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From Ornstein-Uhlenbeck dynamics to long-memory processes and fractional Brownian motion

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This article establishes a natural physical path leading from "regular" Ornstein-Uhlenbeck dynamics to "anomalous" long-memory processes and, thereafter, to fractional Brownian motion. Considering a system composed of n different parts—each part conducting its own Ornstein-Uhlenbeck dynamics, and all parts being perturbed by a common external Lévy noise—we show that the collective system-dynamics, in the limit $n \to \infty$, converges to a temporal moving-average of the driving noise. The limiting moving-average process, in turn, can posses a long memory—in which case, when observed over large time scales, further yields fractional Brownian motion. The temporal correlation structure of the limiting moving-average process turns out to be determined by the structural statistical variability of the system's composing parts. Thus, the emergence of a long memory is a consequence of the intrinsic "quenched disorder" present at the system's formation epoch rather than the consequence of the external annealed disorder carried in continuously by the driving noise.

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I. INTRODUCTION

In this article we establish a natural physical path leading from Ornstein-Uhlenbeck (OU) dynamics to long-memory processes and, thereafter, to fractional Brownian motion (FBM). The OU stochastic differential equation, i.e., the Langevin equation with linear restoring force, is a "cornerstone" equation in physics [1,2]. In its most abstract form, the OU equation describes the stochastic evolution of a system undergoing an exponential relaxation while, simultaneously, being perturbed by an external random noise. The combination of an exponential relaxation on the one hand and random perturbations on the other, is ubiquitous in both natural and designed systems—consequently rendering the OU equation a focal role in physics, chemistry, biology, and engineering.

In case the external random noise is white—the temporal derivative of Brownian motion—then the OU equation describes the continuous motion of diffusion in a harmonic potential well [2]. In case the external random noise is Poissonian—the temporal derivative of a compound Poisson process—then the OU equation describes the discontinuous motion of shot noise [3,4]. Both white and Poissonian noises are special cases of Lévy noises—the temporal derivatives of Lévy processes (namely, processes with stationary and independent increments [5,6]). OU and Langevin equations driven by general Lévy noises attracted major interest in recent years, and were studied via different perspectives and approaches [7–15].

An OU equation driven by a general finite variance Lévy noise yields an output process which is Markov and stationary, and whose autocovariance function is exponential. In particular, the OU output process possesses a short memory. On the other hand, the sciences are abundant with processes possessing long memories. Namely, finite-variance stationary processes with slowly decaying autocovariance functions [16–18].

This apparent dichotomy between the ubiquitous appearance of the OU equation on the one hand and the abundance of processes possessing long memories on the other leads us to the question: Can a long memory emerge from a system whose elemental dynamics are OU? The answer, as shown in Refs. [19–21] and as to be demonstrated in this research, is affirmative. A special case of the model presented in this article was studied in Ref. [19] (see Sec. III A for the details), and superpositions of independent OU processes were explored in Refs. [19–21] (see also references therein).

We introduce and study the class of composite Ornstein-Uhlenbeck (COU) systems. A general COU system is composed of many parts—the system's "atoms." The parts are different, the dynamics of each part are governed by its own OU equation, and all parts are perturbed by a common external driving noise—an arbitrary finite-variance Lévy noise. The output of the COU system is the averaged aggregate of the output of its "OU atomic parts."

COU systems may be considered as a conceptual hydrological model of river flows. Consider a river basin composed of many different water catchments and "fed" by common rainfall events. Each water catchment produces its own output flow, and the many outputs are aggregated-up into the collective river flow. A COU model of such hydrological systems is as follows: The water catchments are the system parts, the rainfall events are a common external Poissonian noise affecting all parts, and the parts' outputs are (nonidentical) shot noise processes driven by the common "rainfall noise."

Our analysis shows that in the case of a large COU system—i.e., when the number of "atoms" tends to infinity—the system's output process converges to a temporal moving-average (MA) of the external driving noise. The functional structure of the temporal averaging—referred to as the system's memory function—turns out to be deter-

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mined by the underlying statistical variability of the system's atomic parts.

The "quenched" disorder present at the COU system-formation epoch governs the statistical variability of the system's atomic parts and, in turn, determines the system's memory function. The "annealed" disorder carried in continuously by the external noise is averaged out temporally by the COU system's memory function—yielding the system's MA output process.

Contrary to OU processes, MA processes are not Markov. Pending on the decay of its memory function, a MA process may posses either a short memory or a long memory. Consequently, the output of a large COU system, pending on the statistical variability of its atomic parts, may posses either a short memory or a long memory. Further analysis conducted in this research pinpoints precisely when a long memory is attained.

So far, we described how a transition from the microscopic OU atom-level to the macroscopic COU system-level may give rise to the emergence of a long memory. But what happens if a COU system is observed macroscopically also along the time axis, i.e., if it is observed over large time scales? This question leads to the analysis of the temporal scaling limits of COU output processes. We prove that, in the presence of a long memory, the universal scaling limit of COU output processes is fractional Brownian noise (FBN)—the temporal derivative of FBM.

FBM—a generalization of Brownian motion which was first introduced by Mandelbrot and Van Ness [22]—is the quintessential example of a random motion with long memory. FBM is a Gaussian and self-similar process, its sample-path trajectories are continuous, and its increments are stationary and possess a long memory [23]. FBM is characterized by a single real-valued parameter—the Hurst exponent—which governs its process statistics [23].

In physics, FBM was shown to emerge from Hamiltonian dynamics leading to generalized Langevin equations: (i) Heat baths with random-matrix interactions [24]; (ii) Kac-Zwanzig heat baths with random initial conditions [25]. In mathematics, abstract limit theorems leading to FBM were devised (see Refs. [23,26], and references therein), and independent superpositions of simple random processes were proved to converge to FBM: (i) Renewal processes [27,28]; (ii) on-off processes [29]; (iii) persistent random walks [30]; (iv) OU processes [19–21].

Two different micro-to-macro transitions of COU systems are addressed. The first transition is a structural one, transcending from the microscopic atom level to the macroscopic system level. This transition takes us from Markov OU processes to non-Markov MA processes. The degrees of freedom of the resulting MA processes are (i) the functional structure of the temporal averaging, i.e., the system's memory function; (ii) the statistical structure of external driving noise, i.e., the noise's Lévy statistics.

The second transition is a temporal one, applied via temporal scaling limits. This transition takes us from non-Markov MA processes to FBM. The degree of freedom of the resulting FBM is its Hurst exponent. Thus, the second transition collapses the MA degrees of freedom to a single one-dimensional parameter—the Hurst exponent.

These two micro-to-macro transitions—both natural and physically intuitive—established a direct path leading from OU dynamics to MA processes, and thereafter to FBM. FBM, in turn, is the universal scaling limit of long-memory COU systems.

The reminder of the article is organized as follows. Lévy noises and OU dynamics are reviewed in Sec. II. COU systems are introduced and studied in Sec. III. The scaling limits of COU systems are investigated in Sec. IV. A concluding example is presented in Sec. V. For a short exposition of the main results of this research the readers are referred to Ref. [31].

II. PRELIMINARIES

A. Lévy noises

In this subsection we concisely review the notion of Lévy noises. For further details the readers are referred to Refs. [5,6].

1. Lévy noises: Fourier characterization

A random noise process $\dot{N}=[\dot{N}(t)]_t$ is a Lévy noise with Lévy exponent $\mathcal{L}(\cdot)$ if its Fourier transform admits the functional form

$$\mathbf{E}\left[\exp\left\{i\int_{-\infty}^{\infty}\theta(t)\dot{N}(t)dt\right\}\right]=\exp\left\{-\int_{-\infty}^{\infty}\mathcal{L}[\theta(t)]dt\right\}, \tag{1}$$

where the "Fourier variable" $\theta(\cdot)$ is an arbitrary test function for which the integral appearing on the right-hand-side of Eq. (1) converges. The Lévy exponent $\mathcal{L}(\cdot)$ characterizes the Lévy noise $\dot{N}$ and its general form is given by the celebrated Lévy-Khinchin formula.

Lévy noises are not processes per se. Rather, they are noises—meaning that they can be measured only via their "actions" on test functions. This is well exemplified by Eq. (1) which provides us with the Fourier transform of the random variable $\int_{-\infty}^{\infty}\theta(t)\dot{N}(t)dt$—the "action" of the Lévy noise $\dot{N}$ on the test function $\theta(\cdot)$.

Considering the Lévy noise $\dot{N}$ as a random velocity yields the induced Lévy motion $N=[N(t)]_{t\geqslant 0}$ given by $N(t)=\int_0^t\dot{N}(t')dt'$. Contrary to Lévy noises, Lévy motions are well-defined processes. Moreover, Lévy motions constitute the class of random processes with stationary and independent increments.

Examples of Lévy noises (Lévy motions) include the following. (i) White noise (Brownian motion): characterized by the Lévy exponent $\mathcal{L}_{\mathrm{WN}}(\theta)=\frac{1}{2}|\theta|^2$. (ii) Stable noises (Stable motions): characterized by the Lévy exponent $\mathcal{L}_{\mathrm{SN}}(\theta)=|\theta|^{\alpha}$, the exponent α taking values in the range $0<\alpha<2$. (iii) Variance-gamma noise (Variance-gamma process): characterized by the Lévy exponent $\mathcal{L}_{\mathrm{VGN}}(\theta)=\ln(1+\frac{1}{2}|\theta|^2)$. (iv) Compound Poisson noises (compound Poisson processes): characterized by the Lévy exponent $\mathcal{L}_{\mathrm{CPN}}(\theta)=1-\hat{F}(\theta)$, the function $\hat{F}(\cdot)$ being the Fourier transform of an (arbitrary) probability distribution F defined on the real line.[1]

2. Lévy noises: Mean, variance, and covariance

The mean, variance, and covariance functionals corresponding to the Lévy noise $\dot{N}$ follow straightforwardly from the Fourier transform of equation (1). Mean:

$$\mathbf{E}\left[\int_{-\infty}^{\infty}\theta(t)\dot{N}(t)dt\right]=i\mathcal{L}'(0)\int_{-\infty}^{\infty}\theta(t)dt \tag{2}$$

provided that the Lévy exponent $\mathcal{L}(\cdot)$ is differentiable at the origin, and that the test function $\theta(\cdot)$ is integrable on the real line. Variance:

$$\mathrm{Var}\left[\int_{-\infty}^{\infty}\theta(t)\dot{N}(t)dt\right]=\mathcal{L}''(0)\int_{-\infty}^{\infty}|\theta(t)|^2dt \tag{3}$$

provided that the Lévy exponent $\mathcal{L}(\cdot)$ is twice differentiable at the origin, and that the test function $\theta(\cdot)$ is square integrable on the real line. Covariance:

$$\mathrm{Cov}\left[\int_{-\infty}^{\infty}\theta_1(t)\dot{N}(t)dt,\int_{-\infty}^{\infty}\theta_2(t)\dot{N}(t)dt\right]$$
$$=\mathcal{L}''(0)\int_{-\infty}^{\infty}\theta_1(t)\theta_2(t)dt \tag{4}$$

provided that the Lévy exponent $\mathcal{L}(\cdot)$ is twice differentiable at the origin, and that the test functions $\theta_1(\cdot)$ and $\theta_2(\cdot)$ are square integrable on the real line.

