Relativistic diffusion

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We discuss relativistic diffusion in proper time in the approach of Schay (Ph.D. thesis, Princeton University, Princeton, NJ, 1961) and Dudley [Ark. Mat. **6**, 241 (1965)]. We derive (Langevin) stochastic differential equations in various coordinates. We show that in some coordinates the stochastic differential equations become linear. We obtain momentum probability distribution in an explicit form. We discuss a relativistic particle diffusing in an external electromagnetic field. We solve the Langevin equations in the case of parallel electric and magnetic fields. We derive a kinetic equation for the evolution of the probability distribution. We discuss drag terms leading to an equilibrium distribution. The relativistic analog of the Ornstein-Uhlenbeck process is not unique. We show that if the drag comes from a diffusion approximation to the master equation then its form is strongly restricted. The drag leading to the Tsallis equilibrium distribution satisfies this restriction whereas the one of the Jüttner distribution does not. We show that any function of the relativistic energy can be the equilibrium distribution for a particle in a static electric field. A preliminary study of the time evolution with friction is presented. It is shown that the problem is equivalent to quantum mechanics of a particle moving on a hyperboloid with a potential determined by the drag. A relation to diffusions appearing in heavy ion collisions is briefly discussed.

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I. INTRODUCTION

There were many attempts to generalize the diffusion in a way respecting relativistic invariance and causality [1–5]; for a review and further references, see [6,7]. In this paper we develop the approach initiated by Schay [1] and Dudley [2]. The diffusion process which should be considered as a relativistic analog of the Brownian motion is uniquely defined by the requirement that this is the diffusion whose four-momentum stays on the mass shell. We discuss the Ito stochastic differential equation [8] defined as a perturbation of the relativistic dynamics on the phase space. As an example we consider the motion in an electromagnetic field [9] (its generalization, a motion in the Yang-Mills field [10,11], could be treated in a similar way). Relativistic stochastic dynamics preserving the particle's mass has no normalizable Lorentz invariant equilibrium measure. We discuss drags which lead to an equilibrium probability measure for a large time but violate the Lorentz invariance. A stochastic process with such a drag would be an analog of the Ornstein-Uhlenbeck process. Covariant drags describing a relativistic particle in a medium moving with a velocity V are discussed in [4]. In such a case the Lorentz transformation of the friction is compensated by the transformation of the velocity V.

A diffusion process can be considered as a relativistic approximation to more complex many particle processes. In particular, a motion of a heavy particle in an environment of a gas of light particles, described in a Markovian approximation by the master equation [12], could be approximated by a diffusion process [13]. Such an approximation is applied in a description of the quark-gluon plasma [11,14–18], for an electron in a background cosmic radiation [19,20] or a particle moving in a fluctuating metric [21]. We show that an equilibrium distribution consistent with the diffusion approximation to the master equation is severely restricted. The Tsallis [22] distribution satisfies this restriction whereas the Jüttner distribution [23] and quantum distributions do not. We discuss the form of the relativistic diffusion equation and compare it to the relativistic diffusions discussed in heavy ion collisions.

We explicitly work out solutions to the stochastic dynamics and its transition function in various coordinate systems. We determine the momentum distribution of a relativistic particle in external electromagnetic fields. We discuss the time evolution with a friction leading to an equilibrium. We show that such a dynamics is equivalent to an imaginary time evolution of a quantum mechanical particle moving on the hyperboloid in a potential determined by the drag. We expect that standard quantum mechanics methods can be applied for a detailed approximation of the diffusive evolution. The explicit formulas may be useful for a comparison of theoretical predictions with experimental results of ultra-relativistic collisions when a gas of relativistic particles is formed.

II. RELATIVISTIC DYNAMICS

We are interested in random perturbations of the dynamics of relativistic particles of mass m. On the Minkowski space the dynamics of a relativistic particle is described by the equations [9]

$$\frac{dx^{\mu}}{d\tau}=\frac{1}{m}p^{\mu}, \tag{1}$$

$$\frac{dp^{\mu}}{d\tau}=K^{\mu}, \tag{2}$$

where $\mu=0,1,2,3$ and K^{μ} is a force. The four-momentum $p(\tau)$ of a relativistic particle defines the mass by the relation

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$$p^2=p_0(\tau)^2-p_1(\tau)^2-p_2(\tau)^2-p_3(\tau)^2=m^2c^2. \quad (3)$$

Equation (3) (together with the positivity of energy) says that the momenta stay on the upper half $\mathcal{H}_+$ (defined by $p_0\geqslant 0$) of the four-dimensional hyperboloid $\mathcal{H}$. If Eq. (3) is satisfied then from Eq. (1) it follows that τ has the meaning of the proper time. If Eq. (3) is to be true then the force K^μ must satisfy the subsidiary condition

$$p_\mu K^\mu=0 \quad (4)$$

(we use the convention of a summation over repeated indices). We add a random force $k_\mu d\tau=dp_\mu^H$ to Eq. (2) writing it in the form

$$dp_\mu=K_\mu d\tau+dp_\mu^H. \quad (5)$$

The diffusion $p^H(\tau)$ on the hyperboloid $\mathcal{H}_+$ is uniquely defined. It is generated by the Laplace-Beltrami operator on $\mathcal{H}$,

$$\Delta_H=\frac{1}{\sqrt{g}}\partial_j g^{jk}\sqrt{g}\partial_k. \quad (6)$$

Here, $g=\det(g_{jk})$ and g_{jk} is the metric on $\mathcal{H}$. If we define the expectation value over the sample paths [starting from (x,p)] of the diffusion process $\phi_\tau(x,p)=E[\phi(x(\tau),p(\tau))]$ (we denote the expectation values by $E[\cdots]$) then ϕ_τ satisfies the diffusion equation

$$\partial_\tau\phi_\tau=\left(\frac{p^\mu}{m}\frac{\partial}{\partial x^\mu}+K^\mu\frac{\partial}{\partial p^\mu}+\frac{\gamma^2}{2}\Delta_H\right)\phi_\tau \quad (7)$$

with the initial condition ϕ. γ^2 has the meaning of a diffusion constant. We write it in the form

$$\gamma=mc\kappa. \quad (8)$$

Then, κ^{-2} has the dimension of time.

