### A. Average junction degree for the Peano river network

The relations mentioned in the previous paragraph can also be determined for the Peano river network (see Fig. 2). This deterministic network, as mentioned above, is construed through an iterative procedure: for $\omega=1$, the trivial prefractal case is represented by a single-segment stream with only two nodes; for $\omega=2$, two segments are crossed and four streams are obtained with four exterior and one interior nodes. At the arbitrary stage of generation, $\omega$, we have a total number of $4^{\omega-1}$ links with a number of exterior nodes (the source nodes and outlet), $n^e(\omega)$, given by the relation

TABLE I. Numerical values of $\langle k(\omega)\rangle_P$ for the Peano river network of sixth order.

| Order $\omega$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\langle k(\omega)\rangle_P$ | 1 | 1.6 | 1.88235 | 1.96923 | 1.99222 | 1.99805 |

$$n^e(\omega) = \frac{8}{3}(4^{\omega-2} - 1) + 4 \tag{10}$$

and with a number of interior nodes, $n^i(\omega)$, equal to

$$n^i(\omega) = \frac{1}{3}(4^{\omega-1} - 1). \tag{11}$$

Relations (10) and (11) are obtained directly through geometric series and therefore in consideration of Eq. (5) the total number of nodes (interior plus exterior) is equal to $4^{\omega-1}+1$. According to (8), the average junction degree of the Peano river network is given by

$$\langle k_n(\omega)\rangle_P = \frac{1 \times n^e(\omega) + 4 \times n^i(\omega)}{4^{\omega-1} + 1} = \frac{2^{2\omega+1}}{4 + 4^{\omega}}. \tag{12}$$

In Eq. (12) the integer factors 1 and 4 are the junction degrees of the external and interior nodes, respectively. In Table I are shown (for example), for the Peano river network of sixth order, the values of $\langle k_n(\omega)\rangle_P$ obtained through the above equation. By considering the integer function $\langle k_n(\omega)\rangle_P$ as a function of one real (complex) variable, we find that it satisfies the initial-value problem for the Bernoulli differential equation of first order:

$$\frac{d\langle k_n(\omega)\rangle_P}{d\omega} - \lambda\langle k_n(\omega)\rangle_P = -\lambda\beta\langle k_n(\omega)\rangle_P^2,$$

$$\langle k_n(1)\rangle_P = 1, \tag{13}$$

in correspondence with the values of the parameters

$$\lambda = \ln(4), \quad \beta = \frac{1}{2}. \tag{14}$$

The graph of the integer function $\langle k_n(\omega)\rangle_P$ for a sixth-order Peano river network is shown in Fig. 3.

In general—that is, in the case of a generic river network—the index $\langle k_n\rangle$ of a specific substructure of order $\omega$ depends only on the number of source nodes of the substructure and enough from the order, unless for a minimum value

$$\frac{(2^{\omega-1}+1) \times 1 + (2^{\omega-1}-1) \times 3}{2^{\omega}} = \frac{2 \times 2^{\omega-1} - 1}{2^{\omega-1}},$$

which corresponds to the minimum number of source nodes, $2^{\omega-1}$, that realizes a substructure of order $\omega$. On the contrary the regularity of the iterative scheme which generates the Peano river network (see Fig. 2) links the index $\langle k_n\rangle$ to the order of the substructure as well as to the number of its source nodes. In this case, in fact, each substructure of order $\omega$ has exactly a fixed number of source nodes which can be calculated by the relation (10).

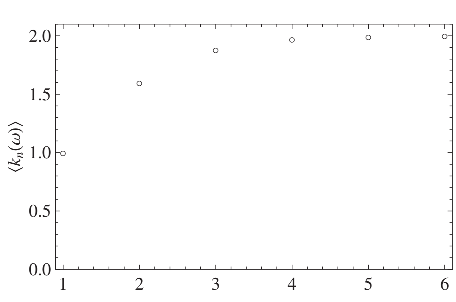


FIG. 3. $\langle k_n(\omega)\rangle$ versus $\omega$ for the Peano river network.

TABLE II. Number $\tilde{N}_\omega$ of substructures of the same order $\omega$ extracted from the Corace river network according to the Horton-Strahler hierarchy.

| Order $\omega$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Number $\tilde{N}_\omega$ | 3277 | 803 | 178 | 39 | 7 | 1 |

### B. Average junction degree for the Horton-Strahler hierarchy

In order to numerically compare a generic river network with the Peano river network in relation to the totality of their substructures of a fixed order, we introduce a descriptor. This descriptor is neither a local descriptor nor a global descriptor; in fact, it is a *partial* descriptor and it is defined as follows.