Note that the variance equation (3) is, in fact, a special case of the covariance equation (4) [with $\theta_1(\cdot)=\theta_2(\cdot)=\theta(\cdot)$]. Henceforth, we shall refer to a Lévy noise $\dot{N}$ as centered if it has zero mean $\mathcal{L}'(0)=0$ and unit variance $\mathcal{L}''(0)=1$.

B. Ornstein-Uhlenbeck dynamics

In this subsection we review the notion of OU dynamics. For further details the readers are referred to the sources cited herein.

1. General Ornstein-Uhlenbeck processes: Dynamics and integral representation

A stochastic process $\xi=[\xi(t)]_t$ is OU if its dynamics are governed by the OU stochastic differential equation

$$\dot{\xi}=-x\xi+y\dot{N}, \tag{5}$$

where x and y are positive parameters, and where $\dot{N}=[\dot{N}(t)]_t$ is a driving noise.

The OU equation (5) is a Langevin equation with linear restoring force. It describes a system undergoing an exponential relaxation while, simultaneously, being perturbed by an external noise. The exponential relaxation and the perturbing noise are antithetical—the former pushing the system towards equilibrium, while the latter driving the system away from equilibrium. The system parameters x and y represent, respectively, the amplitude of the exponential relaxation (x) and the amplitude of the perturbing noise (y).

In case the driving noise is white then the OU equation (5) describes the continuous motion of diffusion in a harmonic potential well [1,2]. In case the driving noise is compound Poisson then the OU equation (5) describes the discontinuous motion of shot noise [3,4]. OU processes driven by general Lévy noises attracted considerable interest in recent years, and were studied via different perspectives and approaches [7–15].

The general solution of the OU equation (5), over the entire real line $-\infty<t<\infty$, is given by

$$\xi(t)=\int_{-\infty}^{t}[y\exp\{-x(t-t')\}]\dot{N}(t')dt'. \tag{6}$$

The solution appearing in Eq. (6) represents a linear integral transformation mapping "input" noises ($\dot{N}$) to "output" OU processes (ξ).

2. Ornstein-Uhlenbeck processes driven by centered Lévy noises: Correlation structure

Consider now the case where the OU driving noise is Lévy. In this case the OU process ξ is stationary and Markov. Moreover, the OU process ξ is of finite variance if and only if the Lévy noise $\dot{N}$ is such. If the Lévy noise $\dot{N}$ is centered then the OU process ξ has zero mean, and its autocovariance function $\mathbf{R}_{\mathrm{OU}}(\cdot)$ is given by

$$\mathbf{R}_{\mathrm{OU}}(t)=\frac{y^2}{2x}\exp\{-x|t|\} \tag{7}$$

(t real). In particular, the variance of the OU process ξ is given by $\mathbf{R}_{\mathrm{OU}}(0)=y^2/2x$.

The power-spectrum function $\mathbf{S}_{\mathrm{OU}}(\cdot)$ corresponding to the autocovariance function $\mathbf{R}_{\mathrm{OU}}(\cdot)$ is given, in turn, by[2]

$$\mathbf{S}_{\mathrm{OU}}(\omega)=\frac{y^2}{2\pi}\frac{1}{x^2+\omega^2} \tag{8}$$

(ω real).

The derivation of the autocovariance function $\mathbf{R}_{\mathrm{OU}}(\cdot)$ follows from Eqs. (4) and (6) via a straightforward calculation. The derivation of the power-spectrum function $\mathbf{S}_{\mathrm{OU}}(\cdot)$ follows from Eq. (7) via a straightforward Fourier calculation. Note how the exponential relaxation of the OU dynamics manifests itself in the exponential decay of the autocovariance function $\mathbf{R}_{\mathrm{OU}}(\cdot)$ (temporal domain) and in the quadratic power-law decay of the power-spectrum function $\mathbf{S}_{\mathrm{OU}}(\cdot)$ (frequency domain).

[1]Namely, $\hat{F}(\theta)=\int_{-\infty}^{\infty}\exp\{i\theta x\}F(dx)$ (θ real).

[2]The power-spectrum function $\mathbf{S}(\cdot)$ of a finite-variance stationary stochastic process with autocovariance function $\mathbf{R}(\cdot)$ is defined as the autocovariance's inverse Fourier transform $\mathbf{S}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{-i\omega t\}\mathbf{R}(t)dt$ (ω real).

III. COMPOSITE ORNSTEIN-UHLENBECK SYSTEMS

The OU stochastic differential equation (5) describes the dynamics of a system undergoing exponential relaxation while, simultaneously, being perturbed by an external noise. In this section we introduce and study the class of composite Ornstein-Uhlenbeck (COU) systems—systems whose "atomic parts" are governed by elemental OU dynamics, yet whose collective system behavior is of higher complexity. We begin with a presentation of the COU system model, and then turn to analyze the structure—both on the microscopic atom level and on the macroscopic system level—of COU systems.

A. The COU system model

1. Model description

Consider a system composed of n different atomic parts, all parts undergoing exponential relaxations and being perturbed by a common external noise $\dot{N}$. The dynamics of part k are governed by the OU equation

$$\dot{\xi}_k = -X_k \xi_k + Y_k \dot{N}, \tag{9}$$

X_k and Y_k being the OU amplitudes of part k $(k=1,\ldots,n)$. The processes $\xi_1,\ldots,\xi_n$ are coupled via the common external noise $\dot{N}$.

The COU system's output process $\Xi_n = [\Xi_n(t)]_t$ is the averaged aggregate of the output processes $\xi_1,\ldots,\xi_n$ of its atomic parts. Namely:

$$\Xi_n(t) := \frac{1}{n}\sum_{k=1}^{n} \xi_k(t). \tag{10}$$

Our main focus shall be the case of large COU systems in which $n \gg 1$. In this case it is natural to assume that the variability of the amplitude pairs $(X_1,Y_1),\ldots,(X_n,Y_n)$ obeys some statistical regularity. The specific model assumptions regarding the amplitude-pair statistics and the external noise statistics are as follows.

Amplitude pairs. The amplitude pairs $(X_1,Y_1),\ldots,(X_n,Y_n)$ are independent and identically distributed copies of a "generic" random amplitude pair (X,Y) satisfying the integrability condition

$$\mathbf{E}\left[\frac{Y^2}{2X}\right] < \infty. \tag{11}$$

Noise. The external driving noise $\dot{N}$ is centered Lévy, and is independent of the amplitude pairs $(X_1,Y_1),\ldots,(X_n,Y_n)$.

We emphasize that, in accordance with the fluctuation-dissipation theorem [32], the 'generic' amplitudes X and Y may certainly be dependent random variables (as demonstrated in the concluding example presented in Sec. V). Moreover, we shall also show that the requirement that the random amplitude pairs $(X_1,Y_1),\ldots,(X_n,Y_n)$ be independent is not necessary and can be relaxed to allow for interdependencies (between the amplitude pairs).

A special case of the COU model—in which the generic amplitude X is gamma distributed, the noise-amplitudes are degenerate and deterministic (specifically, $Y\equiv 1$), and the driving noise $\dot{N}$ is white—was studied in Ref. [19]. Superpositions of independent OU processes—where each OU process is driven by its own noise process, and the driving noises are independent—were explored in Refs. [19–21] (see also references therein).

2. The memory and base functions

Before continuing on to analyze COU systems, let us first introduce notation that will accompany us henceforth. We denote by $\psi(x,y)$ $(x,y>0)$ the probability density function governing the probability distribution of the generic random amplitude pair (X,Y), and define the following associated functions. Memory function:

$$\phi(\tau) := \int_0^\infty \int_0^\infty y \exp\{-\tau x\}\psi(x,y)dxdy \;\; (\tau \geq 0), \tag{12}$$

base functions:

$$\psi_m(x) := \int_0^\infty y^m \psi(x,y)dy \;\; (x>0; m=1,2). \tag{13}$$

Note that the memory function $\phi(\cdot)$ satisfies $\phi(\tau) = \mathbf{E}[Y\exp\{-\tau X\}]$ $(\tau \geq 0)$, and that it is the Laplace transform of the base function $\psi_1(\cdot)$.[3]

B. COU analysis

A large COU system is composed of many atomic parts, and produces an output signal which conveys the system's collective stochastic behavior. Observing the COU system on its microscopic atom-level yields the stochastic process ξ_k—k being the specific atomic part observed. Observing the COU system on a macroscopic system level, on the other hand, yields the stochastic process Ξ_n—the averaged aggregate of the system's atomic parts. In this subsection we analyze COU systems both microscopically and macroscopically.

1. Microscopic analysis

Consider a microscopic observation focusing on the COU system's kth atomic part—yielding the stochastic process ξ_k. Applying the general OU solution of Eq. (6) to the OU dynamics of Eq. (9) we obtain that the "microscopic process" ξ_k is given by

$$\xi_k(t) = \int_{-\infty}^{t} [Y_k \exp\{-X_k(t-t')\}]\dot{N}(t')dt' \tag{14}$$

(t real).

We emphasize that the microscopic process ξ_k is conditional OU. Namely, given the specific realization of the amplitude pair (X_k,Y_k), the microscopic process ξ_k is OU with amplitudes $x=X_k$ and $y=Y_k$. However, the microscopic process ξ_k per se is not OU.

