

## Equilibrium long-ranged charge correlations at the surface of a conductor coupled to electromagnetic radiation. II.

Bernard Jancovici\* and Ladislav Šamaj†

Laboratoire de Physique Théorique, Université de Paris-Sud, 91405 Orsay Cedex, France‡

(Received 3 November 2008; revised manuscript received 25 December 2008; published 10 February 2009)

Results of a previous paper with the same title are retrieved by a different method. A one-component plasma is bounded by a plane surface. The plasma is fully coupled to the electromagnetic field, therefore, the charge correlations are retarded. The quantum correlation function of the surface charge densities, at times different by  $t$ , at asymptotical large distances  $R$ , at inverse temperature  $\beta$ , decays as  $-1/(8\pi^2\beta R^3)$ , a surprisingly simple result: The decay is independent of Planck's constant  $\hbar$ , the time difference  $t$ , and the velocity of light  $c$ . The present paper is based on the analysis of the collective vibration modes of the system.

DOI: 10.1103/PhysRevE.79.021111

PACS number(s): 05.30.-d, 52.40.Db, 73.20.Mf, 05.40.-a

### I. INTRODUCTION

In a previous paper [1], we studied the asymptotic form of the two-point correlation function of the surface charge densities on a plane wall bounding a conductor [in the special case when the conductor is a one-component plasma (OCP), also called jellium], taking into account the retardation and the quantum nature of both the one-component plasma and the radiation. The special feature was the retardation; instead of assuming as an interaction the Coulomb potential only, in [1] the coupling was through the full electromagnetic radiation. In other words instead of assuming the velocity of light  $c$  to be infinite, the full Maxwell equations were used.

The previous paper [1] used the elaborate formalism of Rytov [2], presented also in [3]. This formalism is macroscopic, using frequency-dependent dielectric functions. In the present paper, we retrieve the same results by a simpler method, based on the analysis of the collective vibration modes of the system. This method is partially microscopic. It has already been used in the nonretarded case [4], Sec. IV.

We use Gaussian units. The OCP is made of nonrelativistic point particles of charge  $e$ , mass  $m$ , and number density  $n$ , immersed in a uniform neutralizing background of charge density  $-ne$ . We recall the geometry (Fig. 1). We use Cartesian coordinates; a point is  $\mathbf{r}=(x, y, z)$ . The OCP occupies the half-space  $\Lambda_1=\{x>0\}$ , the half-space  $\Lambda_2=\{x<0\}$  is vacuum; the two half-spaces are separated by a plane wall, impenetrable to the jellium, at  $x=0$ . A point on the wall is  $\mathbf{R}=(y, z)$ .

After long calculations, a very simple result was found in [1],

$$\begin{aligned} \beta S(t, \mathbf{R}) &\equiv \beta \frac{1}{2} \langle \sigma(t, \mathbf{R}) \sigma(0, \mathbf{0}) + \sigma(0, \mathbf{0}) \sigma(t, \mathbf{R}) \rangle^T \\ &\sim -\frac{1}{8\pi^2} \frac{1}{R^3}, \quad R \rightarrow \infty, \end{aligned} \quad (1)$$

where  $\beta$  is the inverse temperature,  $\langle \dots \rangle^T$  represents a truncated statistical average,  $\sigma(t, \mathbf{R})$  is the surface charge density at time  $t$  and at point  $\mathbf{R}$  on the surface. This value (1) surprisingly is independent of  $t$ ,  $\hbar$ , and  $c$ , boiling down to the classical result at time difference zero without retardation.

We define the Fourier transform of a function  $f(\mathbf{R})$  as

$$f(\mathbf{q}) = \int d^2R \exp(i\mathbf{q} \cdot \mathbf{R}) f(\mathbf{R}). \quad (2)$$

A result equivalent to (1) is that the Fourier transform of its left-hand side (lhs),  $\beta S(t, \mathbf{q})$ , has a kink singularity at  $\mathbf{q}=\mathbf{0}$ , behaving at small  $q$  like  $q/(4\pi)$ .