For a generic river network, we denote by $n(\omega,i)$ the number of all substructures of order $\omega$ with $i$ source nodes. We denote by $\mathbf{n}(\omega)=[\cdot]_\omega$ the vector of occurrences to the order $\omega$ of a partitioned river network according to the Horton-Strahler hierarchy:

$$\mathbf{n}(\omega) = [n(\omega,2^{\omega-1}), n(\omega,2^{\omega-1}+1), \ldots, n(\omega,2^{\omega-1}+L-1)]_\omega,$$

$$n(\omega,2^{\omega-1}+L-1) \neq 0$$

and

$$n(\omega,j) = 0$$

for any

$$j \geqslant 2^{\omega-1} + L. \tag{15}$$

In the definition of $\mathbf{n}(\omega)$, the values $n(\omega,i)$, $1 \leqslant i < 2^{\omega-1}$, as well as the values $n(\omega,j)$, $j \geqslant 2^{\omega-1}+L$, do not appear since they are all vanishing and not meaningful. $L$ is the length of the vector $\mathbf{n}(\omega)$—i.e., the number of its components.

The average junction degree of a substructure of order $\omega$ with $i$ source nodes is

$$\langle k_n(\omega,i)\rangle = \frac{2i-1}{i}. \tag{16}$$

We introduce the average junction degree of the river network to the order $\omega=1,\ldots,\Omega$ through the equation

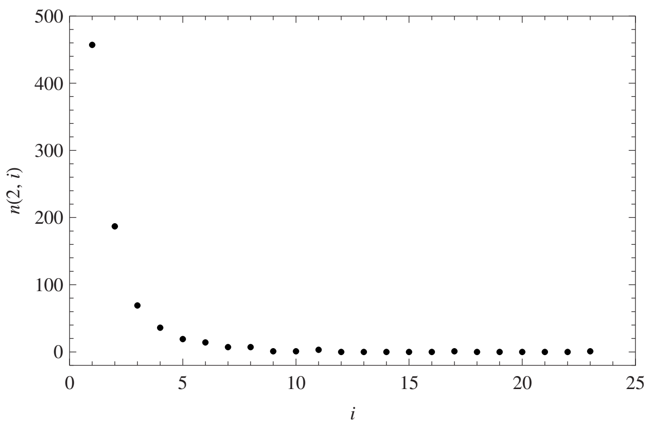


FIG. 4. Occurrences $n(2,i)$ for the Corace river network.

$$\langle k_n(\omega)\rangle := \frac{\displaystyle\sum_{i=2^{\omega-1}}^{2^{\omega-1}+L-1} n(\omega,i)\frac{2i-1}{i}}{\displaystyle\sum_{i=2^{\omega-1}}^{2^{\omega-1}+L-1} n(\omega,i)}. \tag{17}$$

We note that $\langle k_n(\Omega)\rangle=\langle k_n\rangle$ due to the requirement of maximality in the definition of substructures; in general, for each $\omega=1,\ldots,\Omega$, $\langle k_n(\omega)\rangle$ is a rational number which satisfies the following properties:

(a) $\frac{2\times 2^{\omega-1}-1}{2^{\omega-1}} \leqslant \langle k_n(\omega)\rangle < 2$,

(b) $\sup_{\mathbf{n}(\omega)}\langle k_n(\omega)\rangle = 2$.

In statement (b) we assume that the vector of occurrences, $\mathbf{n}(\omega)$, varies in the totality of the Hortonian structures. In order to verify property (a), we note that, for a network whose vector of occurrences to the order $\omega$ is $[1]_\omega$, it yields

$$\langle k_n(\omega)\rangle = \frac{2\times 2^{\omega-1}-1}{2^{\omega-1}}.$$

On the other hand, if the vector of occurrences to order $\omega$ is $[0,\ldots,0,1]_\omega$ of length $L$, then

$$\langle k_n(\omega)\rangle = \frac{2\times 2^{\omega-1}+2L-3}{2^{\omega-1}+L-1} \geqslant \frac{2\times 2^{\omega-1}-1}{2^{\omega-1}}.$$