III. COORDINATES ON $\mathcal{H}$

A proper choice of coordinates may be useful for a solution of differential equations. The momenta (3) $\mathbf{p}$ (with $p_0=\sqrt{\mathbf{p}^2+m^2c^2}$) could be used as coordinates on $\mathcal{H}_+$. Then, the metric tensor can be obtained from the embedding of $\mathcal{H}$ in R^4. We obtain [expressing dp_0 by dp_k from Eq. (3)]

$$g_{jk}=\delta^{jk}-p_0^{-2}p^jp^k. \quad (9)$$

Then, $g=m^2c^2p_0^{-2}$ and

$$g^{jk}=\delta^{jk}+(mc)^{-2}p^jp^k. \quad (10)$$

Hence

$$\Delta_H=\partial_1^2+\partial_2^2+\partial_3^2+(mc)^{-2}p^jp^k\partial_j\partial_k+(mc)^{-2}3p^k\partial_k, \quad (11)$$

where $k=1,2,3$ and $\partial_j=\frac{\partial}{\partial p^j}$. We prefer another choice of coordinates (p_+,p_a) (where $a=1,2$),

$$p_+=p_0+p_3.$$

In such a case $g_{++}=p_+^{-2}(p_ap_a+m^2c^2)$, $g_{+a}=-p_ap_+^{-1}$, and $g_{ab}=\delta_{ab}$. Then, $g=m^2c^2p_+^{-2}$, $g^{++}=(mc)^{-2}p_+^2$, $g^{+a}=(mc)^{-2}p_+p_a$, $g^{aa}=1+(mc)^{-2}p_a^2$, $g^{12}=(mc)^{-2}p_1p_2$. Hence

$$\begin{aligned}\Delta_H&=\partial_1^2+\partial_2^2+(mc)^{-2}p_+^2\partial_+^2+(mc)^{-2}p_1^2\partial_1^2\\&\quad+(mc)^{-2}p_2^2\partial_2^2+2(mc)^{-2}p_+p_a\partial_a\partial_++2(mc)^{-2}p_1p_2\partial_1\partial_2\\&\quad+(mc)^{-2}3p_+\partial_++(mc)^{-2}3p_a\partial_a,\end{aligned} \quad (12)$$

where $\partial_a=\frac{\partial}{\partial p_a}$. The formula (12) can be rewritten in a Lorentz invariant form,

$$\Delta_H=-3(mc)^{-2}p^\mu\partial_\mu+(mc)^{-2}(p^\mu p^\nu-\eta^{\mu\nu}p^2)\partial_\mu\partial_\nu, \quad (13)$$

where $\eta^{\mu\nu}$ is the Minkowski metric. In order to derive Eq. (12) from Eq. (13), assume that $p^2=m^2c^2$ and the function ϕ in Eq. (7) is expressed as a function of p_+, p_1, p_2.

It is instructive to compare $\mathbf{p}$ with some other widely used coordinates (a change of coordinates in the relativistic diffusion equation is also discussed in [6,24]). First, let us consider the analogues of spherical coordinates,

$$p_0=mc\cosh\alpha,$$

$$p_1=mc\sinh\alpha\cos\phi\sin\theta,$$

$$p_2=mc\sinh\alpha\sin\phi\sin\theta,$$

$$p_3=mc\sinh\alpha\cos\theta. \quad (14)$$

In these coordinates the metric is expressed as

$$ds^2=(mc)^2[d\alpha^2+(\sinh\alpha)^2ds_2^2], \quad (15)$$

where

$$ds_2^2=d\theta^2+(\sin\theta)^2d\phi^2 \quad (16)$$

is the metric on the sphere S_2. The Laplace-Beltrami operator reads

$$(mc)^2\Delta_H=(\sinh\alpha)^{-2}\partial_\alpha(\sinh\alpha)^2\partial_\alpha+(\sinh\alpha)^{-2}\Delta_{S_2}, \quad (17)$$

where Δ_{S_2} is the Laplace-Beltrami operator on the sphere. Next, let us consider the Poincaré coordinates (q_1,q_2,q_3) on $\mathcal{H}_+$ which are related to momenta p as follows:

$$p_3+p_0=\frac{mc}{q_3},$$

$$p_3-p_0=-\frac{mc}{q_3}(q_1^2+q_2^2+q_3^2),$$

$$p_1=\frac{mcq_1}{q_3},$$

$$p_2=\frac{mcq_2}{q_3}, \quad (18)$$

where $q_3\geqslant 0$. Then, the metric is

$$ds^2=(mc)^2q_3^{-2}(dq_1^2+dq_2^2+dq_3^2) \quad (19)$$

and

$$(mc)^2\Delta_H=q_3^2(\partial_1^2+\partial_2^2+\partial_3^2)-q_3\partial_3. \quad (20)$$