In the present paper, we consider a fluctuation of the surface charge density  $\sigma$ , of wave number  $\mathbf{q}$  and frequency  $\omega$ , such that  $\sigma$  is of the form

$$\sigma(\mathbf{R}, t) = \sigma_{\mathbf{q}\omega}(t) \exp(i\mathbf{q} \cdot \mathbf{R}) + \text{c.c.}, \quad (3)$$

where c.c. means complex conjugate.  $\sigma_{\mathbf{q}\omega}(t)$  is a complex quantity, vibrating at frequency  $\omega$ . This surface charge is viewed as an external excitation, while the response of the plasma is treated in the linear regime. Various domains of frequencies must be considered. The total energy (electromagnetic energy plus the kinetic energy of the particles) is obtained as a function of  $\sigma_{\mathbf{q}\omega}(t)$  and  $\dot{\sigma}_{\mathbf{q}\omega}(t)$  in the form of a harmonic oscillator. The final step deals with equilibrium statistical averages, which involve standard calculations for these oscillators, combined with frequency integrations.

The present paper is organized as follows. Section II is a general exposition of the formalism of vibration modes. Section III describes the contribution of surface modes (which are localized on both sides of the wall). Section IV describes

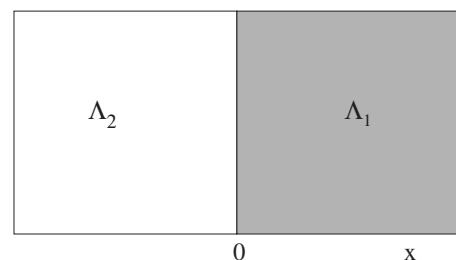


FIG. 1. The geometry.

\*bernard.jancovici@th.u-psud.fr

†On leave from Institute of Physics, Slovak Academy of Sciences, Bratislava; ladislav.samaj@savba.sk

‡Unité Mixte de Recherche No. 8627-CNRS.

the contribution of transverse modes delocalized on the vacuum side. Section V describes the contribution of transverse modes delocalized on both sides. In Section VI, we recall the contribution of the longitudinal modes. Section VII is the Conclusion.

## II. COLLECTIVE VIBRATION MODES

The emitted radiation corresponding to the source (3) is described by Maxwell equations (we use the microscopic ones, involving only the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{B}$ , averaged in a suitable way [5]). The charge density is  $\rho$ , the electric current density is  $\mathbf{J}$ , the velocity of light is  $c$ . These Maxwell equations are

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{J}, \quad (4)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (5)$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad (6)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (7)$$

In the quantum case, the quantities appearing in these equations are operators. In region  $\Lambda_2$ ,  $\rho=0$  and  $\mathbf{J}=0$ , of course.

Furthermore, in region  $\Lambda_1$ , the Maxwell equations (4)–(7) must be supplemented by

$$\mathbf{E} = \frac{4\pi}{\omega_p^2} \frac{\partial \mathbf{J}}{\partial t}, \quad (8)$$

where  $\omega_p = (4\pi ne^2/m)^{1/2}$  is the plasma frequency. This Eq. (8) is obtained from assuming that the OCP can be considered as a collection of nonrelativistic free charges and the dynamics can be linearized. The velocity of a particle,  $\mathbf{v}(\mathbf{r}, t)$  obeys the Newton equation  $m\partial\mathbf{v}/\partial t = e\mathbf{E}$ . The term  $e(\mathbf{v}/c) \times \mathbf{B}$  in the Lorentz force is suppressed because for a nonrelativistic plasma  $v/c$  is negligible (the condition that the plasma is nonrelativistic is  $\beta mc^2 \gg 1$ ). The current density is  $\mathbf{J} = env$  (the density is the constant  $n$ , because of the linearization).

It should be remarked that in Eq. (8), there is no damping term; this absence of damping term is valid for the small wave numbers which will be considered, and is a property special to the OCP.