Now, let us suppose that the vector of occurrences to the order $\omega$ is the generic $[n(\omega,2^{\omega-1}),n(\omega,2^{\omega-1}+1),\ldots,n(\omega,2^{\omega-1}+L-1)]_\omega$. In this case from the definition of $\langle k_n(\omega)\rangle$ we have

$$\langle k_n(\omega)\rangle \geqslant \sum_{i=2^{\omega-1}}^{2^{\omega-1}+L-1}\left(\frac{n(\omega,i)}{\displaystyle\sum_{i=2^{\omega-1}}^{2^{\omega-1}+L-1} n(\omega,i)}\right)\frac{2\times 2^{\omega-1}-1}{2^{\omega-1}} = \frac{2\times 2^{\omega-1}-1}{2^{\omega-1}} \tag{18}$$

and

FIG. 5. Hortonian substructures of the Corace river network for the order $\omega=2$.

$$\langle k_n(\omega)\rangle \leqslant \sum_{i=2^{\omega-1}}^{2^{\omega-1}+L-1}\left(\frac{n(\omega,i)}{\displaystyle\sum_{i=2^{\omega-1}}^{2^{\omega-1}+L-1} n(\omega,i)}\right)\frac{2\times 2^{\omega-1}+2L-3}{2^{\omega-1}+L-1} = \frac{2\times 2^{\omega-1}+2L-3}{2^{\omega-1}+L-1} < 2. \tag{19}$$

That is, statement (a) holds. Statement (b) follows from (19) and from the fact that if the vector of occurrences to order $\omega$ is $[0,\ldots,0,1]_\omega$ of length $L$, we find

$$\lim_{L\to\infty}\langle k_n(\omega)\rangle = \lim_{L\to\infty}\frac{2\times 2^{\omega-1}+2L-3}{2^{\omega-1}+L-1} = 2.$$

## III. APPLICATION TO THE CORACE RIVER NETWORK: ANALYSIS OF THE RESULTS

In the case of a natural river network of order $\Omega$ and for each order $\omega\leqslant\Omega$, the value of $\langle k_n(\omega)\rangle$ can be compared with the value $\langle k_n(\omega)\rangle_P$ given by Eq. (12) which is the average junction degree to the order $\omega$ for a Peano river network of order $\Omega$.

By means of the new descriptor, we have compared the Corace river network with the Peano river network. More precisely, we have digitalized on a 1:25 000 scale the Corace

TABLE III. Numerical results of $\langle k_n(\omega)\rangle_C$ for the Corace river network and their comparisons with the Peano river network.

| Order $\omega$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\langle k(\omega)\rangle_P$ | 1 | 1.600 | 1.882 | 1.969 | 1.992 | 1.998 |
| $\langle k(\omega)\rangle_C$ | 1 | $1.598\pm0.013$ | $1.887\pm0.02$ | $1.971\pm0.01$ | $1.995\pm0.002$ | $1.999\pm0.001$ |
| $\Delta$ | 0 | 0.001 | 0.003 | 0.001 | 0.001 | 0.001 |

river network maps produced by the Italian Military Geographic Institute [Istituto Geografico Militare (IGM)] through the aerial photo restitution technique. In these maps, which are in fact two-dimensional projections of the real three-dimensional structure of the basin, the river network is represented with the blue lines (drainage channel networks). The blue lines have been digitalized with an average spatial resolution corresponding to about 5 m at the real scale. After this preliminary operation the river network was partitionated through the Horton-Strahler hierarchy criterion. By this procedure we found that, for the Corace river network, $\Omega=6$; afterward, we computed the numbers $\tilde{N}_\omega$ of all substructures of order $\omega=1,\ldots,\Omega$ as shown in Table II and the vectors of occurrences, $\mathbf{n}_C(\omega)$, to the orders $\omega=1,\ldots,\Omega$ as shown in Fig. 4 for the special case of $\omega=2$. In Fig. 5 are represented, as an example, the 803 substructures at second order with the corresponding subareas of drainage. The computation of the average junction degrees $\langle k_n(\omega)\rangle_C$ for the Corace river network has been carried out for any set of substructures of order $\omega=1,\ldots,6$ through Eq. (17). The numerical results are shown in Table III, where the values of $\langle k_n(\omega)\rangle_C$ are presented with their respective errors and with the relative differences $\Delta$ from the corresponding values $\langle k_n(\omega)\rangle_P$ for the Peano river network. The errors in $\langle k_n(\omega)\rangle_C$ are computed by means a Poissonian distribution of the counting $n(\omega,i)$, with a corresponding error $\sqrt{n(\omega,i)}$. Such errors are then propagated through standard methods. From the analysis of the results presented in Table III, we note that the order of magnitude of the relative differences is $10^{-3}$. The values of $\langle k_n(\omega)\rangle_C$ are then close to the exact values of $\langle k_n(\omega)\rangle_P$. In Fig. 6 these values are graphically compared for all orders $\omega$.