IV. STOCHASTIC EQUATIONS

A solution of the diffusion equation as well as expectation values of observables can be expressed by an expectation value over the solution of stochastic equations [8]. We have discussed stochastic equations corresponding to the diffusion on $\mathcal{H}_+$ and their solutions in [25,26]. The stochastic equations on $\mathcal{H}_+$ are also discussed in [6,27,24]. We solve these equations in the case of the free motion ($K=0$) using the Poincaré coordinates or the light-cone coordinates. In the Poincaré coordinates the diffusion process is a solution of the linear stochastic differential equations,

$$dq_a = \kappa q_3 db_a, \tag{21}$$

$a=1,2$,

$$dq_3 = -\frac{\kappa^2}{2} q_3 d\tau + \kappa q_3 db_3 = -\kappa^2 q_3 d\tau + \kappa q_3 \circ db_3,$$

where Stratonovitch differentials are denoted by a circle and the Ito stochastic differentials without the circle (the notation is the same as in [8]). The Brownian motion appearing on the right-hand side of Eqs. (21) is defined as the Gaussian process with the covariance

$$E[b_a(\tau) b_c(s)] = \delta_{ac} \min(\tau, s). \tag{22}$$

The solution of Eqs. (21) is

$$q_3(\tau) = \exp[-\kappa^2 \tau + \kappa b_3(\tau)] q_3 \tag{23}$$

and

$$q_a(\tau) = q_a + \kappa \int_0^\tau q_3(s) db_a(s) \tag{24}$$

for $a=1,2$. The solution could be applied for a calculation of correlation functions and the transition function. The transition function P_τ of the diffusion is a solution of the equation

$$\partial_\tau P = \frac{\gamma^2}{2} \Delta_H P \tag{25}$$

with the initial condition $P_0(q,q') = g^{-1/2} \delta(q-q')$. We have [8] (we calculated the transition function from the solution of the stochastic equations in [26])

$$P_\tau(\sigma) = (2\pi\kappa^2\tau)^{-3/2} \sigma (\sinh \sigma)^{-1} \exp\left(-\frac{\kappa^2 \tau}{2} - \frac{\sigma^2}{2\kappa^2 \tau}\right), \tag{26}$$

where the geodesic distance σ in the Poincaré coordinates can be expressed in the form

$$\cosh \sigma = 1 + (2 q_3 q_3')^{-1} [(q_1 - q_1')^2 + (q_2 - q_2')^2 + (q_3 - q_3')^2]. \tag{27}$$

Using Eqs. (18) we can derive the differentials dp_μ^H. In the light-cone coordinates

$$p_\pm = p_0 \pm p_3 \tag{28}$$

we have

$$dp_+ = \kappa^2 p_+ d\tau + \kappa p_+ \circ db_+ = \frac{3\kappa^2}{2} p_+ d\tau + \kappa p_+ db_+ \tag{29}$$

(here we denoted b_3 by b_+),

$$dp_a = \frac{3}{2} \kappa^2 p_a d\tau + \kappa p_a db_+ + \gamma db_a, \tag{30}$$

where $a=1,2$.

p_- can be obtained from the formula

$$p_- = (m^2 c^2 + p_a p_a) p_+^{-1}. \tag{31}$$

Then

$$dp_- = \kappa^2 \left(2p_- - \frac{3m^2c^2}{p_+}\right) d\tau + \kappa \left(p_- \frac{-2m^2c^2}{p_+}\right) \circ db_+ + \frac{2\gamma}{p_+} p_a \circ db_a.$$

Let

$$\phi_\tau(p) = E[\phi(p(\tau))], \tag{32}$$

where $p(\tau)$ is the solution of stochastic equations (29) and (30) with the initial condition p. Then,

$$\partial_\tau \phi_\tau = \frac{\gamma^2}{2} \Delta_H \phi_\tau, \tag{33}$$

where Δ_H is defined in Eq. (12).

The spatial momenta which are useful for a physical interpretation of the relativistic diffusion lead to a nonlinear (Ito) Langevin equation,

$$dp^j = \frac{3}{2} \kappa^2 p^j d\tau + e_n^j(\mathbf{p}) db^n, \tag{34}$$

where

$$g^{jk} = e_n^j e_n^k \tag{35}$$

with

$$e_n^j = \delta^{jn} + (p_0 - mc)(mc)^{-1} \mathbf{p}^{-2} p^j p^n. \tag{36}$$

These equations have been derived earlier in [6]. We can solve these equations by means of a change of coordinates (18) applying the solutions (23) and (24) or Eqs. (29) and (30).

V. PHASE-SPACE EVOLUTION IN AN ELECTROMAGNETIC FIELD

The evolution of coordinates can be obtained as an integral over the proper time,

$$x_\mu(\tau) = x_\mu + \frac{1}{m} \int_0^\tau p_\mu(s) ds. \tag{37}$$

In an electromagnetic field the momentum satisfies the equation

$$dp_\mu = \frac{e}{mc} F_{\mu\nu} p^\nu d\tau + dp^H_\mu. \tag{38}$$

Here, by p^H we denote the diffusion on the hyperboloid defined in Eqs. (29) and (30). In general, we obtain nonlinear stochastic differential equations from Eq. (38) [because Eq. (31) for p_- is nonlinear in momentum]. Equation (38) is a linear equation if the equation for p_+ does not involve p_- on the right-hand side. The only case which leads to linear stochastic differential equations describes constant parallel electric and magnetic fields (or a special case when one of them is zero). Let

$$\alpha = \frac{e}{mc}. \tag{39}$$

Assume that the only components of F are $F_{12}=B$ and $F_{30}=E$. In such a case Eqs. (38) read

$$dp_1 = \alpha B p_2 d\tau + \frac{3}{2}\kappa^2 p_1 d\tau + \kappa p_1 db_+ + \gamma db_1, \tag{40}$$

$$dp_2 = -\alpha B p_1 d\tau + \frac{3}{2}\kappa^2 p_2 d\tau + \kappa p_2 db_+ + \gamma db_2, \tag{41}$$

$$dp_+ = \alpha E p_+ d\tau + \kappa^2 p_+ d\tau + \kappa p_+ \circ db_+. \tag{42}$$