Macroscopic Maxwell equations equivalent to (4)–(8) can also be obtained. Combining (4) and (8), at frequency  $\omega$ , we obtain

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial(\epsilon\mathbf{E})}{\partial t}, \quad (9)$$

where the frequency-dependent dielectric function is

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad (10)$$

which is of the Drude form, without dissipation. Multiplying (6), where  $\rho=0$ , by  $\epsilon$  does not change anything if  $\epsilon \neq 0$  (the

case  $\epsilon=0$  will be studied later). For a nonrelativistic plasma, we have neglected the magnetic force in (8); therefore, it is consistent to take the magnetic permeability as  $\mu=1$ . Thus we have obtained the macroscopic Maxwell equations [5], with now  $\mathbf{J}=0$ .

There are solutions to these Maxwell equations which are superpositions of transverse waves, which we shall study first. In region  $\Lambda_1$ , for these transverse waves,  $\rho=0$ . The wave-number vector has components  $(k_1, \mathbf{q})$  in region  $\Lambda_1$  and  $(k_2, \mathbf{q})$  in region  $\Lambda_2$ , where  $k_1$  and  $k_2$  are the  $x$  components [the wave numbers have the same components parallel to the surface  $\mathbf{q}=(q_y, q_z)$  as a consequence of the boundary conditions, as will be shown later].

From the Maxwell equations (4)–(8), the dispersion relations are

$$\omega^2 = c^2(q^2 + k_2^2) \quad (11)$$

in region  $\Lambda_2$ , and

$$\omega^2 = \omega_p^2 + c^2(q^2 + k_1^2) \quad (12)$$

in region  $\Lambda_1$ . Depending on the value of  $\omega$ ,  $k_1$  or  $k_2$  are real or pure imaginary, as shown from (11) and (12). If  $\omega^2 < (cq)^2$ , both  $k_1$  and  $k_2$  are pure imaginary. If  $(cq)^2 < \omega^2 < \omega_p^2 + (cq)^2$ ,  $k_1$  is pure imaginary and  $k_2$  is real. If  $\omega^2 > \omega_p^2 + (cq)^2$ , both  $k_1$  and  $k_2$  are real.

In addition to these transverse plane waves, in region  $\Lambda_1$ , the Maxwell equations have solutions with frequency  $\omega_p$  (then  $\epsilon=0$ ) which are longitudinal waves, without magnetic field [5,6]. They have already been studied in [4]. The equations (4)–(8) reduce to  $\nabla \times \mathbf{E} = 0$ ,  $\nabla \cdot \mathbf{E} = 4\pi\rho$ ,  $\mathbf{E} = (4\pi/\omega_p^2)(\partial\mathbf{J}/\partial t)$ , supplemented by the continuity equation  $(\partial\rho/\partial t) + \nabla \cdot \mathbf{J} = 0$ .

The energy density in region  $\Lambda_2$  is [5]

$$U_2 = \frac{1}{8\pi}(E^2 + B^2); \quad (13)$$

the energy density in region  $\Lambda_1$  is

$$U_1 = \frac{1}{8\pi}(E^2 + B^2) + \frac{2\pi}{\omega_p^2} J^2, \quad (14)$$

where the last term in (14) is the kinetic energy. A derivation of (14) is presented in the Appendix, as well as another form of it [6].

At  $x=0$ , the existence of a surface charge density  $\sigma$  is associated with a discontinuity of the  $x$  component of the electrical field

$$4\pi\sigma = E_x^+ - E_x^-, \quad (15)$$

where the superscripts plus or minus mean approaching the plane  $x=0$  from the regions  $\Lambda_1$  or  $\Lambda_2$ , respectively. The other conditions at  $x=0$  are the continuity of the components of the electric and magnetic fields parallel to the surface. From the conditions at  $x=0$ , one easily deduces that the  $(y, z)$  components of the wave numbers indeed are  $\mathbf{q}$ . Since the system is invariant by rotations around the  $x$  axis, general results can be obtained by choosing  $\mathbf{q}$  along the  $y$  axis:  $\mathbf{q}=(q_y, 0)$ ; then  $q_y^2 = q^2$ . We consider only modes which contribute to  $\sigma$ ; these modes are TM waves, with the electric field in the plane

determined by the wave vector  $\mathbf{q}$  and the normal to the surface.