The validity of relation (17) can be further tested. This formula allows for the computation of the average junction degree to a given order $\omega$ by using the information provided only through the source nodes of the substructures of that order. On the other hand, for any given order $\omega$ and for any given vector of non-negative integers $\mathbf{n}=(n_1,n_2,\ldots,n_L)$, there exists a network with $n_1$ substructures of order $\omega$ each with $2^{\omega-1}$ source nodes (the minimum number), $n_2$ substructures of order $\omega$ each with $2^{\omega-1}+1$ source nodes, and so on; by applying formula (17) to the vector $\mathbf{n}$, the number $\langle k_n(\omega)\rangle$ for that network can be computed. In the framework of random graph theory [14] and in consideration of the Shreve theory [15], the vector $\mathbf{n}$ can be obtained in a random way through the relation $\mathbf{n}=\mathrm{round}(M*\mathrm{random}(L))$, where $M$ is a positive real number which represents the maximum magnitude of the component data of $\mathbf{n}$ and $L$ is the number of components of $\mathbf{n}$. In Table IV the relative differences $\Delta$ between $\langle k_n(\omega)\rangle$ for the vector $\mathbf{n}$ and $\langle k_n(\omega)\rangle_P$ are shown, in correspondence with the orders $\omega=2,\ldots,\Omega-1$ and the specified values of $M$ and $L$. The values of $M$ and $L$ have been chosen as much as possible to be in agreement with the values provided by the partition of the Corace river network given in Table II according to the following criteria: for a given order $\omega=2,\ldots,\Omega-1$, $L$ is the number of components of the vector of occurrences, $\mathbf{n}_C(\omega)$; $M_1$ is the maximum of the components of $\mathbf{n}_C(\omega)$; and $M_2$ is chosen to minimize the differences between the averages of the components of the random vector $\mathbf{n}$ and the vector $\mathbf{n}_C(\omega)$. The analysis of the results presented in Table IV shows a greater difference with the Peano network for lower values of $\omega$ and, for higher values of $\omega$, a better agreement of results, if compared with those of Table III.

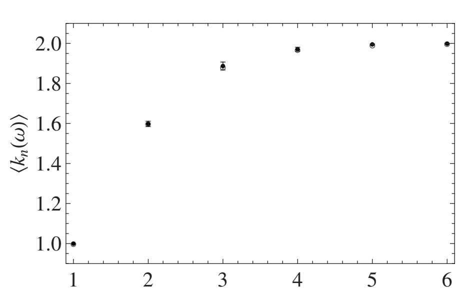

FIG. 6. $\langle k_n(\omega)\rangle$ versus $\omega$ for the Corace river network with error bars and Peano river network (circles), respectively.

Taking into account the Corace case, we note that the behavior of $\langle k_n(\omega)\rangle$ for a generic fluvial river network can be analyzed directly. In fact, the discrete data $\langle k_n(\omega)\rangle$, $\omega=1,\ldots,\Omega$, can be computed; these data are then used as input in a procedure of nonlinear best-fitting approximation with a suitable family of functions. The family of functions

TABLE IV. The relative differences $\Delta$ between $\langle k_n(\omega)\rangle$ for the vector $\mathbf{n}$ and $\langle k_n(\omega)\rangle_P$, in correspondence with the orders $\omega=2,\ldots,\Omega-1$ and the specified values of $M$ and $L$. For a given order $\omega=2,\ldots,\Omega-1$, $L$ is the number of components of the vector of occurrences, $\mathbf{n}_C(\omega)$; $M_1$ is the maximum of the components of $\mathbf{n}_C(\omega)$; and $M_2$ is chosen to minimize the differences between the averages of the components of the random vector $\mathbf{n}$ and the vector $\mathbf{n}_C(\omega)$.

| Order $\omega$ | $L$ | $M_1$ | $\Delta$ | $M_2$ | $\Delta$ |
|---|---|---|---|---|---|
| 2 | 23 | 457 | 0.172 | 69.83 | 0.172 |
| 3 | 40 | 19 | 0.028 | 8.90 | 0.032 |
| 4 | 201 | 3 | 0.007 | 0.62 | 0.008 |
| 5 | 539 | 1 | 0.007 | 0.51 | 0.002 |

is chosen according to the exact case provided by the Peano network: these functions satisfy the differential equation (13) with specific parameter values $\beta$, $\lambda$, and $\gamma$:

$$\langle k_n(\omega)\rangle = \frac{1}{\beta + \gamma e^{-\lambda\omega}}. \tag{20}$$