The COU model assumptions imply that the microscopic

[3] Namely, $\phi(\tau) = \int_0^\infty \exp\{-\tau x\}\psi_1(x)dx$ $(\tau \geq 0)$.

process ξ_k has *zero mean*, and that its autocovariance function $\mathbf{R}_{\text{micro}}(\cdot)$ is given by

$$\mathbf{R}_{\text{micro}}(t) = \int_0^\infty \exp\{-|t|x\}\left(\frac{\boldsymbol{\psi}_2(x)}{2x}\right)dx \tag{15}$$

(t real). In particular, the variance of the microscopic process ξ_k is given by $\mathbf{R}_{\text{micro}}(0)=\mathbf{E}[Y^2/2X]$.

The power-spectrum function $\mathbf{S}_{\text{micro}}(\cdot)$ corresponding to the autocovariance function $\mathbf{R}_{\text{micro}}(\cdot)$ is given, in turn, by

$$\mathbf{S}_{\text{micro}}(\omega) = \int_0^\infty \frac{1}{\omega^2+x^2}\left(\frac{\boldsymbol{\psi}_2(x)}{2\pi}\right)dx \tag{16}$$

(ω real). The derivation of Eqs. (15) and (16) is given in AppendixA.

Note the profound difference between the OU autocovariance $\mathbf{R}_{\text{OU}}(\cdot)$ [Eq. (7)] and the "microscopic autocovariance" $\mathbf{R}_{\text{micro}}(\cdot)$, and between the OU power spectrum $\mathbf{S}_{\text{OU}}(\cdot)$ [Eq. (8)] and the "microscopic" power spectrum $\mathbf{S}_{\text{micro}}(\cdot)$. The randomization of the OU amplitude pairs yields microscopic correlation structures which are no longer confined to an exponential decay in the temporal domain, and to a quadratic power-law decay in the frequency domain.

This fact was exploited, in the context of shot noise, to produce "$1/f^\alpha$ noise:" Various researchers randomized the x amplitude of OU processes driven by compound Poisson noise so that to obtain a power spectrum which, across a wide range of frequencies, behaves similar to a power-law with exponent $0<\alpha<2$ (see, for example, Refs. [33–35]). Equations (15) and (16) imply that the generation of "$1/f^\alpha$ noise" is far more robust. Indeed, any OU process driven by a centered Lévy noise can be turned into $1/f^\alpha$ noise by an appropriate randomization of its amplitude pair (x,y)—the resulting power spectrum being contingent on the randomization's base function $\boldsymbol{\psi}_2(\cdot)$.

2. Macroscopic analysis: Moving-average limit

Let us transcend from the microscopic observation to a macroscopic one, and consider the COU system's output process Ξ_n—conveying the system's collective stochastic behavior. Substituting the solutions of the microscopic processes $\xi_1,\ldots,\xi_n$ [given by Eq. (14)] into Eq. (10) we obtain that the "macroscopic process" Ξ_n is given by

$$\Xi_n(t) = \int_{-\infty}^t \left(\frac{1}{n}\sum_{k=1}^n Y_k \exp\{-X_k(t-t')\}\right)\dot{N}(t')dt' \quad (t \text{ real}). \tag{17}$$

As the COU system-size grows to infinity ($n\to\infty$), the law of large numbers implies that the integrand of Eq. (17) converges to the deterministic limit $\boldsymbol{\phi}(t-t')$—the convergence holding with probability 1 for any (fixed) $t'\leq t$, and $\boldsymbol{\phi}(\cdot)$ being the memory function defined in Eq. (12). Hence, in the asymptotic limit $n\to\infty$ we expect to obtain the limiting macroscopic process $\Xi=[\Xi(t)]_t$ given by

$$\Xi(t) = \int_{-\infty}^t \boldsymbol{\phi}(t-t')\dot{N}(t')dt'. \tag{18}$$

The macroscopic process Ξ is a temporal moving-average (MA) of the driving noise $\dot{N}=(\dot{N}(t))_t$, where the temporal averaging is performed by the memory function $\boldsymbol{\phi}(\cdot)$. In signal processing the system described by Eq. (18) is referred to as a casual convolution filter with impulse-response function $\boldsymbol{\phi}(\cdot)$ [36].

Equation (18) is a generalization of Eq. (6)—replacing the exponential memory function of Eq. (6) by the general memory function of Eq. (18). Analogous to the case of Eq. (6), Eq. (18) represents a linear integral transformation mapping input noises ($\dot{N}$) to output MA processes (Ξ). A rigorous analysis of the asymptotic limit $n\to\infty$ is provided by the following proposition.

Proposition 1. The sequence of macroscopic processes $\{\Xi_n\}_{n=1}^\infty$ converges, in L^2 norm, over any time interval, to the limiting macroscopic process Ξ. Namely,

$$\lim_{n\to\infty} \boldsymbol{E}\left[\int_a^b |\Xi_n(t)-\Xi(t)|^2 dt\right] = 0, \tag{19}$$

the limit holding for all $-\infty<a<b<\infty$. The proof of Proposition 1 is given in Appendix B.

3. Macroscopic analysis: Correlation structure

The macroscopic process Ξ of proposition 1 is stationary. The fact that the driving noise $\dot{N}$ is centered Lévy implies that the macroscopic process Ξ has zero mean, and that its autocovariance function $\mathbf{R}_{\text{macro}}(\cdot)$ is given by

$$\mathbf{R}_{\text{macro}}(t) = \int_0^\infty \boldsymbol{\phi}(x)\boldsymbol{\phi}(x+|t|)dx = \int_0^\infty \int_0^\infty \exp\{-|t|x\}\frac{\boldsymbol{\psi}_1(x)\boldsymbol{\psi}_1(y)}{x+y}dxdy, \tag{20}$$

(t real). In particular, the variance of the macroscopic process Ξ is given by

$$\mathbf{R}_{\text{macro}}(0) = \int_0^\infty \boldsymbol{\phi}(x)^2 dx = \int_0^\infty \int_0^\infty \frac{\boldsymbol{\psi}_1(x)\boldsymbol{\psi}_1(y)}{x+y}dxdy. \tag{21}$$

The power-spectrum function $\mathbf{S}_{\text{macro}}(\cdot)$ corresponding to the autocovariance function $\mathbf{R}_{\text{macro}}(\cdot)$ is given, in turn, by

$$\mathbf{S}_{\text{macro}}(\omega) = \frac{1}{2\pi}\left|\int_0^\infty \exp\{i\omega x\}\boldsymbol{\phi}(x)dx\right|^2 = \frac{1}{2\pi}\left|\int_0^\infty \frac{\boldsymbol{\psi}_1(x)}{x+i\omega}dx\right|^2 \quad (\omega \text{ real}). \tag{22}$$

The derivation of the middle term of Eq. (20) follows from Eqs. (4) and (6) via a straightforward calculation. The derivation of the middle term of Eq. (22) follows from the middle term of Eq. (20) via a straightforward Fourier calculation. The right-hand terms of Eqs. (20) and (22) are obtained, following simple calculations, by the Laplace substitution $\boldsymbol{\phi}(\tau)=\int_0^\infty \exp\{-\tau x\}\boldsymbol{\psi}_1(x)dx$ into their middle terms.

As in the case of the microscopic analysis, the macroscopic correlation structures obtained are not confined to an exponential decay in the temporal domain, and to a quadratic power-law decay in the frequency domain. Rather, the "macroscopic autocovariance" $\mathbf{R}_{\text{macro}}(\cdot)$ and the "macroscopic power spectrum" $\mathbf{S}_{\text{macro}}(\cdot)$ may assume various decay forms—contingent on the base function $\boldsymbol{\psi}_1(\cdot)$.

C. Discussion

1. Quenched disorder vs annealed disorder

A COU system has two sources of underlying randomness: (i) A "quenched" disorder governing the realizations of the OU amplitude pairs of the system's atomic parts and (ii) an "annealed" disorder governing the sample-path realization of the system's external driving noise. The "quenched" disorder is internal and static. It influences the system at its formation epoch by "molding," once and for all, the values of the OU amplitude pairs of the system's atomic parts. These amplitude-pairs, in turn, are the COU system's intrinsic characteristics. Counterwise, the "annealed" disorder is external and dynamic. It influences the COU system from after its formation, perturbing the system by external "shocks" admitted to it continuously in time.

On the atom-level the microscopic autocovariance function $\mathbf{R}_{\text{micro}}(\cdot)$ and the microscopic power-spectrum function $\mathbf{S}_{\text{micro}}(\cdot)$ turn out to be contingent on the base function $\boldsymbol{\psi}_2(\cdot)$, whereas on the system level the macroscopic autocovariance function $\mathbf{R}_{\text{macro}}(\cdot)$ and the macroscopic power-spectrum function $\mathbf{S}_{\text{macro}}(\cdot)$ turn out to be contingent on the base function $\boldsymbol{\psi}_1(\cdot)$. The base functions $\boldsymbol{\psi}_1(\cdot)$ and $\boldsymbol{\psi}_2(\cdot)$, in turn, are functionals of the "quenched" disorder statistics [characterized by the probability density function $\boldsymbol{\psi}(\cdot,\cdot)$].

On the other hand, the microscopic and the macroscopic autocovariance and power-spectrum function turn out to be independent of the statistics of the "annealed" disorder [characterized by the Lévy exponent $\mathcal{L}(\cdot)$]. Hence, the correlation structure of COU systems—both on the microscopic atom level and on the macroscopic system-level—is determined solely by the "quenched" disorder present at the system's formation epoch, and is unaffected by the 'annealed' disorder carried in continuously by the external driving noise.

2. Microscopic level vs macroscopic level

COU systems display markedly different stochastic behaviors on the microscopic atom-level and on the macroscopic system level. This difference is vividly clear by comparing the integral representations of the microscopic and macroscopic processes ξ_k and Ξ [given, respectively, by Eqs. (14) and (18)]. Moreover, the COU microscopic correlation structure is governed by the base function $\boldsymbol{\psi}_2(\cdot)$, whereas the COU macroscopic correlation structure is governed by the base function $\boldsymbol{\psi}_1(\cdot)$.