### III. $\omega^2 < (cq)^2$ (SURFACE POLARITONS)

The surface polaritons are the only modes in this frequency range. They are TM waves [7];  $k_1$  and  $k_2$  are pure imaginary. These solutions of the Maxwell equations are of the form (the subscript  $j=1,2$  denotes the region  $\Lambda_1$  or  $\Lambda_2$ )

$$\mathbf{E}_j = [a_j(t), b_j(t), 0] \exp(iq_y y - \kappa_j |x|) + \text{c.c.}, \quad (16)$$

$$\mathbf{B}_j = [0, 0, d_j(t)] \exp(iq_y y - \kappa_j |x|) + \text{c.c.}, \quad (17)$$

where, from (11) and (12),  $c\kappa_2 = \sqrt{(cq)^2 - \omega^2}$  and  $c\kappa_1 = \sqrt{\omega_p^2 + (cq)^2 - \omega^2}$ . Since all fields are localized near the surface, these modes are called surface plasmons; polariton is also used when retardation, as here, is taken into account.

From the Maxwell equations and the conditions that  $E_y$  and  $B_z$  are continuous at  $x=0$ , in this section there is a relation between  $\omega$  and  $\mathbf{q}$ ,

$$\omega^2 = \omega_p^2/2 + (cq)^2 - \sqrt{(\omega_p^2/2)^2 + (cq)^4}. \quad (18)$$

Therefore, in this section,  $\sigma_{\mathbf{q}\omega}$  will be called  $\sigma_{\mathbf{q}}$ .  $\omega$  has its nonretarded value  $\omega_p/\sqrt{2}$  only at large  $q$ . For the small values of  $q$ , that we are interested in,  $\omega$  behaves as  $cq$ . From (18) follows  $\kappa_1\kappa_2 = q^2$ , a useful relation in the calculations (the detail of which is omitted) which follow.

Taking into account (15) and the continuity of  $E_y$  and  $B_z$ , the Maxwell equations (5), (6) [ $\rho=0$  in (6)] give the prefactors  $a_i, b_i, d_i$  as functions of  $\sigma_{\mathbf{q}}$ . One finds

$$\begin{aligned} a_1(t) &= \frac{\kappa_2}{\kappa_1 + \kappa_2} 4\pi\sigma_{\mathbf{q}}(t), & a_2(t) &= -\frac{\kappa_1}{\kappa_1 + \kappa_2} 4\pi\sigma_{\mathbf{q}}(t), \\ b_1(t) &= b_2(t) = -i\frac{q_y}{\kappa_1 + \kappa_2} 4\pi\sigma_{\mathbf{q}}(t), \\ d_1(t) &= d_2(t) = \frac{iq_y}{c\kappa_2\kappa_1 + \kappa_2} 4\pi\dot{\sigma}_{\mathbf{q}}(t). \end{aligned} \quad (19)$$

In region  $\Lambda_1$ ,  $\mathbf{J}$  can be obtained from (4). One finds

$$\begin{aligned} J_{1x} &= -\exp(iq_y y - \kappa_1 x) \dot{\sigma}_{\mathbf{q}}(t) + \text{c.c.}, \\ J_{1y} &= \frac{iq_y}{\kappa_2} \exp(iq_y y - \kappa_1 x) \dot{\sigma}_{\mathbf{q}}(t) + \text{c.c.} \end{aligned} \quad (20)$$

From (19) and (20), one computes the energy densities (13) and (14) and the total energy  $H_{\mathbf{q}}$ . Later, we shall consider that the large area  $A$  of the wall goes to infinity; therefore, the oscillatory terms  $\exp(\pm 2iq_y y)$  do not contribute to the integral on  $\mathbf{R}$ . One finds

$$\begin{aligned} H_{\mathbf{q}} &\equiv \int_A d^2R \left( \int_0^\infty dx U_1 + \int_{-\infty}^0 dx U_2 \right) \\ &= A 2\pi \left( \frac{\kappa_1}{\kappa_2} + \frac{\kappa_2}{\kappa_1} \right) \frac{1}{\kappa_1 + \kappa_2} \left( |\sigma_{\mathbf{q}}(t)|^2 + \frac{1}{\omega^2} |\dot{\sigma}_{\mathbf{q}}(t)|^2 \right) \\ &\equiv A C_{\mathbf{q}} \left( |\sigma_{\mathbf{q}}(t)|^2 + \frac{1}{\omega^2} |\dot{\sigma}_{\mathbf{q}}(t)|^2 \right). \end{aligned} \quad (21)$$