It is clear that the linear equations (40)–(42) can explicitly be solved. The solution of Eq. (42) is an elementary function,

$$p_+(\tau) = p_+ \exp[\alpha E\tau + \kappa^2\tau + \kappa b_+(\tau)]. \tag{43}$$

The environment of electromagnetic waves or (in a quantized form) photons can be another source of diffusion. In Eq. (2) let $K_\mu=(F_{\mu\nu}+Q_{\mu\nu})p^\nu$ where Q is a Gaussian electromagnetic field (depending only on the proper time) with the covariance

$$E[Q_{\mu\nu}(\tau)Q_{\sigma\rho}(\tau')] = (\eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\sigma})\delta(\tau-\tau'). \tag{44}$$

Let $p(\tau)$ be the solution of Eq. (2) with an external (deterministic) electromagnetic field F and a random (or quantum) electromagnetic field Q. Then, $\phi_\tau(p)=E[\phi(p(\tau;p)\,]$ is the solution of Eq. (7).

In order to derive the nonrelativistic limit we assume that

$$p_+ = mc = \text{const} \tag{45}$$

in the stochastic equations (29) and (30). Then,

$$\partial_+ = \partial_3$$

in the diffusion equation (33). In the nonrelativistic limit Eqs. (38) for a constant electromagnetic field become linear. The solution is expressed by the Ornstein-Uhlenbeck process [28].

VI. THE MOMENTUM DISTRIBUTION

We are interested in a distribution of momenta of particles coming out from a gas formed after heavy ion collisions. For this purpose we express the transition function (26) by the momenta. The relativistic invariant formula reads

$$\cosh\sigma = \frac{1}{2}(mc)^{-2}pp'. \tag{46}$$

In terms of the (p_+,p_1,p_2) coordinates we have

$$2\cosh\sigma \equiv 2a = m^{-2}c^{-2}p_+^{-1}p_+'^{-1}[(p_1p_+' - p_1'p_+)^2 + (p_2p_+' - p_2'p_+)^2 + m^2c^2p_+'^2 + m^2c^2p_+^2]. \tag{47}$$

As a function of a the geodesic distance σ has the form

$$\sigma = \ln(a + \sqrt{a^2-1}). \tag{48}$$

The time evolution in the momentum coordinates is

$$\phi_\tau(p) \equiv T_\tau\phi(p) = \int d\mu(p')P_\tau(p,p')\phi(p'), \tag{49}$$

where $d\mu=d^3pp_0^{-1}mc$ is the relativistic invariant volume measure μ. We express Eq. (49) in various coordinate systems inserting the transition function (26) in Eq. (49) with proper volume elements $d\mu$. In the coordinates (14) the Riemannian volume element is

$$d\mu = (mc)^3 d\alpha d\theta d\phi \sinh^2\alpha \sin\theta. \tag{50}$$

In the Poincaré coordinates

$$d\mu = (mc)^3 dq_1 dq_2 dq_3 q_3^{-3}. \tag{51}$$

From the invariance under Lorentz transformations Λ,

$$\cosh\sigma(p,p') = \cosh\sigma(\Lambda p, \Lambda p'). \tag{52}$$

We can choose

$$\Lambda p = (p_0, 0, 0, p_3). \tag{53}$$

Let us define the rapidity y by

$$p_0 \pm p_3 = m_T c \exp(\pm y), \tag{54}$$

where

$$m_T^2c^2 = m^2c^2 + p_T^2 = m^2c^2 + p_1^2 + p_2^2. \tag{55}$$

The rapidity transforms in a simple way under the Lorentz boost (with the velocity v) in the (0, 3) plane,

$$\tilde{y} = y + \frac{v}{c}. \tag{56}$$

Then, in the frame where $p_T=0$,

$$\sigma(p,p') = y - y'. \tag{57}$$

The rapidity is also closely related to the variable α in the coordinates (14). We have

$$\cosh\sigma = \cosh\alpha\cosh\alpha' - \sinh\alpha\sinh\alpha'\cos\sigma_2,$$

where σ_2 is the geodesic distance on the unit sphere. Hence in the Lorentz frame (53) if $\sigma_2=0$ then $\sigma=y-y'=\alpha-\alpha'$.

From Eqs. (29) and (43) we can see that the process $p_+(\tau)$ is an exponential of a Gaussian process. Hence its probability distribution should be the log-normal distribution. We could calculate it from the general formula [using the transition function (26)]

$$\phi_\tau(p_+) = \int dp_1' dp_2' dp_+' p_+'^{-1} P_\tau(p,p')\phi(p_+') \equiv \int dp_+' P_\tau^{(+)}(p_+,p_+')\phi(p_+'). \quad (58)$$

However, it is easier to derive it directly from the solution (43). So, for the diffusion in the electric field (40)–(42) we obtain

$$P_\tau^{(+)}(p_+,p_+')_E = (2\pi\kappa^2\tau)^{-1/2}(p_+')^{\kappa^{-2}\alpha E} p_+^{-1-\kappa^{-2}\alpha E} \times \exp\left[-\frac{\tau}{2\kappa^2}(\alpha E+\kappa^2)^2 - \frac{1}{2\kappa^2\tau}\left(\ln\left(\frac{p_+'}{p_+}\right)\right)^2\right]. \quad (59)$$