When transcending from the microscopic atom level to the macroscopic system level the "quenched" disorder "solidifies" into the deterministic memory function $\boldsymbol{\phi}(\cdot)$. This "solidification" induces a reduction of variance: The variance of the macroscopic process Ξ [i.e., the macroscopic variance $\mathbf{R}_{\text{macro}}(0)$] is bounded from above by the variance of the microscopic process ξ_k [i.e., the microscopic variance $\mathbf{R}_{\text{micro}}(0)$]. Namely,

$$\mathbf{R}_{\text{macro}}(0) \leq \mathbf{E}\left[\frac{Y^2}{2X}\right] = \mathbf{R}_{\text{micro}}(0). \tag{23}$$

The derivation of the left-hand-side bound of Eq. (23) is given in part *4* of the proof of proposition 1 (see the Appendix B).

3. Macroscopic stationary distribution and Markov breaking

OU processes driven by Lévy noises are stationary and Markov. On the other hand, MA processes driven by Lévy noises are stationary but, in general, are non-Markov. Specifically, a Lévy-driven MA process is Markov if and only if its memory function is exponential.

The Fourier transform of the stationary distribution of the macroscopic process Ξ is given by

$$\mathbf{E}[\exp\{i\theta\Xi(t)\}] = \exp\left\{-\int_0^\infty \mathcal{L}[\theta\boldsymbol{\phi}(\tau)]d\tau\right\}\ (\theta \text{ real}). \tag{24}$$

Equation (24) follows straightforwardly from Eqs. (1) and (18).

The transition from the OU process ξ of Eq. (6) to the macroscopic COU process Ξ of Eq. (18) preserves stationarity, but losses the Markov property. Hence, replacing a single OU process by the averaged aggregate of many different and coupled OU processes renders the resulting output a memory which extends beyond the output's current position.

This "Markov-breaking" phenomena is a consequence of the statistical heterogeneity and of the averaging taking place in the COU system model. As shall be demonstrate in the sequel, Markov breaking is precisely what enables the emergence of a long memory—which, in turn, will further yield the FBM scaling limit.

4. Model robustness

In the COU system model the OU amplitude pairs of the system's atomic parts are assumed independent and identically distributed random variables. Proposition 1, however, holds also in the presence of dependence.

Indeed, if the amplitude pairs $(X_1,Y_1),\ldots,(X_n,Y_n)$ are identically distributed—yet dependent—random variables, then the condition

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k,l=1}^{n}\mathbf{E}\left[\frac{Y_kY_l}{X_k+X_l}\right] = 0 \tag{25}$$

is sufficient in order ensure that the result of Proposition 1 remains valid. The proof of this assertion is given at the end of the proof of proposition 1 (see the Appendix).

IV. LONG MEMORY AND THE FRACTIONAL BROWNIAN MOTION SCALING LIMIT

In this section we study the emergence of a long memory in the collective stochastic behavior of large COU systems. We begin with a quantitative definition of the notion of "long memory" and with the analysis of long memory in large COU systems. Thereafter, we consider the temporal scaling of macroscopic COU processes, and prove FBM to be the universal scaling limit of long-memory macroscopic COU processes.

A. Long memory

1. Long memory: Definition

A finite-variance stationary stochastic process, with autocovariance function $\mathbf{R}(\cdot)$ and power spectrum function $\mathbf{S}(\cdot)$, is said to posses a long memory [16–18] if either of the following equivalent asymptotic conditions holds.[4]

Autocovariance. $\mathbf{R}(t) \sim \mathbf{r}(t)t^{-\alpha}$, as $t \to \infty$, where the exponent α is in the range $0 < \alpha < 1$, and the function $\mathbf{r}(\cdot)$ is slowly varying at infinity.

Power spectrum. $\mathbf{S}(\omega) \sim \mathbf{s}(|\omega|)|\omega|^{-\beta}$, as $|\omega| \to 0$, where the exponent β is in the range $0 < \beta < 1$, and the function $\mathbf{s}(\cdot)$ is slowly varying at the origin.

The autocovariance characterization of long memory is in the temporal domain, whereas the power-spectrum characterization of long memory is in the frequency domain. The equivalence of the two characterizations is as follows (see, for example, theorem 4.10.3 in Ref. [37]).

The autocovariance condition implies the power-spectrum condition with exponent $\beta = 1 - \alpha$ and slowly varying function $\mathbf{s}(|\omega|) \sim \mathbf{r}(1/|\omega|)/c_\alpha$ (as $|\omega| \to 0$), where $c_\alpha = 2\Gamma(\alpha)\cos(\pi\alpha/2)$. Conversely, the power-spectrum condition implies the auto-covariance condition with exponent $\alpha = 1 - \beta$ and slowly varying function $\mathbf{r}(t) \sim c_\beta \mathbf{s}(1/t)$ (as $t \to \infty$), where $c_\beta = 2\Gamma(1-\beta)\sin(\pi\beta/2)$. The phenomena of long memory was coined by Mandelbrot and Wallis the "Joseph effect" [38]—the etymology of the term stems from the biblical story of Joseph's prophesy: "… there came seven years of great plenty throughout the land of Egypt. And there shall arise after them seven years of famine…" Genesis, 41: 29, 30.

2. Long memory: COU systems

Long memory in the case of the macroscopic COU process Ξ—whose autocovariance function $\mathbf{R}_{\text{macro}}(\cdot)$ and power-spectrum function $\mathbf{S}_{\text{macro}}(\cdot)$ are given, respectively, by Eqs. (20) and (22)—is the issue of the following proposition.

Proposition 2. Assume that the base function $\boldsymbol{\psi}_1(\cdot)$ satisfies $\boldsymbol{\psi}_1(x) \sim \mathbf{l}(x)x^{p-1}$, as $x \to 0$, where the exponent p is in the range $1/2 < p < 1$, and the function $\mathbf{l}(\cdot)$ is slowly varying at the origin. Then we have the following.

(1) Memory function:

$$\boldsymbol{\phi}(\tau) \underset{\tau\to\infty}{\sim} \Gamma(p)\frac{\mathbf{l}(1/\tau)}{\tau^p}. \tag{26}$$

(2) Macroscopic autocovariance function:

$$\mathbf{R}_{\text{macro}}(t) \underset{t\to\infty}{\sim} \left(\frac{\pi\Gamma(2p-1)}{\sin(\pi p)}\right)\frac{\mathbf{l}(1/t)^2}{t^{2p-1}}. \tag{27}$$

(3) Macroscopic power-spectrum function:

$$\mathbf{S}_{\text{macro}}(\omega) \underset{\omega\to 0}{\sim} \left(\frac{\pi/2}{\sin(\pi p)^2}\right)\frac{\mathbf{l}(|\omega|)^2}{|\omega|^{2-2p}}. \tag{28}$$

Namely, the macroscopic COU process Ξ posses a long memory with "temporal exponent" $\alpha = 2p - 1$ and "frequency exponent" $\beta = 2 - 2p$.

The proof of proposition 2 is given in Appendix C. Note that the exponents $\alpha = 2p - 1$ and $\beta = 2 - 2p$, appearing in proposition 2, indeed take values in the range $0 < \alpha, \beta < 1$, and indeed sum up to unity (i.e., $\alpha + \beta = 1$).

B. Temporal scaling

We turn now to consider the temporal scaling of the macroscopic COU process Ξ. Given a positive constant c we speed-up time by the factor c and rescale the macroscopic COU process Ξ by the positive factor $\boldsymbol{\sigma}(c)$—the function $\boldsymbol{\sigma}(\cdot)$ being an arbitrary positive-valued scaling function. The output of this temporal scaling procedure is the scaled process $Z_c = [Z_c(t)]_t$ given by

$$Z_c(t) = \frac{1}{\boldsymbol{\sigma}(c)}\Xi(ct) \ (t \text{ real}). \tag{29}$$

1. Temporal scaling: Correlation-structure limit

Let $\mathbf{R}_c(\cdot)$ and $\mathbf{S}_c(\cdot)$ denote, respectively, the autocovariance and power-spectrum functions of the scaled process Z_c. The limiting correlation structure of the scaled process Z_c—in the scaling limit $c \to \infty$, and in the presence of long memory—is the issue of the following proposition.

Proposition 3. Assume that the long-memory condition of proposition 2 is satisfied, and set $\boldsymbol{\sigma}(c) \sim \sqrt{c}\,\boldsymbol{\phi}(c)$ (as $c \to \infty$).

(1) *Autocovariance limit*

$$\mathbf{R}_\infty(t) \coloneqq \lim_{c\to\infty} \mathbf{R}_c(t) = \left(\frac{\Gamma(1-p)\Gamma(2p-1)}{\Gamma(p)}\right)\frac{1}{|t|^{2p-1}}. \tag{30}$$

(2) Power-spectrum limit

$$\mathbf{S}_\infty(\omega) \coloneqq \lim_{c\to\infty} \mathbf{S}_c(\omega) = \left(\frac{\Gamma(1-p)^2}{2\pi}\right)\frac{1}{|\omega|^{2-2p}}. \tag{31}$$

(3) Plugging the base function $\boldsymbol{\psi}_1(x) = x^{p-1}/\Gamma(p)$ and its Laplace transform—the memory function $\boldsymbol{\phi}(\tau) = \tau^{-p}$—into Eqs. (20) and (22) yields, respectively, the limits $\mathbf{R}_\infty(\cdot)$ and $\mathbf{S}_\infty(\cdot)$.

The proof of proposition 3 is given in Appendix D. Note that the power-spectrum limit $\mathbf{S}_\infty(\cdot)$ obtained admits the form of a pure "$1/f^{2-2p}$ noise."

Equations (30) and (31) are, respectively, the "distilled" versions of Eqs. (27) and (28)—admitting a "pure" power-law form, rather than an asymptotic one. We emphasize, however, that the autocovariance limit $\mathbf{R}_\infty(\cdot)$ and the power-spectrum limit $\mathbf{S}_\infty(\cdot)$ are not, respectively, admissible autocovariance and power-spectrum functions [this is due to the divergence of the limit $\mathbf{R}_\infty(\cdot)$ at the origin].

[4]A real function $\mathbf{l}(\cdot)$ is said to be slowly varying at the limit point x_* if the limit $\lim_{x\to x_*} \mathbf{l}(cx)/\mathbf{l}(x) = 1$ holds for all positive constants c [37]. The class of slowly varying functions generalizes the class of asymptotically constant functions (at the limit point x_*) and includes constants, logarithms, iterated logarithms, and powers of logarithms.