One sees that  $H_{\mathbf{q}}$  is the energy of a two-dimensional harmonic oscillator [two-dimensional because  $\sigma_{\mathbf{q}}(t)$  is a complex quantity]. For such a quantum oscillator, where the variable  $\sigma_{\mathbf{q}}(t)$  plays the role of the position variable, the contribution to  $\beta S(t, \mathbf{q})$  is

$$\beta S_A(t, \mathbf{q}) = \frac{1}{C_{\mathbf{q}}} f(\omega) \cos(\omega t), \quad (22)$$

where

$$f(\omega) = \frac{\beta \hbar \omega}{2} \coth \frac{\beta \hbar \omega}{2}. \quad (23)$$

Here, for  $q \rightarrow 0$ ,  $\omega \sim cq$ ,  $C_{\mathbf{q}}$  is proportional to  $1/q^2$ , and (22) behaves like  $q^2$ . Therefore, the polaritons do not contribute to the retarded asymptotic form of  $S(t, \mathbf{R})$ .

On the contrary, in the nonretarded case  $c \rightarrow \infty$ ,  $\omega = \omega_s \equiv \omega_p/\sqrt{2}$ , and (22) becomes

$$\beta S_A(t, q) = \frac{q}{2\pi} f(\omega_s) \cos(\omega_s t), \quad (24)$$

in agreement with [4].

### IV. $(cq)^2 < \omega^2 < \omega_p^2 + (cq)^2$

Now  $k_1$  is pure imaginary while  $k_2$  is real. With  $\mathbf{q} = (q_y, 0)$ , in region  $\Lambda_2$  the general form of component  $E_x$  must be

$$E_{2x} = \exp(iq_y y) [a(t) \cos(k_2 x) + b(t) \sin(k_2 x)] + \text{c.c.} \quad (25)$$

Using the Maxwell equations (5) and (6) gives the other nonzero components of the fields in region  $\Lambda_2$  as

$$E_{2y} = \exp(iq_y y) \frac{ik_2}{q_y} [-a(t) \sin(k_2 x) + b(t) \cos(k_2 x)] + \text{c.c.}, \quad (26)$$

$$B_{2z} = -\exp(iq_y y) \frac{1}{cq_y} [\dot{a}(t) \cos(k_2 x) + \dot{b}(t) \sin(k_2 x)] + \text{c.c.} \quad (27)$$

In region  $\Lambda_1$ , since  $k_1$  is pure imaginary, the fields depend on  $x$  like  $\exp(-\kappa_1 x)$ . The continuity of  $E_y$  and  $B_z$  at  $x=0$  determine the coefficients in function of  $a$  and  $b$ , and (3) and (15) give

$$a(t) = \frac{\omega^2 - \omega_p^2}{\omega_p^2} 4\pi\sigma_{\mathbf{q}\omega}(t), \quad b(t) = -\frac{\kappa_1 \omega^2}{k_2 \omega_p^2} 4\pi\sigma_{\mathbf{q}\omega}(t). \quad (28)$$

Let  $\Lambda_2$  be the region  $0 > x > -L_2$  (with a large  $L_2$ , which at the end will be taken as infinite). Since  $U_1$  has an exponential factor  $\exp(-2\kappa_1 x)$ , the region  $\Lambda_1$  does not contribute to the total energy in this limit. The total energy only is  $H_{\mathbf{q}\omega} = A \int_{-L_2}^0 dx U_2$ . Using (25)–(28) in (13) gives