VII. THE EQUILIBRIUM DISTRIBUTION

Let us define the time evolution of an expectation value of an observable ϕ in a state ρ (a measure on the phase space) by

$$\langle\phi\rangle_\rho^\tau = \int d\rho_\tau \phi \equiv \int d\rho\phi_\tau. \quad (60)$$

We say that a measure ν is the invariant measure for the diffusion process (see [8]) if the expectation value in Eq. (60) is time independent, i.e.,

$$\int d\nu(p,x)\phi_\tau(p,x) = \text{const}. \quad (61)$$

Assume that (in general) the diffusion equation reads

$$\partial_\tau\phi_\tau = \mathcal{G}\phi_\tau,$$

where

$$\mathcal{G} = \frac{\gamma^2}{2}\Delta_H + Y \quad (62)$$

and

$$Y = R^j\frac{\partial}{\partial p^j} + \frac{p^\mu}{m}\frac{\partial}{\partial x^\mu} \quad (63)$$

is the generator of the deterministic flow in the coordinates (10) or

$$Y = R_a\frac{\partial}{\partial p_a} + R_-\frac{\partial}{\partial p_+} + \frac{p^\mu}{m}\frac{\partial}{\partial x^\mu} \quad (64)$$

in the coordinates (12). Let us write

$$d\rho_\tau = dx d^3p\Phi_\tau. \quad (65)$$

Then, from Eq. (60),

$$\partial_\tau\Phi_\tau = \mathcal{G}^*\Phi_\tau, \quad (66)$$

where in the coordinates (10)

$$\mathcal{G}^* = \frac{\gamma^2}{2}\Delta_H^* - \frac{\partial}{\partial p_j}R_j - \frac{p^\mu}{m}\frac{\partial}{\partial x^\mu} \quad (67)$$

and

$$\Delta_H^* = \partial_j g^{jk}\sqrt{g}\partial_k\frac{1}{\sqrt{g}}. \quad (68)$$

Let us write the invariant measure in the form

$$d\nu = d^3p dx\sqrt{g}\Phi_R \equiv d^3p dx\Phi_0\Phi_R. \quad (69)$$

Here, $d^3p\Phi_0 = d^3p\sqrt{g}$ is the [not normalizable; g is calculated below Eq. (9)] equilibrium measure for Δ_H, i.e.,

$$\Delta_H^*\Phi_0 = 0. \quad (70)$$

Then, the invariant measure ν for the diffusion (62) is determined by the solution Φ_R of the equation [obtained by differentiating Eq. (61) over τ]

$$\mathcal{G}^*\Phi_0\Phi_R = 0 \quad (71)$$

or

$$\tilde{\mathcal{G}}\Phi_R = 0, \quad (72)$$

where

$$\tilde{\mathcal{G}} = \frac{\gamma^2}{2}\Delta_H - p_0\left(\frac{\partial}{\partial p_j}R_j + \frac{\partial}{\partial x^\mu}\frac{p^\mu}{m}\right)p_0^{-1}. \quad (73)$$

It can easily be seen that if the (nonzero) limit of ρ_τ (as $\tau \to \infty$) exists, then

$$\lim_{\tau\to\infty} d\rho_\tau = dx d^3p p_0^{-1}\Phi_R \quad (74)$$

(in the weak sense of the convergence of measures). We can express Eq. (72) as an evolution equation in time x^0,

$$\partial_0\Phi_R = \frac{\kappa^2}{2}mp_0^{-1}\Delta_H\Phi_R - m\left(\frac{\partial}{\partial p_j}R_j + p_j\frac{\partial}{\partial x^j}\right)p_0^{-1}\Phi_R. \quad (75)$$

If R^j does not depend on x^0 then Eqs. (66) and (72) may have the same time-independent solutions determining the static equilibrium distribution Φ_E. In general, Eq. (75) is the transport equation for Φ_R. When $x^0 \to \infty$ then Φ_R tends to the x^0-independent equilibrium distribution Φ_E,

$$\lim_{x_0\to\infty}\Phi_R = \Phi_E, \quad (76)$$

solving both Eqs. (66) and (75).

Equation (72) can be considered as an equation for the drag if the equilibrium measure Φ_E is fixed. It is not possible to obtain the equilibrium measure which is normalizable, Lorentz invariant, and at the same time concentrated on the mass shell ($p^2 = m^2c^2$). We relinquish the explicit Lorentz invariance. If we still require the rotation invariance then it is useful to work in the spherical coordinates (14). In these coordinates Eq. (72) for Φ_E reads (we restrict ourselves to the momentum dependence of Φ_E)

$$\frac{\gamma^2}{2}\Delta_H\Phi_E = p_0\partial_\alpha(\omega\Phi_E), \quad (77)$$

where

$$Y=\omega(\alpha)\frac{\partial}{\partial\alpha} \tag{78}$$

of Eq. (62) in the coordinates (14) has only one component ω. From Eq. (77) we obtain

$$\omega=\frac{1}{2}\gamma^2\partial_\alpha \ln \Phi_E. \tag{79}$$

If

$$\Phi_E=\exp(-\beta c p_0)=\exp(-\beta mc^2\cosh\alpha) \tag{80}$$

then

$$\omega=-\frac{1}{2}\gamma^2 mc^2\beta\sinh\alpha. \tag{81}$$

For the Bose-Einstein distribution

$$\Phi_E=[\exp(\beta mc^2\cosh\alpha)-1]^{-1} \tag{82}$$

we have

$$\omega=-\frac{1}{2}\gamma^2 mc^2\beta\sinh\alpha[1-\exp(-\beta mc^2\cosh\alpha)]^{-1}. \tag{83}$$

The drifts can be inserted (after a change of coordinates) into the diffusion equation (7) or the stochastic equations (29) and (30) in order to determine the diffusive dynamics.