2. Temporal scaling: Fractional Brownian noise limit

With proposition 3 at hand it is "tempting" to infer that the scaled process Z_c converges—in the scaling limit $c \to \infty$, and in the presence of long memory—to a limiting MA process with memory function $\boldsymbol{\phi}(\tau)=\tau^{-p}$. The precise description of the stochastic process-limit obtained is provided by the following proposition.

Proposition 4. Assume that the long-memory condition of proposition 2 is satisfied, and set $\boldsymbol{\sigma}(c) \sim \sqrt{c}\,\boldsymbol{\phi}(c)$ (as $c \to \infty$). Then, the scaled process Z_c converges, in the scaling limit $c \to \infty$, in law, to a limiting noise process $Z=[Z(t)]_t$. Formally, the limiting noise process Z is a MA of white noise[5] $\dot{N}_{\mathrm{WN}}$ with memory function $\boldsymbol{\phi}(\tau)=\tau^{-p}$:

$$Z(t) = \int_{-\infty}^{t} (t-t')^{-p} \dot{N}_{\mathrm{WN}}(t')dt'. \tag{32}$$

The proof of proposition 4 is given in Appendix E. We emphasize that the limit Z is ill defined as a process per se, and is well defined as a noise—which is measured via its "action" on test functions. See the proof of proposition 4 for an explanation of this delicate, yet important, point.

The limit Z obtained is fractional Brownian noise (FBN)—the "velocity" of FBM. We shall elaborate on this in the proceeding subsection.

3. Temporal scaling: Universality

The most distinctive feature of the FBN scaling limit Z of proposition 4 is its universality—i.e., the invariance of the FBN scaling limit Z to the details of the COU system it stems from. The statistics of a COU system are governed by two infinite-dimensional parameters: (i) The probability density function $\boldsymbol{\psi}(\cdot,\cdot)$ which characterizes the statistical variability of the OU amplitude pairs of the system's atomic parts and (ii) the Lévy exponent $\mathcal{L}(\cdot)$ which characterizes the statistics of the system's external driving noise. These two infinite-dimensional parameters—when the COU system is observed over large time scales, and in the presence of long memory—collapse to a single real-valued parameter: The system's "long memory exponent" p. When observed on large time scales, long memory COU systems will always yield the FBN scaling limit Z which is a MA of white noise $\dot{N}_{\mathrm{WN}}$ with a power-law memory function—the limit's only degree of freedom being the exponent p of its power-law memory function.

C. Fractional Brownian Motion

1. Fractional Brownian Motion: Construction

In Sec. II A we described how considering Lévy noises as random velocities results in Lévy motions—which, contrary to Lévy noises, turn out to be well-defined random processes. In this section we repeat this very procedure to the FBN scaling limit Z of proposition 4.

Considering a given MA process Ξ as a random velocity yields the induced motion $M_{\Xi}=[M_{\Xi}(t)]_{t\geq 0}$ given by $M_{\Xi}(t) =\int_0^t \Xi(t')dt'$. Using the MA representation of Eq. (18), further yields the representation of the induced motion M_{Ξ} as an integral with respect to the underlying driving noise $\dot{N}$:

$$M_{\Xi}(t) = \int_{-\infty}^{0} [\Phi(t-t') - \Phi(0-t')]\dot{N}(t')dt' + \int_{0}^{t} \Phi(t-t')\dot{N}(t')dt', \tag{33}$$

where $\Phi(\cdot)$ is the integrated memory function $\boldsymbol{\phi}(\cdot)$, i.e., $\Phi(t) \coloneqq \int_0^t \boldsymbol{\phi}(t')dt'$.

In particular, the motion M_Z induced by the FBN scaling limit Z of proposition 4—a MA of white noise $\dot{N}_{\mathrm{WN}}$ with memory function $\boldsymbol{\phi}(\tau)=\tau^{-p}$—admits the integral representation

$$M_Z(t) = \int_{-\infty}^{0} [(t-t')^{1-p} - (0-t')^{1-p}]\dot{N}_{\mathrm{WN}}(t')dt' + \int_{0}^{t} (t-t')^{1-p}\dot{N}_{\mathrm{WN}}(t')dt' \tag{34}$$

[with no loss of generality, we eliminated the multiplicative factor $1/(1-p)$].

The integral representation of the stochastic process M_Z implies that it is a FBM with Hurst exponent

$$H = 3/2 - p \tag{35}$$

[23]. The Hurst exponent H takes values in the range $1/2 < H < 1$. Contrary to the FBN scaling limit Z, the induced FBM M_Z is a well-defined random process.

2. Fractional Brownian motion: Properties

FBM—a generalization of Brownian motion which was first introduced by Mandelbrot and Van Ness [22]—is the quintessential example of a random motion with continuous sample-path trajectories and long memory. We mention three key features of the FBM M_Z (for further details the readers are referred to Ref. [23]).[6]

Gaussianity. The FBM M_Z is a Gaussian process. Namely, the probability distribution of any finite-dimensional vector $[M_Z(t_1),\ldots,M_Z(t_n)]$ is multivariate Gaussian. A Gaussian process is characterized by its mean and its autocovariance. The FBM M_Z has zero mean and autocovariance

$$\mathrm{Cov}[M_Z(t_1),M_Z(t_2)] = \frac{C_H}{2}\{|t_1|^{2H} + |t_2|^{2H} - |t_1-t_2|^{2H}\} \times (t_1,t_2 \geq 0). \tag{36}$$

Long memory. The increments of the FBM M_Z are stationary and posses a long memory:

[5]Recall from Sec. III A that a white noise $\dot{N}_{\mathrm{WN}}$ is a Lévy noise characterized by the Lévy exponent $\mathcal{L}_{\mathrm{WN}}(\theta)=\frac{1}{2}|\theta|^2$.

[6]The coefficient C_H appearing in Eqs. (36) and (37) is a constant depending on the value of the Hurst exponent H. Explicitly, $C_H =[\Gamma(\frac{1}{2}+H)\Gamma(2-2H)]/[2H\Gamma(\frac{3}{2}-H)]$.

$$\mathrm{Cov}[M_Z(t_1+\Delta)-M_Z(t_1),M_Z(t_2+\Delta)-M_Z(t_2)]$$

$$\underset{|t_1-t_2|\to\infty}{\sim} C_H H(2H-1)\frac{\Delta^2}{|t_1-t_2|^{2-2H}} \ (t_1,t_2\geqslant 0 \text{ and } \Delta>0). \tag{37}$$

Note that the exponent $2-2H$ takes values in the range $0<2-2H<1$.

Self-similarity. The FBM M_Z is a self-similar process with self-similarity exponent H. Namely, for any positive constant c the rescaled process $[c^{-H}M_Z(ct)]_{t\geqslant 0}$ is equal, in law, to the "original" process $[M_Z(t)]_{t\geqslant 0}$. Self-similarity implies that, statistically, the sample-path trajectories of the FBM M_Z are fractal objects.

V. AN EXAMPLE

As an illustrative example consider a COU system whose OU amplitude pairs are drawn from a probability distribution with probability density function

$$\boldsymbol{\psi}(x,y)=\frac{1}{C_g}\frac{g(x)}{x}\exp\left\{-\frac{y}{\sqrt{x}}\right\} \tag{38}$$

$(x,y>0)$, where $g(\cdot)$ is an arbitrary non-negative valued function satisfying the integrability condition $C_g:=\int_0^\infty(g(x)/\sqrt{x})dx<\infty$. The probability density function of Eq. (38) satisfies the finite-mean condition of Eq. (11) (with $\mathbf{E}[Y^2/2X]=1$), and yields the base functions

$$\boldsymbol{\psi}_1(x)=\frac{1}{C_g}g(x) \quad \text{and} \quad \boldsymbol{\psi}_2(x)=\frac{2}{C_g}g(x)\sqrt{x}. \tag{39}$$

Hence, this example facilitates a "reverse engineering" of the COU system under consideration—telling us what probability density function $\boldsymbol{\psi}(\cdot,\cdot)$ is required in order to yield, up to a multiplicative factor, a desired "target" base function [$\boldsymbol{\psi}_1(\cdot)$ for the macroscopic system-level and $\boldsymbol{\psi}_2(\cdot)$ for the microscopic atom level].

As an example of a COU system with long memory consider the choice $g(x)=\exp\{-x\}x^{p-1}$, where the exponent p takes values in the range $1/2<p<1$. On the microscopic atom-level this choice yields the microscopic autocovariance function

$$\mathbf{R}_{\mathrm{micro}}(t)=\frac{1}{(1+|t|)^p} \tag{40}$$

(t real). On the macroscopic system-level this choice yields the memory function

$$\boldsymbol{\phi}(\tau)=\frac{c_p}{(1+\tau)^p} \ (\tau\geqslant 0), \tag{41}$$

and Fourier transform

$$\mathbf{E}[\exp\{i\theta\Xi(t)\}]=\exp\left\{-\frac{\theta^{1/p}}{p}\int_0^\theta\frac{\mathcal{L}(c_p u)}{u^{1+1/p}}du\right\} \ (\theta \text{ real}) \tag{42}$$

of the stationary distribution of the macroscopic process Ξ [the coefficient c_p appearing in Eqs. (41) and (42) is given by $c_p=\Gamma(p)/\Gamma(p-\frac{1}{2})$].

VI. CONCLUSIONS

In this article we introduced and explored the class of composite Ornstein-Uhlenbeck (COU) systems—systems whose elemental dynamics are Ornstein-Uhlenbeck (OU), yet whose collective system-behavior is of higher complexity. A generic COU system is composed of n different "atomic parts," where each atomic part conducts its own OU dynamics, and where all atomic parts are perturbed by a common external Lévy noise. The output process of the COU system is the averaged aggregate of the OU processes of its atomic parts.

In the case of a large COU system—i.e., in the limit $n\to\infty$—the COU output process was shown to converge to a limiting macroscopic process which is a temporal moving-average (MA) of the driving Lévy noise. The transcendence from the microscopic atom-level to the macroscopic system-level resulted in a Markov-breaking phenomena: Replacing the Markovian OU dynamics by the non-Markov MA dynamics. Moreover, in this transcendence the "quenched randomness" governing the statistical variability of the COU system's atomic parts "solidified" into the memory function of the MA dynamics.