$$\begin{aligned} H_{\mathbf{q}\omega} &= 2\pi AL_2 \frac{\omega^2(\omega_+^2 - \omega^2)(\omega^2 - \omega_-^2)}{\omega_p^2(cq)^2[\omega^2 - (cq)^2]} \left( |\sigma_{\mathbf{q}\omega}(t)|^2 + \frac{1}{\omega^2} |\dot{\sigma}_{\mathbf{q}\omega}(t)|^2 \right) \\ &\equiv AL_2 C_{\mathbf{q}\omega} \left( |\sigma_{\mathbf{q}\omega}(t)|^2 + \frac{1}{\omega^2} |\dot{\sigma}_{\mathbf{q}\omega}(t)|^2 \right), \end{aligned} \quad (29)$$

where  $\omega_-^2$  is given by (18) and  $\omega_+^2$  is given by a modified (18) with a plus sign in front of the square root. Therefore, the contribution of this mode to  $\beta S(t, \mathbf{q})$  is

$$\beta S_{B\omega}(t, \mathbf{q}) = \frac{1}{L_2 C_{\mathbf{q}\omega}} f(\omega) \cos(\omega t). \quad (30)$$

There are an infinity of modes of this kind, labeled by their frequency  $\omega$ , or, equivalently, by  $k_2$ . For obtaining the total contribution of these modes,  $\beta S_B(t, \mathbf{q})$  to  $\beta S(t, \mathbf{q})$ , one must take the sum on  $k_2$  of (30). Since  $L_2$  is large,  $\sum_{k_2} \dots = (L_2/\pi) \int dk_2 \dots$ . Furthermore,  $dk_2 = d(\omega^2)/(2c^2 k_2)$ . Therefore,

$$\begin{aligned} \beta S_B(t, \mathbf{q}) &= \frac{1}{\pi} \int_{(cq)^2}^{\omega_p^2 + (cq)^2} d(\omega^2) \frac{1}{2c\sqrt{\omega^2 - (cq)^2} C_{\mathbf{q}\omega}} \\ &\times f(\omega) \cos(\omega t). \end{aligned} \quad (31)$$

After the change of variable  $\omega^2 = (cq)^2(1+u)$ , (31) becomes, in the limit  $q \rightarrow 0$ ,

$$\beta S_B(t, \mathbf{q}) \sim \frac{q}{4\pi^2} f(0) \cos 0 \int_0^\infty \frac{du}{\sqrt{u(1+u)}} = \frac{q}{4\pi}. \quad (32)$$

Equation (32) will be found as the only contribution of order  $q$  to  $\beta S(t, \mathbf{q})$ .

### V. $\omega^2 > \omega_p^2 + (cq)^2$

Now,  $k_1$  and  $k_2$  are both real. The general form of the components  $x$  of the electrical fields are

$$E_{jx} = \exp(iq_y y) [a_j(t) \cos(k_j x) + b_j(t) \sin(k_j x)] + \text{c.c.} \quad (33)$$

Equation (33) involves four coefficients, instead of three in  $E_{ix}$  of Sec. IV (where  $E_{ix}$ , not written explicitly, involves only one coefficient). Therefore, the equations which were used in Sec. IV are not enough for determining all the coefficients in the fields as functions of  $\sigma_{\mathbf{q}\omega}(t)$ . Fortunately, parts of the fields are uncoupled to  $\sigma$  and, for describing the modes coupled to  $\sigma$ , it is enough to choose in (33)  $b_i = 0$ . Then, the Maxwell equations and the conditions at  $x=0$  determine all the fields as functions of  $\sigma_{\mathbf{q}\omega}(t)$ . Assuming region  $\Lambda_i$  to be of large length  $L_i$  in  $x$ , one finds the total energy  $H = H_{1\mathbf{q}\omega} + H_{2\mathbf{q}\omega}$  where the energy in region  $\Lambda_1$  is

$$\begin{aligned} H_{1\mathbf{q}\omega} &= AL_1 \frac{2\pi\omega^2}{\omega_p^4(cq)^2} \omega^2(\omega^2 - \omega_p^2) \left( |\sigma_{\mathbf{q}\omega}(t)|^2 + \frac{1}{\omega^2} |\dot{\sigma}_{\mathbf{q}\omega}(t)|^2 \right) \\ &\equiv AL_1 C_{1\mathbf{q}\omega} \left( |\sigma_{\mathbf{q}\omega}(t)|^2 + \frac{1}{\omega^2} |\dot{\sigma}_{\mathbf{q}\omega}(t)|^2 \right), \end{aligned} \quad (34)$$