The equilibrium distribution (80) determines the diffusion generator in spherical coordinates

$$\mathcal{G}=\frac{\gamma^2}{2m^2c^2}p^0u^{-2}\frac{\partial}{\partial u}p^0u^2\frac{\partial}{\partial u}+\frac{\gamma^2}{2u^2}\Delta_{S_2}-\frac{1}{2}\kappa^2\beta c p^0 u\frac{\partial}{\partial u}, \tag{84}$$

where $u=|\mathbf{p}|$ and $p^0=\sqrt{m^2c^2+u^2}$. If the drift R^j for Φ_E (80) is derived from Eq. (72) in the $\mathbf{p}$ coordinates (10) then we obtain

$$\mathcal{G}=\frac{\gamma^2}{2}\Delta_H-\frac{1}{2}\kappa^2\beta c p_0 p^j\frac{\partial}{\partial p^j}. \tag{85}$$

VIII. DIFFUSION EQUATION AS AN APPROXIMATION TO MASTER EQUATION

The diffusion equation (7) could be considered as an approximation to the dynamics of a heavy particle embedded in a gas of light particles. The kinetic equation describing the flow conservation under a Markovian scattering process reads [12,13] (here $t=\frac{x_0}{c}$)

$$\left(\partial_t+cp^jp_0^{-1}\frac{\partial}{\partial x^j}\right)\rho(\mathbf{p},x)=\tilde{\kappa}^2\int d^3k[w(\mathbf{p}+\mathbf{k},\mathbf{k})\rho(\mathbf{p}+\mathbf{k},x)-w(\mathbf{p},\mathbf{k})\rho(\mathbf{p},x)], \tag{86}$$

where $\int d^3k w(\mathbf{p},\mathbf{k})=1$ and $\tilde{\kappa}^2 w(\mathbf{p},\mathbf{k})$ is the probability that in the unit of time the momentum $\mathbf{p}$ of the heavy particle is changed to $\mathbf{p}-\mathbf{k}$ through scattering on light particles (we could assume $\tilde{\kappa}=\kappa$ but this is not necessary). The diffusion equation can be obtained by means of the Taylor expansion in $\mathbf{k}$ [13]. In such a case the diffusion coefficients can be calculated using the formulas

$$C^j=\tilde{\kappa}^2\int d^3k w(\mathbf{p},\mathbf{k})k^j\equiv\tilde{\kappa}^2\langle k^j\rangle, \tag{87}$$

$$\frac{1}{2}D^{ij}=\frac{1}{2}\tilde{\kappa}^2\int d^3k w(\mathbf{p},\mathbf{k})k^ik^j. \tag{88}$$

Then,

$$\frac{1}{2}D^{ij}=\frac{1}{2}\tilde{\kappa}^2\int d^3k w(\mathbf{p},\mathbf{k})(k^i-\langle k^i\rangle)(k^j-\langle k^j\rangle)+\frac{1}{2}\tilde{\kappa}^2\langle k^i\rangle\langle k^j\rangle \equiv\frac{1}{2}M^{ij}+\frac{1}{2}\tilde{\kappa}^{-2}C^iC^j. \tag{89}$$

It follows that if $w(\mathbf{p},\mathbf{k})\geqslant 0$ then the matrix

$$M^{ij}=D^{ij}-\tilde{\kappa}^{-2}C^iC^j \tag{90}$$

must be positive definite.

The drift and diffusion coefficients have the simplest meaning in the coordinates (10). In these coordinates, comparing Eqs. (7), (11), (75), and (86) we obtain the diffusion equation for $\Phi_R=\rho$ with

$$C^j=\frac{mc}{p_0}\left(\frac{3}{2}\kappa^2p^j+R^j\right). \tag{91}$$

From Eqs. (10) and (90)

$$M^{ij}=\frac{mc}{p_0}g^{ij}-\tilde{\kappa}^{-2}C^iC^j \tag{92}$$

must be a positive definite matrix [the metric tensor g^{ij} is defined in Eq. (10)]. Let us assume that

$$\Phi_E=\exp(f(\beta p_0 c)). \tag{93}$$

In such a case, from Eq. (72) we obtain

$$R_j=\frac{1}{2}p_jp_0\beta\kappa^2cf'(\beta p_0c). \tag{94}$$

Hence,

$$M_{jk}=\gamma^2(\delta_{jk}-p_jp_k\mathbf{p}^{-2})\frac{mc}{p_0}+p_jp_k\gamma^2(mc)^{-1}p_0^{-2}\mathbf{p}^{-2}\times\left[p_0^3-mc\kappa^2\tilde{\kappa}^{-2}\mathbf{p}^2\left(\frac{3}{2}+\frac{1}{2}p_0\beta cf'\right)^2\right]. \tag{95}$$

For Jüttner [23] as well as Bose-Einstein equilibrium distributions the longitudinal term in Eq. (95) becomes negative at large momenta. Hence at large energies the diffusion equation could not be a good approximation to the master equation. However, for the Tsallis distribution [22]

$$\Phi_E(x)=[1+(q-1)x]^{-1/(q-1)} \tag{96}$$

we have

$$R_j = -\frac{1}{2}p_j p_0 \beta\kappa^2 c[1 + (q-1)\beta c p_0]^{-1}. \tag{97}$$

$xf'(x)$ in Eq. (94) is a bounded function. As a consequence the longitudinal term in Eq. (95) is positive definite. In such a case we can apply the diffusion equation as an approximation of the master equation at high momenta as well. The Tsallis distribution appears in many models ranging from turbulence [29] to heavy ion collisions [17,30,31].