MA processes, pending on their memory functions, may posses a long memory. Namely, long temporal correlations characterized by slowly decaying autocovariances. Thus, the transition from simple OU systems to more complex COU systems may certainly result in the emergence of a long memory. The emergence of a long memory, in turn, is determined solely by the "quenched randomness" of the COU system under consideration, and is unaffected by the "annealed randomness" carried in continuously by the driving Lévy noise.

When observed on large time scales, the macroscopic MA outputs of large COU systems with long memory were shown to converge to a universal scaling limit: Fractional Brownian noise (FBN)—a temporal moving average of white noise, with a power-law memory function. FBN is the "velocity" of Fractional Brownian motion (FBM)—a Gaussian process with continuous and self-similar ("fractal") sample-path trajectories, whose increments are stationary and posses a long memory. The FBN scaling limit is characterized by a single degree of freedom—its Hurst exponent—which turned out to be determined solely by the "quenched randomness" of the underlying COU system considered. This article thus established a natural physical path leading from OU dynamics—fundamental in physics, chemistry, biology, and engineering—to long-memory MA processes, and, thereafter, to FBM.

APPENDIX A: MICROSCOPIC AUTOCOVARIANCE AND POWER SPECTRUM

1. Microscopic autocovariance: Eq. (15)

$$\mathbf{R}_{\text{micro}}(t) = \text{Cov}[\xi_k(0), \xi_k(t)] \tag{A1}$$

[conditioning on the realizations of the random amplitude-pair (X_k, Y_k)]

$$= \mathbf{E}[\text{Cov}[\xi_k(0), \xi_k(t)|X_k, Y_k]] + \text{Cov}[\mathbf{E}[\xi_k(0)|X_k, Y_k], \mathbf{E}[\xi_k(t)|X_k, Y_k]] \tag{A2}$$

[using the fact that—given the realization of the amplitude pair (X_k, Y_k)—the process ξ_k is OU with zero mean and autocovariance function given by Eq. (7) with amplitudes $x = X_k$ and $y = Y_k$]

$$= \mathbf{E}\left[\frac{Y_k^2}{2X_k}\exp\{-X_k|t|\}\right] + \text{Cov}[0,0] \tag{A3}$$

[using the probability density function $\boldsymbol{\psi}(\cdot,\cdot)$ of the random amplitude-pair (X_k, Y_k), and thereafter the base function $\boldsymbol{\psi}_2(\cdot)$]

$$= \int_0^\infty \int_0^\infty \left(\frac{y^2}{2x}\exp\{-x|t|\}\right)\boldsymbol{\psi}(x,y)dxdy$$

$$= \int_0^\infty \exp\{-|t|x\}\left(\frac{\boldsymbol{\psi}_2(x)}{2x}\right)dx. \tag{A4}$$

2. Microscopic power spectrum: Eq. (16)

$$\mathbf{S}_{\text{micro}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-i\omega t\}\mathbf{R}_{\text{micro}}(t)dt \tag{A5}$$

[using Eq. (15)]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-i\omega t\}\left(\int_0^\infty \exp\{-|t|x\}\left(\frac{\boldsymbol{\psi}_2(x)}{2x}\right)dx\right)dt \tag{A6}$$

(changing the order of integration)

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{-i\omega t\}\left(\frac{1}{2x}\exp\{-|t|x\}\right)dt\right]\boldsymbol{\psi}_2(x)dx \tag{A7}$$

[using Eqs. (7) and (7) with $y=1$]

$$\int_0^\infty \left(\frac{1}{2\pi}\frac{1}{\omega^2+x^2}\right)\boldsymbol{\psi}_2(x)dx = \int_0^\infty \frac{1}{\omega^2+x^2}\left(\frac{\boldsymbol{\psi}_2(x)}{2\pi}\right)dx. \tag{A8}$$

APPENDIX B: PROOF OF PROPOSITION 1

We split the proof into four steps, and thereafter address the case of dependent amplitude pairs.

Step 1. Set

$$\boldsymbol{\phi}_n(\tau) = \frac{1}{n}\sum_{k=1}^{n} Y_k \exp\{-X_k\tau\} \ (\tau \geq 0; n = 1,2,\ldots). \tag{B1}$$

The amplitude pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent and identically distributed copies of the "generic" random amplitude pair (X, Y). This implies that the random variables $Y_1\exp\{-X_1\tau\}, \ldots, Y_n\exp\{-X_n\tau\}$ are independent and identically distributed copies of the "generic" random variable $Y\exp\{-X\tau\}$. Hence, (i) The mean of $\boldsymbol{\phi}_n(\tau)$ is given by

$$\mathbf{E}[\boldsymbol{\phi}_n(\tau)] = \mathbf{E}[Y\exp\{-X\tau\}] \coloneqq \boldsymbol{\phi}(\tau). \tag{B2}$$

(ii) The variance of $\boldsymbol{\phi}_n(\tau)$ is bounded by

$$\text{Var}[\boldsymbol{\phi}_n(\tau)] = \frac{1}{n}\text{Var}[Y\exp\{-X\tau\}] \leq \frac{1}{n}\mathbf{E}[(Y\exp\{-X\tau\})^2] = \frac{1}{n}\mathbf{E}[Y^2\exp\{-2X\tau\}]. \tag{B3}$$

Equation (B3), in turn, implies that

$$\int_0^\infty \text{Var}[\boldsymbol{\phi}_n(\tau)]d\tau \leq \int_0^\infty \frac{1}{n}\mathbf{E}[Y^2\exp\{-2X\tau\}]d\tau = \frac{1}{n}\mathbf{E}\left[Y^2\int_0^\infty \exp\{-2X\tau\}d\tau\right] = \frac{1}{n}\mathbf{E}\left[\frac{Y^2}{2X}\right]. \tag{B4}$$

Step 2.

$$\mathbf{E}[|\Xi_n(t) - \Xi(t)|^2] \tag{B5}$$

[using Eqs. (17), (B1), and (18)]

$$= \mathbf{E}\left[\left|\int_{-\infty}^{t}[\boldsymbol{\phi}_n(t-t') - \boldsymbol{\phi}(t-t')]\dot{N}(t')dt'\right|^2\right] \tag{B6}$$

[using "Ito's isometry"—see, for example, Eq. (2.14) in Ref. [39]]

$$= \int_{-\infty}^{t} \mathbf{E}[|\boldsymbol{\phi}_n(t-t') - \boldsymbol{\phi}(t-t')|^2]dt' \tag{B7}$$

[using equations (B2) and (B4)]

$$= \int_{-\infty}^{t}\text{Var}[\boldsymbol{\phi}_n(t-t')]dt' = \int_0^\infty \text{Var}[\boldsymbol{\phi}_n(\tau)]d\tau \leq \frac{1}{n}\mathbf{E}\left[\frac{Y^2}{2X}\right]. \tag{B8}$$

Step 3. Step 2 yields the bound

$$\mathbf{E}[|\Xi_n(t) - \Xi(t)|^2] \leq \frac{1}{n}\mathbf{E}\left[\frac{Y^2}{2X}\right]. \tag{B9}$$

Given $-\infty < a < b < \infty$, Eq. (B9) further yields the bound

$$\mathbf{E}\left[\int_a^b |\Xi_n(t)-\Xi(t)|^2 dt\right] = \int_a^b \mathbf{E}[|\Xi_n(t)-\Xi(t)|^2]dt \leq \frac{b-a}{n}\mathbf{E}\left[\frac{Y^2}{2X}\right]. \quad (B10)$$

With Eq. (B10) at hand, we conclude that

$$\lim_{n\to\infty} \mathbf{E}\left[\int_a^b |\Xi_n(t)-\Xi(t)|^2 dt\right] = 0. \quad (B11)$$

Step 4.

$$\mathrm{Var}[\Xi(t)] = \mathrm{Var}\left[\int_{-\infty}^t \boldsymbol{\phi}(t-t')\dot{N}(t')dt'\right] \quad (B12)$$

[using Eqs. (3) and (12)]

$$= \int_{-\infty}^t \boldsymbol{\phi}(t-t')^2 dt' = \int_0^\infty \boldsymbol{\phi}(\tau)^2 d\tau = \int_0^\infty \mathbf{E}[Y\exp\{-X\tau\}]^2 d\tau \quad (B13)$$

[using Jensen's inequality—see, for example, Eq. (8.6) in Ref. [40]; and repeating the calculation conducted in Eq. (B4)]

$$\leq \int_0^\infty \mathbf{E}[(Y\exp\{-X\tau\})^2]d\tau = \mathbf{E}\left[\frac{Y^2}{2X}\right]. \quad (B14)$$

Dependence. In the case the amplitude-pairs $(X_1,Y_1),\ldots,(X_n,Y_n)$ are identically distributed—yet dependent—random variables, we have the following counterparts of equations (B3) and (B4):

$$\mathrm{Var}[\boldsymbol{\phi}_n(\tau)] = \frac{1}{n^2}\sum_{k,l=1}^n \mathrm{Cov}[Y_k\exp\{-X_k\tau\}, Y_l\exp\{-X_l\tau\}] \leq \frac{1}{n^2}\sum_{k,l=1}^n \mathbf{E}[Y_k\exp\{-X_k\tau\}Y_l\exp\{-X_l\tau\}] \quad (B15)$$

and

$$\int_0^\infty \mathrm{Var}[\boldsymbol{\phi}_n(\tau)]d\tau \leq \int_0^\infty \frac{1}{n^2}\sum_{k,l=1}^n \mathbf{E}[Y_k\exp\{-X_k\tau\}Y_l\exp\{-X_l\tau\}]d\tau = \frac{1}{n^2}\sum_{k,l=1}^n \mathbf{E}\left[Y_kY_l\int_0^\infty \exp\{-(X_k+X_l)\tau\}d\tau\right] = \frac{1}{n^2}\sum_{k,l=1}^n \mathbf{E}\left[\frac{Y_kY_l}{X_k+X_l}\right]. \quad (B16)$$

Consequently, the counterpart of Eq. (B10) is

$$\mathbf{E}\left[\int_a^b |\Xi_n(t)-\Xi(t)|^2 dt\right] \leq \frac{b-a}{n^2}\sum_{k,l=1}^n \mathbf{E}\left[\frac{Y_kY_l}{X_k+X_l}\right]. \quad (B17)$$