Note that, in force of the initial condition at $\omega=1$, $\beta$, $\gamma$, and $\lambda$ are related by the equation

$$\beta + \gamma e^{-\lambda} = 1. \tag{21}$$

By using a nonlinear regression, we found for the Corace case the following set of values: $\beta=0.499\,361$, $\gamma=2.000\,53$, and $\lambda=1.383\,99$ with an estimated variance of $5.86\times 10^{-6}$. In this case we also found $\beta+\gamma e^{-\lambda}=1.0006$. Therefore, we suggest that the relation (20) can be used as a general law for a description of the dependences of $\langle k_n(\omega)\rangle$ on the Hortonian order $\omega$.

## IV. CONCLUSIONS

The proposed analysis shows a correspondence between the structures of the Corace river network and the Peano river network: lower values of the hierarchical order produce numerical results with differences of order $10^{-3}$. Although the analyzed case is quite specific, we suggest that the average junction degree, introduced here, is a robust descriptor for the physical characterization, at the basin scale, of the relevant transport processes, independently of the knowledge of scaling laws, and thus in general. Consequently it can be integrated with the recent theories on transport-invasion processes to provide a more accurate modeling of them. This analysis can be also used in order to characterize, at the channel scale, other hydraulic and fluvial dynamics seen as manifestations of complex systems.

## ACKNOWLEDGMENTS

The authors wish to thank the anonymous reviewers for their constructive criticism and suggestions, from which this article has benefited richly.

[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[2] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[3] R. E. Horton, Trans., Am. Geophys. Union **13**, 450 (1932); Geol. Soc. Am. Bull. **56**, 275 (1945).
[4] A. N. Strahler, Geol. Soc. Am. Bull. **63**, 1117 (1952); Trans., Am. Geophys. Union **38**, 913 (1957); Geol. Soc. Am. Bull. **69**, 279 (1958).
[5] I. Rodriguez-Iturbe and A. Rinaldo, *Fractal River Basins. Chance and Self-Organization* (Cambridge University Press, New York, 1997); A. Rinaldo, I. Rodriguez-Iturbe, and R. Rigon, Annu. Rev. Earth Planet Sci. **26**, 289 (1998); P. S. Dodds and D. H. Rothman, Phys. Rev. E **59**, 4865 (1999); Annu. Rev. Earth Planet Sci. **28**, 571 (2000); Phys. Rev. E **63**, 016115 (2000); Phys. Rev. E **63**, 016116 (2000); **63**, 016117 (2000); S. G. De Bartolo, L. Primavera, R. Gaudio, A. D'Ippolito, and M. Veltri, *ibid.* **74**, 026101 (2006).
[6] B. B. Mandelbrot, *Fractals* (Freeman, San Francisco, 1977).
[7] A. Marani, R. Rigon, and A. Rinaldo, Water Resour. Res. **27**, 3041 (1991).
[8] A. Flammini and F. Colaiori, J. Phys. A **29**, 6701 (1996).
[9] S. A. Veitzer and V. K. Gupta, Adv. Water Resour. **24**, 955 (2001).
[10] B. M. Troutman and T. M. Over, Adv. Water Resour. **24**, 967 (2001).
[11] L. T. Tay, B. S. D. Sagar, and H. T. Chuah, Water Resour. Res. **42**, W06502 (2006).
[12] D. Campos, J. Fort, and V. Méndez, Theor Popul. Biol. **69**, 88 (2006).
[13] E. Bertuzzo, A. Maritan, M. Gatto, I. Rodriguez-Iturbe, and A. Rinaldo, Water Resour. Res. **43**, W04419 (2007); E. Bertuzzo, S. Azaele, A. Maritan, M. Gatto, I. Rodriguez-Iturbe, and A. Rinaldo, *ibid.* **44**, W01424 (2008).
[14] P. Erdős and A. Renyi, Publ. Math. (Debrecen) **6**, 290 (1959); Publ. Math., Inst. Hautes Etud. Sci. **5**, 17 (1960); Bull. Internat. Statist. Inst. **38**, 343 (1961).
[15] R. L. Shreve, J. Geol. **74**, 17 (1966); **75**, 178 (1967); **77**, 397 (1969).