We return to the proper time evolution of the relativistic diffusion in an electromagnetic field of Sec. V. We restrict ourselves to the static case without a magnetic field. Then, the electromagnetic potential is $A=(A_0,0,0,0)$. We notice that $\mathcal{G}$ in Eq. (62) consists of two parts: the diffusion in momentum and the drag which is a sum of the dynamical piece and a dissipative one. It is easy to see that in Eq. (72) for the equilibrium measure the dissipative and dynamical parts must separately vanish (this is a version of the fluctuation-dissipation theorem). Hence

$$d\nu = d^3x d^3p p_0^{-1} \exp[f(\beta(cp_0 + eA_0))] \tag{98}$$

is the equilibrium measure in a time-independent electric field $\mathbf{E}=-\nabla A_0$. The drag term of Eq. (62) reads

$$Y = \frac{1}{2}\kappa^2\beta c p_0 p^j f'(\beta p_0 c + \beta e A_0)\frac{\partial}{\partial p^j} - \frac{e}{c}\partial^j A_0 \frac{\partial}{\partial p^j} + \frac{p^\mu}{m}\frac{\partial}{\partial x^\mu}. \tag{99}$$

If we suggest that the master equation (86) is an equation in the proper time τ (instead of $t=\frac{x_0}{c}$) then there would be no damping factor $\frac{mc}{p_0}$ in the drift (91). In such a case the longitudinal term would become negative for large momenta. As a consequence the relativistic diffusion equation (7) could not be a consistent approximation at large energies to the proper time master equation.

IX. EVOLUTION WITH FRICTION

In the case of time-independent drags (as discussed in Sec. VII) the search of equilibrium can equivalently be treated as a study of either proper time evolution of x_0-independent densities Φ or x_0-independent solutions of the transport equation (75). Without friction the proper time evolution is expressed explicitly by Eqs. (49) and (26). With friction we do not expect explicit solutions. We must rely on approximate methods or computer simulations. There is no normalizable invariant measure for the relativistic diffusion generated by Δ_H because the process is fast growing as can be seen from the solution (43) (without the electric field),

$$E[p_+^2(\tau)] = p_+^2 \exp(4\kappa^2\tau). \tag{100}$$

The friction is damping the time evolution. Without the noise term the proper time evolution is determined by the solution of the equation

$$\frac{d\mathbf{p}}{d\tau} = \mathbf{R}(\mathbf{p}). \tag{101}$$

generated by the flow Y (63). The evolution in x_0 is defined by the drag of Eq. (75),

$$\frac{d\mathbf{p}}{dx_0} = \frac{m}{p_0}\mathbf{R}(\mathbf{p}). \tag{102}$$

As an example, for the generator (85) (without the diffusion) we have

$$\mathbf{p}(x_0) = \exp(-\kappa^2\beta m c x_0)\mathbf{p}.$$

The evolution in proper time (101) is also expressed by an elementary function which has the asymptotic behavior (coinciding with the one in the time $\frac{x^0}{c}$)

$$|\mathbf{p}(\tau)| \simeq |\mathbf{p}(\tau)|\exp(-\kappa^2 mc^2\beta\tau). \tag{103}$$

The dynamical systems (101) and (102) have a trivial limit when time tends to infinity. The diffusive spreading makes it nontrivial. The evolution with friction is described by the semigroup $\exp[\tau(\frac{1}{2}\Delta_H+Y)]$. A rough approximation of this evolution as a product $\exp(\tau\frac{1}{2}\Delta_H)\exp(\tau Y)$ can be expressed as a diffusion acting on the deterministic flow (101). Such an approximation is reliable only for a small time.

Before we propose an exact method to approach the time evolution with friction let us explain it in the well-known case of the Ornstein-Uhlenbeck (OU) process. The evolution is generated by

$$\mathcal{G}_{OU} = \frac{1}{2}\frac{d^2}{d\xi^2} - \omega\xi\frac{d}{d\xi}. \tag{104}$$

Applying the invariant measure for the OU process we can transform the generator (104) into the Hamiltonian of the harmonic oscillator,

$$H_{osc} = -\frac{1}{2}\frac{d^2}{d\xi^2} + \frac{\omega^2}{2}\xi^2 - \frac{\omega}{2} = -\exp\left(-\frac{\omega}{2}\xi^2\right)\mathcal{G}_{OU}\exp\left(\frac{\omega}{2}\xi^2\right). \tag{105}$$

We apply the method to the diffusion with friction. We have for the generator (62) [with the friction (94)]

$$-\exp(f)\mathcal{G}\exp(-f) = -\frac{\gamma^2}{2}\Delta_H + V, \tag{106}$$

where

$$V = \frac{\kappa^2}{2}(c\beta)^2\mathbf{p}^2(f'^2 + f'') + \frac{3}{2}\kappa^2\beta c p_0 f'. \tag{107}$$