Hence,

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k,l=1}^n \mathbf{E}\left[\frac{Y_kY_l}{X_k+X_l}\right] = 0 \Rightarrow \lim_{n\to\infty}\mathbf{E}\left[\int_a^b |\Xi_n(t)-\Xi(t)|^2 dt\right] = 0. \quad (B18)$$

APPENDIX C: PROOF OF PROPOSITION 2

1. Memory function

Since the memory function $\boldsymbol{\phi}(\cdot)$ is the Laplace transform of the base function $\boldsymbol{\psi}_1(\cdot)$ we have

$$\boldsymbol{\phi}(\tau) = \int_0^\infty \exp\{-\tau x\}\boldsymbol{\psi}_1(x)dx \quad (C1)$$

(using the change of variables $y=\tau x$)

$$= \int_0^\infty \exp\{-y\}\boldsymbol{\psi}_1\left(\frac{1}{\tau}y\right)\frac{1}{\tau}dy = \frac{\boldsymbol{\psi}_1\left(\frac{1}{\tau}\right)}{\tau}\int_0^\infty \exp\{-y\}\frac{\boldsymbol{\psi}_1\left(\frac{1}{\tau}y\right)}{\boldsymbol{\psi}_1\left(\frac{1}{\tau}\right)}dy \quad (C2)$$

[using the fact that $\boldsymbol{\psi}_1(x) \sim \mathbf{l}(x)x^{p-1}$, as $x\to 0$, the function $\mathbf{l}(\cdot)$ being slowly varying at the origin]

$$\underset{\tau\to\infty}{\sim} \frac{\mathbf{l}\left(\frac{1}{\tau}\right)\left(\frac{1}{\tau}\right)^{p-1}}{\tau}\int_0^\infty \exp\{-y\}y^{p-1}dy = \frac{\mathbf{l}\left(\frac{1}{\tau}\right)}{\tau^p}\Gamma(p). \quad (C3)$$

2. Macroscopic autocovariance

With no loss of generality, consider $t>0$. Applying Eq. (20) we have

$$\mathbf{R}_{\mathrm{macro}}(t) = \int_0^\infty \boldsymbol{\phi}(x)\boldsymbol{\phi}(x+t)dx \quad (C4)$$

(using the change of variables $y=x/t$)

$$= \int_0^\infty \boldsymbol{\phi}(ty)\boldsymbol{\phi}[t(1+y)]tdy = t\boldsymbol{\phi}(t)^2\int_0^\infty \frac{\boldsymbol{\phi}(ty)}{\boldsymbol{\phi}(t)}\frac{\boldsymbol{\phi}[t(1+y)]}{\boldsymbol{\phi}(t)}dy \quad (C5)$$

[using Eq. (26)]

$$\underset{t\to\infty}{\sim} t\left(\Gamma(p)\frac{\mathbf{l}(1/t)}{t^p}\right)^2\int_0^\infty \frac{1}{y^p(1+y)^p}dy \quad (C6)$$

[using Eq. (4.3) in Ref. [40]]

$$=t\left(\Gamma(p)\frac{\mathbf{l}(1/t)}{t^{p}}\right)^{2}\frac{\Gamma(1-p)\Gamma(2p-1)}{\Gamma(p)} \quad (C7)$$

[using the identity $\Gamma(p)\Gamma(1-p)=\pi/\sin(\pi p)$, $0<p<1$]

$$=\left(\frac{\pi\Gamma(2p-1)}{\sin(\pi p)}\right)\frac{\mathbf{l}(1/t)^{2}}{t^{2p-1}}. \quad (C8)$$

The result proved in this part is valid for exponents p taking values in the range $1/2<p<1$. This, in turn, corresponds to exponents $\alpha=2p-1$ taking values in the range $0<\alpha<1$.

3. Macroscopic power spectrum

With no loss of generality, consider $\omega>0$. Applying Eq. (22) we have

$$\mathbf{S}_{\mathrm{macro}}(\omega)=\frac{1}{2\pi}\left|\int_{0}^{\infty}\exp\{i\omega x\}\boldsymbol{\phi}(x)dx\right|^{2} \quad (C9)$$

(using the change of variables $y=\omega x$),

$$=\frac{1}{2\pi}\left|\int_{0}^{\infty}\exp\{iy\}\boldsymbol{\phi}\left(\frac{1}{\omega}y\right)\frac{1}{\omega}dy\right|^{2}$$

$$=\frac{1}{2\pi}\frac{\boldsymbol{\phi}\left(\frac{1}{\omega}\right)^{2}}{\omega^{2}}\left|\int_{0}^{\infty}\exp\{iy\}\frac{\boldsymbol{\phi}\left(\frac{1}{\omega}y\right)}{\boldsymbol{\phi}\left(\frac{1}{\omega}\right)}dy\right|^{2} \quad (C10)$$

[using Eq. (26)],

$$\underset{\omega\to0}{\sim}\frac{1}{2\pi}\frac{\left(\Gamma(p)\frac{\mathbf{l}(\omega)}{\omega^{-p}}\right)^{2}}{\omega^{2}}\left|\int_{0}^{\infty}\exp\{iy\}y^{-p}dy\right|^{2} \quad (C11)$$

[using Eq. (6) of Table 1 (p. 503) in Ref. [40]],

$$=\frac{1}{2\pi}\left(\frac{\Gamma(p)\mathbf{l}(\omega)}{\omega^{1-p}}\right)^{2}|\Gamma(1-p)i^{1-p}|^{2}=\frac{1}{2\pi}\left(\frac{\Gamma(p)\mathbf{l}(\omega)}{\omega^{1-p}}\right)^{2}\Gamma(1-p)^{2} \quad (C12)$$

[using the identity $\Gamma(p)\Gamma(1-p)=\pi/\sin(\pi p)$, $0<p<1$],

$$=\left(\frac{\pi/2}{\sin(\pi p)^{2}}\right)\frac{\mathbf{l}(\omega)^{2}}{\omega^{2-2p}}. \quad (C13)$$

The result proved in this part is valid for exponents p taking values in the range $0<p<1$. This, in turn, corresponds to exponents $\beta=2-2p$ taking values in the range $0<\beta<2$. Power-spectrum exponents β, however, correspond to admissible autocovariance functions if and only if they are in the range $0<\beta<1$.

APPENDIX D: PROOF OF PROPOSITION 3

1. Autocovariance limit

$$\mathbf{R}_{c}(t)=\mathrm{Cov}[Z_{c}(0),Z_{c}(t)] \quad (D1)$$

[using Eq. (29)],

$$=\mathrm{Cov}\left[\frac{1}{\boldsymbol{\sigma}(c)}\Xi(c0),\frac{1}{\boldsymbol{\sigma}(c)}\Xi(ct)\right]=\frac{1}{\boldsymbol{\sigma}(c)^{2}}\mathrm{Cov}[\Xi(0),\Xi(ct)] \quad (D2)$$

[using Eq. (20)],

$$=\frac{1}{\boldsymbol{\sigma}(c)^{2}}\mathbf{R}_{\mathrm{macro}}(ct)=\frac{1}{\boldsymbol{\sigma}(c)^{2}}\int_{0}^{\infty}\boldsymbol{\phi}(x)\boldsymbol{\phi}(x+|ct|)dx \quad (D3)$$

(using the change of variables $y=x/c$),

$$=\frac{1}{\boldsymbol{\sigma}(c)^{2}}\int_{0}^{\infty}\boldsymbol{\phi}(cy)\boldsymbol{\phi}(cy+|ct|)cdy$$

$$=\left(\frac{\sqrt{c}\boldsymbol{\phi}(c)}{\boldsymbol{\sigma}(c)}\right)^{2}\int_{0}^{\infty}\frac{\boldsymbol{\phi}(cy)}{\boldsymbol{\phi}(c)}\frac{\boldsymbol{\phi}[c(y+|t|)]}{\boldsymbol{\phi}(c)}dy \quad (D4)$$

[using Eq. (26) and the scaling $\boldsymbol{\sigma}(c)\sim\sqrt{c}\boldsymbol{\phi}(c)$ (as $c\to\infty$)]

$$\underset{c\to\infty}{\to}\int_{0}^{\infty}y^{-p}(y+|t|)^{-p}dy. \quad (D5)$$

Thus, we have obtained that

$$\mathbf{R}_{\infty}(t):=\lim_{c\to\infty}\mathbf{R}_{c}(t)=\int_{0}^{\infty}x^{-p}(x+|t|)^{-p}dx. \quad (D6)$$

Calculating the right-hand-side integral of Eq. (D6) [using Eq. (4.3) in Ref. [40]] we conclude that

$$\mathbf{R}_{\infty}(t)=\left(\frac{\Gamma(1-p)\Gamma(2p-1)}{\Gamma(p)}\right)\frac{1}{|t|^{2p-1}}. \quad (D7)$$

Comparing Eq. (D6) to Eq. (20) we observe that plugging the memory function $\boldsymbol{\phi}(\tau)=\tau^{-p}$ into Eq. (20) yields the autocovariance function $\mathbf{R}_{\infty}(\cdot)$. The memory function $\boldsymbol{\phi}(\tau)=\tau^{-p}$, in turn, is the Laplace transform of the base function $\psi_{1}(x)=x^{p-1}/\Gamma(p)$. The result proved in this part is valid for exponents p taking values in the range $1/2<p<1$.