and in region  $\Lambda_2$ ,

$$\begin{aligned} H_{2\mathbf{q}\omega} &= AL_2 \frac{2\pi\omega^2}{\omega_p^4(cq)^2} (\omega^2 - \omega_p^2)^2 \left( |\sigma_{\mathbf{q}\omega}(t)|^2 + \frac{1}{\omega^2} |\dot{\sigma}_{\mathbf{q}\omega}(t)|^2 \right) \\ &\equiv AL_2 C_{2\mathbf{q}\omega} \left( |\sigma_{\mathbf{q}\omega}(t)|^2 + \frac{1}{\omega^2} |\dot{\sigma}_{\mathbf{q}\omega}(t)|^2 \right). \end{aligned} \quad (35)$$

For obtaining the total contribution of these modes, of different frequencies  $\omega$ , to  $\beta S(t, \mathbf{q})$ , it would be necessary to sum on  $\omega$  the quantity

$$\beta S_{C\omega}(t, \mathbf{q}) = \frac{1}{L_1 C_{1\mathbf{q}\omega} + L_2 C_{2\mathbf{q}\omega}} f(\omega) \cos(\omega t). \quad (36)$$

We were not able to perform this sum. Fortunately, it turns out that, in the limit of small  $q$ , the sum over  $\omega$  involves only frequencies close to  $\omega_p$ ; for such frequencies, (35) is negligible compared to (34). Thus, we can neglect the term  $L_2 C_{2\mathbf{q}\omega}$  in (36), writing

$$\beta S_{C\omega}(t, \mathbf{q}) = \frac{1}{L_1 C_{1\mathbf{q}\omega}} f(\omega) \cos(\omega t), \quad (37)$$

and perform the sum on  $\omega$  by replacing it by an integral like in Sec. IV. Therefore, the contribution of these modes is

$$\begin{aligned} S_C(t, \mathbf{q}) &= \frac{1}{\pi} \int_{\omega_p^2 + (cq)^2}^\infty d(\omega^2) \\ &\times \frac{\omega_p^4 f(\omega) \cos(\omega t)}{4\pi c \sqrt{\omega^2 - \omega_p^2 - (cq)^2} \omega^4 (\omega^2 - \omega_p^2)}. \end{aligned} \quad (38)$$

After the change of variable  $\omega^2 = \omega_p^2 + (cq)^2(1+u)$ , (38) becomes, in the limit  $q \rightarrow 0$ ,

$$\begin{aligned} S_C(t, \mathbf{q}) &\sim \frac{q}{4\pi^2} f(\omega_p) \cos(\omega_p t) \int_0^\infty \frac{du}{\sqrt{u(1+u)}} \\ &= \frac{q}{4\pi} f(\omega_p) \cos(\omega_p t). \end{aligned} \quad (39)$$

### VI. LONGITUDINAL MODES

In addition to the transverse modes studied up to now, in region  $\Lambda_1$ , there are longitudinal modes of frequency  $\omega_p$  [then  $\epsilon(\omega_p) = 0$ ] [5,6]. They are solutions of Eqs. (4)–(8) with now  $\rho \neq 0$  and  $\mathbf{B} = 0$ ; therefore, they occur also in the nonretarded case already studied in [4]. These longitudinal modes are a superposition of waves with the electric field parallel to the wave-number vector. Here, we summarize the calculations of [4], using a slightly different notation.

Since all modes have the same frequency but differ by the component  $x$  of the wave vector, in (3)  $\sigma_{\mathbf{q}\omega}(t)$  must be replaced by  $\sigma_{\mathbf{q}k}(t)$ . Since  $\mathbf{B} = 0$ , (5) reduces to  $\nabla \times \mathbf{E} = 0$ , and  $\mathbf{E}$  is derivable from a potential. This potential is of the form