We have obtained a Hamiltonian for a particle moving on the hyperboloid (3) in a potential V. We could apply either Hamiltonian methods of quantum mechanics to this model [32] or functional integration. In the latter case, let $\phi=\exp(-f)\psi$, then we have the Feynman-Kac formula

$$\phi_\tau(\mathbf{p},x) = \exp(-f)E\left\{\exp\left[-\int_0^\tau V(\mathbf{p}_s^H)ds\right]\psi(\mathbf{p}_\tau^H,x_\tau)\right\}, \tag{108}$$

where $\mathbf{p}_s^H$ is the stochastic process (34)–(36) on the hyperboloid (3) starting at the point $\mathbf{p}$. Clearly, the solution (108) can also be expressed in the form

$$\phi_\tau(\mathbf{p},x) = E[\phi(\mathbf{p}_\tau,x_\tau)],$$

where $\mathbf{p}_\tau$ is the solution of the equation

$$d\mathbf{p}_\tau = \mathbf{R}d\tau + d\mathbf{p}_\tau^H \qquad (109)$$

with the initial condition $\mathbf{p}$.

Let us note that if f is determined by the relativistic Boltzmann equilibrium distribution (80) (or Bose-Einstein or Jüttner) then the potential V in Eq. (107) is growing quadratically. Such a quadratic growth can substantially change the large time behavior of the solution of the diffusion equation. In the case of the Tsallis distribution the potential V is bounded. The expansion in V of Eq. (108) (which coincides with the Dyson expansion of quantum mechanics) is convergent for an arbitrarily large time. If the evolution of the momentum distribution tends to the Tsallis distribution then this should also be visible from the solution (108) even far from the equilibrium.

X. DISCUSSION

It is expected that the relativistic diffusion will appear in relativistic models of plasma. Some calculations based on the master equation (86) have been performed already in the 1980s [14,15] (for a recent review, see [33]). The scattering probabilities can be calculated in quantum field theory. Another approach applies the Wigner function [34] for a description of the particle phase-space evolution [11,18,35].

The approach based on the master equation leads to the diffusion equation (75) for the probability density. We may write the diffusion part of Eq. (75) (no friction) as

$$\Delta_H^* = \partial_i\partial_j D_{ij} - \partial_i A_i, \qquad (110)$$

where

$$D_{ij} = \gamma^2(\delta_{ij} - p_i p_j \mathbf{p}^{-2})\frac{mc}{p_0} + \gamma^2 p_i p_j \mathbf{p}^{-2}\frac{p_0}{mc} \qquad (111)$$

and

$$A_i = \gamma^2 \frac{3}{2}\frac{1}{p_0 mc}p_i. \qquad (112)$$

The total drift (with the friction) is

$$C_i = \kappa^2 p_i\left(\frac{3}{2} + \frac{1}{2}\beta p_0 c f'(\beta p_0 c)\right)\frac{mc}{p_0}. \qquad (113)$$

The authors [15,17,33,36] write a general form of the diffusion equation without specifying the diffusion coefficients. It follows from our work that the diffusion coefficients D^{ij} are defined by the relativistic invariance. Then, the functions $B_\parallel$ and $B_\perp$ in Eq. (11) of Ref. [17] and (the analogs) B_1 and B_0 in Eq. (25) of Ref. [33] are uniquely determined. Relativistic invariance determines also the coefficient A of the drift in Eq. (10) of Ref. [17] as $\frac{3mc}{2p_0}$ if friction is switched off. The drag defining the friction is in one to one correspondence with the equilibrium measure as discussed already in [17].

We have shown that the equilibrium measures resulting from the diffusion approximation to the master equation (86) which are selected by the additivity of entropy (Jüttner or Bose-Einstein) do not lead to a diffusion equation which could be valid at arbitrarily high energies. The stochastic process with the generator (85) could have been considered as a relativistic generalization of the Ornstein-Uhlenbeck (OU) process (see [4] for another definition of the relativistic OU process). However, the relativistic counterpart of the Maxwell-Boltzmann equilibrium measure does not seem to be restricted to the Jüttner distribution (see the discussion in [17,31,37,38]). For this reason all the stochastic processes with the drifts (113) could be considered as relativistic OU processes (they have the usual OU nonrelativistic limit).

The transport equation resulting from quantum field theory has an application to ultrarelativistic collisions (in particular to heavy ion collisions). The time evolution of observables (denoted by ϕ) or the probability density of the diffusion process (denoted by Φ) has been expressed in this paper by analytic formulas or by equations which could be solved numerically. The results are determined by the relativistic invariance and the form of the invariant measure. In this way the experimental results concerning the particle momentum distribution (available from RHIC [39] and future experiments on LHC) can decide whether the particles coming out from relativistic collisions can be described as diffusing in a gas of light particles. A diffusion equation can also be treated as a tool for a phenomenological description of the scattering data in heavy ion collisions (see [36,40]).

Finally, let us mention that although there is only one diffusion on the hyperboloid (3), nevertheless, there are many Markov processes on this hyperboloid. We have a general Levy-Khintchin representation formula for processes with independent increments. Among these processes we distinguish here the fractional diffusion (with a stable probability distribution) defined by

$$\partial_\tau\phi = (-\Delta_H)^\delta\phi, \qquad (114)$$

where $0<\delta\leq 1$. The soluble case $\delta=\frac{1}{2}$ with the transition function

$$P_\tau(\sigma) = \tau\sqrt{2}\pi^{-3/2}\frac{\sigma}{\sinh\sigma}(\tau^2+\sigma^2)^{-1}K_2(\sqrt{\tau^2+\sigma^2}) \qquad (115)$$

(where K_ν is the Bessel function of the third kind) shows characteristic features of the fractional diffusion. An application of the fractional diffusion in heavy ion collisions has been suggested recently in [41].

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