2. Power-spectrum limit

$$\mathbf{S}_{c}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{-i\omega t\}\mathbf{R}_{c}(t)dt \quad (D8)$$

(using step 1),

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{-i\omega t\}\left(\frac{1}{\boldsymbol{\sigma}(c)^{2}}\mathbf{R}_{\mathrm{macro}}(ct)\right)dt \quad (D9)$$

(using the change of variables $s=ct$),

$$=\frac{1}{c\boldsymbol{\sigma}(c)^{2}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{-i\frac{\omega}{c}s\right\}\mathbf{R}_{\mathrm{macro}}(s)ds$$

$$=\frac{1}{c\boldsymbol{\sigma}(c)^{2}}\mathbf{S}_{\mathrm{macro}}\left(\frac{\omega}{c}\right) \quad (D10)$$

[using Eq. (22)],

$$=\frac{1}{c\boldsymbol{\sigma}(c)^2}\frac{1}{2\pi}\left|\int_0^\infty \exp\left\{i\frac{\omega}{c}x\right\}\boldsymbol{\phi}(x)dx\right|^2 \tag{D11}$$

(using the change of variables $y=x/c$),

$$=\frac{1}{c\boldsymbol{\sigma}(c)^2}\frac{1}{2\pi}\left|\int_0^\infty \exp\{i\omega y\}\boldsymbol{\phi}(cy)cdy\right|^2$$

$$=\left(\frac{\sqrt{c}\boldsymbol{\phi}(c)}{\boldsymbol{\sigma}(c)}\right)^2\frac{1}{2\pi}\left|\int_0^\infty \exp\{i\omega y\}\frac{\boldsymbol{\phi}(cy)}{\boldsymbol{\phi}(c)}dy\right|^2 \tag{D12}$$

[using Eq. (26) and the scaling $\boldsymbol{\sigma}(c)\sim\sqrt{c}\boldsymbol{\phi}(c)$ (as $c\to\infty$)],

$$\underset{c\to\infty}{\to}\frac{1}{2\pi}\left|\int_0^\infty \exp\{i\omega y\}y^{-p}dy\right|^2. \tag{D13}$$

Thus, we have obtained that

$$\mathbf{S}_\infty(\omega) := \lim_{c\to\infty}\mathbf{S}_c(\omega)=\frac{1}{2\pi}\left|\int_0^\infty \exp\{i\omega x\}x^{-p}dx\right|^2. \tag{D14}$$

Calculating the right-hand-side integral of Eq. (D14) [using Eq. (6) of Table 1 (p. 503) in Ref. [40]] we conclude that

$$\mathbf{S}_\infty(\omega)=\left(\frac{\Gamma(1-p)^2}{2\pi}\right)\frac{1}{|\omega|^{2-2p}}. \tag{D15}$$

Comparing Eq. (D14) to Eq. (22) we observe that: Plugging the memory function $\boldsymbol{\phi}(\tau)=\tau^{-p}$ into Eq. (22) yields the power-spectrum function $\mathbf{S}_\infty(\cdot)$. The memory function $\boldsymbol{\phi}(\tau)=\tau^{-p}$, in turn, is the Laplace transform of the base function $\boldsymbol{\psi}_1(x)=x^{p-1}/\Gamma(p)$. The result proved in this part is valid for exponents p taking values in the range $0<p<1$.

APPENDIX E: PROOF OF PROPOSITION 4

We split the proof into three steps, and thereafter explain why the stochastic limit Z is a noise process. Throughout the proof, (i) the function $\theta(\cdot)$ will denote an arbitrary Fourier variable test function and (ii) we shall consider the memory function $\boldsymbol{\phi}(\cdot)$ as defined on the entire real line $(-\infty,\infty)$, and vanishing on the nonpositive half-line $(-\infty,0]$.

Step 1. Using the definition of the scaled process Z_c [Eq. (29)] and the macroscopic COU process Ξ [Eq. (18)] we have

$$\int_{-\infty}^{\infty}\theta(t)Z_c(t)dt=\int_{-\infty}^{\infty}\theta(t)\left(\frac{1}{\boldsymbol{\sigma}(c)}\Xi(ct)\right)dt$$
$$=\int_{-\infty}^{\infty}\theta(t)\left(\frac{1}{\boldsymbol{\sigma}(c)}\int_{-\infty}^{\infty}\boldsymbol{\phi}(ct-t')\dot{N}(t')dt'\right)dt$$
$$=\int_{-\infty}^{\infty}\left(\frac{1}{\boldsymbol{\sigma}(c)}\int_{-\infty}^{\infty}\theta(t)\boldsymbol{\phi}(ct-t')dt\right)\dot{N}(t')dt'. \tag{E1}$$

Applying Eq. (1) to Eq. (E1) further gives

$$\mathbf{E}\left[\exp\left\{i\int_{-\infty}^{\infty}\theta(t)Z_c(t)dt\right\}\right]=\exp\{-\mathcal{J}_c(\theta)\}, \tag{E2}$$

where

$$\mathcal{J}_c(\theta) := \int_{-\infty}^{\infty}\mathcal{L}\left(\frac{1}{\boldsymbol{\sigma}(c)}\int_{-\infty}^{\infty}\theta(t)\boldsymbol{\phi}(ct-t')dt\right)dt'. \tag{E3}$$

Using the change of variables $s=t'/c$ in Eq. (E3) we conclude that

$$\mathcal{J}_c(\theta)=\int_{-\infty}^{\infty}c\mathcal{L}\left(\frac{1}{\boldsymbol{\sigma}(c)}\int_s^{\infty}\theta(t)\boldsymbol{\phi}[c(t-s)]dt\right)ds. \tag{E4}$$

Step 2. Eq. (E4) can be rewritten as

$$\mathcal{J}_c(\theta)=\int_{-\infty}^{\infty}c\mathcal{L}\left(\frac{1}{\sqrt{c}}I_c(s)\right)ds, \tag{E5}$$

where

$$I_c(s) := \left(\frac{\sqrt{c}\boldsymbol{\phi}(c)}{\boldsymbol{\sigma}(c)}\right)\int_s^{\infty}\theta(t)\frac{\boldsymbol{\phi}[c(t-s)]}{\boldsymbol{\phi}(c)}dt \tag{E6}$$

(s real). Equation (26), together with the scaling $\boldsymbol{\sigma}(c)\sim\sqrt{c}\boldsymbol{\phi}(c)$ (as $c\to\infty$), imply that

$$I(s) := \lim_{c\to\infty}I_c(s)=\int_s^{\infty}\theta(t)(t-s)^{-p}dt. \tag{E7}$$

On the other hand,

$$\mathcal{L}\left(\frac{1}{\sqrt{c}}I_c(s)\right)=\mathcal{L}(0)+\mathcal{L}'(0)\left(\frac{1}{\sqrt{c}}I_c(s)\right)+\frac{1}{2}\mathcal{L}''(0)\left(\frac{1}{\sqrt{c}}I_c(s)\right)^2$$
$$+o\left[\left(\frac{1}{\sqrt{c}}I_c(s)\right)^2\right]=\frac{1}{2}\frac{1}{c}I_c(s)^2+o\left[\frac{1}{c}I_c(s)^2\right] \tag{E8}$$

[we used the fact that $\mathcal{L}(0)=0$, as well as the fact that the driving noise $\dot{N}$ is centered—i.e., $\mathcal{L}'(0)=0$ and $\mathcal{L}''(0)=1$]. Hence, combining Eqs. (E7) and (E8) together, we have

$$\lim_{c\to\infty}c\mathcal{L}\left(\frac{1}{\sqrt{c}}I_c(s)\right)=\frac{1}{2}I(s)^2. \tag{E9}$$

Consequently, plugging Eq. (E9) into Eq. (E10) we obtain that

$$\mathcal{J}(\theta) := \lim_{c\to\infty}\mathcal{J}_c(\theta)=\int_{-\infty}^{\infty}\frac{1}{2}\left(\int_s^{\infty}\theta(t)(t-s)^{-p}dt\right)^2ds. \tag{E10}$$

Step 3. Combining Eqs. (E2) and (E10) together implies that

$$\lim_{c\to\infty} \mathbf{E}\left[\exp\left\{i\int_{-\infty}^{\infty}\theta(t)Z_c(t)dt\right\}\right] = \exp\left\{-\int_{-\infty}^{\infty}\frac{1}{2}\left(\int_{s}^{\infty}\theta(t)(t-s)^{-p}dt\right)^2 ds\right\}. \quad \text{(E11)}$$

Consider now the MA process $Z=[Z(t)]_t$ given by

$$Z(t) = \int_{-\infty}^{t}(t-t')^{-p}\dot{N}_{\mathrm{WN}}(t')dt', \quad \text{(E12)}$$

where $\dot{N}_{\mathrm{WN}}$ is a white noise. Equations (E2) and (E4) imply that [setting $c=1=\boldsymbol{\sigma}(c)$ and recalling that $\mathcal{L}_{\mathrm{WN}}(\theta)=\frac{1}{2}|\theta|^2$]:

$$\mathbf{E}\left[\exp\left\{i\int_{-\infty}^{\infty}\theta(t)Z(t)dt\right\}\right] = \exp\left\{-\int_{-\infty}^{\infty}\frac{1}{2}\left(\int_{s}^{\infty}\theta(t)(t-s)^{-p}dt\right)^2 ds\right\}. \quad \text{(E13)}$$

Since the right-hand sides of Eqs. (E11) and (E13) coincide, we arrive at the conclusion that

$$\lim_{c\to\infty} \mathbf{E}\left[\exp\left\{i\int_{-\infty}^{\infty}\theta(t)Z_c(t)dt\right\}\right] = \mathbf{E}\left[\exp\left\{i\int_{-\infty}^{\infty}\theta(t)Z(t)dt\right\}\right]. \quad \text{(E14)}$$

Namely, the scaled process Z_c converges, in the scaling limit $c\to\infty$, in law, to a limiting MA process $Z=[Z(t)]_t$.

The limiting process Z. The MA limit Z is not a "real process" but rather, a noise process—as we now explain. The (one-dimensional) Fourier transform of the random variable $Z(t)$ is given by

$$\mathbf{E}[\exp\{i\omega Z(t)\}] = \mathbf{E}\left[\exp\left\{i\int_{-\infty}^{t}\frac{\omega}{(t-t')^p}\dot{N}_{\mathrm{WN}}(t')dt'\right\}\right] \quad \text{(E15)}$$

[using Eq. (1), and thereafter the change of variables $\tau=t-t'$],

$$=\exp\left\{-\int_{-\infty}^{t}\frac{1}{2}\left(\frac{\omega}{(t-t')^p}\right)^2 dt'\right\} = \exp\left\{-\frac{\omega^2}{2}\int_{0}^{\infty}\frac{1}{\tau^{2p}}d\tau\right\}(\omega\ \text{real}). \quad \text{(E16)}$$

The integral $\int_0^\infty \tau^{-2p}d\tau$ appearing on the right-hand side of Eq. (E16) is divergent. This, in turn, implies that the function $\varphi(t')=\omega/(t-t')^p$ $(t'<t)$ is a nonadmissible integrand with respect to white noise. In other words, the MA process Z is ill defined. On the other hand, the Fourier transform of the process Z appearing in Eq. (E13) is well defined [say, for test functions $\theta(\cdot)$ which are bounded and have a bounded support]. This means that Z is a noise process—ill defined as a process per se but well defined as a noise measured via its "action" on test functions.

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