$$\phi = [a(t)\exp(iq_y y) + \text{c.c.}]\sin(kx); \quad (40)$$

this form ensures that the  $x$  component of the electric field

$$E_{1x} = -k[a(t)\exp(iq_y y) + \text{c.c.}]\cos(kx) \quad (41)$$

is nonzero at  $x=0$ , thus is coupled by (15) to  $\sigma_{\mathbf{q}k}(t)$  (in region  $\Lambda_2$ , there are no longitudinal waves, thus  $\mathbf{E}_2=0$ ). Equation (15) gives  $-ka(t)=4\pi\sigma_{\mathbf{q}k}(t)$ . After  $E_{2y}$  has been expressed from (40), one finds for the energy

$$H_{\mathbf{q}k} = 4\pi AL_1 \frac{k^2 + q^2}{k^2} \left( |\sigma_{\mathbf{q}k}(t)|^2 + \frac{1}{\omega_p^2} |\dot{\sigma}_{\mathbf{q}k}(t)|^2 \right). \quad (42)$$

After an integration on  $k$ , one finds for the contribution of the longitudinal modes

$$\beta S_D(t, \mathbf{q}) = \frac{1}{2\pi^2} f(\omega_p) \cos(\omega_p t) \int_0^K dk \frac{k^2}{k^2 + q^2}, \quad (43)$$

where  $K$  is a cutoff beyond which the use of collective variables breaks down. For small  $q$ , the integral in (43) is  $K - (\pi/2)q + O(q^2)$ ; therefore,  $\beta S_D(t, \mathbf{q})$  has a kink singularity at  $\mathbf{q}=0$  of the form

$$\beta S_D(t, \mathbf{q}) \sim -\frac{q}{4\pi} f(\omega_p) \cos(\omega_p t). \quad (44)$$

## VII. CONCLUSION

Equations (39) and (44) cancel each other. The only contribution of order  $q$  to  $\beta S(t, \mathbf{q})$  is (32). Thus

$$\beta S(t, \mathbf{q}) \sim \frac{q}{4\pi}, \quad (45)$$

which is the classical nonretarded result at  $t=0$ , as announced in the Introduction. This classical nonretarded result at  $t=0$  was found previously [4] as the result of two contributions: The one from the nonretarded surface plasmons  $2q/(4\pi)$  and the one from the longitudinal bulk modes in the plasma  $-q/(4\pi)$ . Things are different for the present retarded result, which comes from the fields in vacuum (in Sec. IV the fields in the plasma do not contribute to the total energy), and a cancellation of the contributions of the transverse waves and the longitudinal waves in the plasma.

The present analysis relies on several assumptions. In particular, the linearization implies that the fluctuations are small (the same assumption is made in [1]). Furthermore, in the derivation of (8), the plasma is treated as a collection of free charges; the frequency-dependent dielectric function (10) is obtained by the same approximation. The validity of

these assumptions should be confirmed by a fully microscopic derivation.

Like in [1], our model is restricted to the OCP. For real conductors, with dissipation, the same simple final result might survive or not; this is an open problem. Another open qualitative problem is to understand by a physical argument why the retarded quantum result at an arbitrary  $t$  is so simple, at least for an OCP, not involving  $\hbar$ , nor  $t$ , nor  $c$ .

## ACKNOWLEDGMENTS

L.Š. is grateful to LPT for its very kind hospitality. The support received from the European Science Foundation (ESF “Methods of Integrable Systems, Geometry, Applied Mathematics”) and Grant No. VEGA 2/0113/2009 is acknowledged.

## APPENDIX

The energy density of a one-component plasma (14) can be obtained from the Maxwell equations (4)–(7) and the force equation (8). Energy conservation can be expressed as [5]

$$\frac{1}{8\pi} \frac{\partial}{\partial t} (E^2 + B^2) + \mathbf{J} \cdot \mathbf{E} = -\nabla \cdot \mathbf{S}, \quad (A1)$$

where

$$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) \quad (A2)$$

is the Poynting vector (energy flow) and  $\mathbf{J} \cdot \mathbf{E}$  is the rate per unit volume of performing work by the electric field. Equation (8) gives

$$\mathbf{J} \cdot \mathbf{E} = \frac{2\pi}{\omega_p^2} \frac{\partial (J^2)}{\partial t}. \quad (A3)$$

The lhs of (A1) is the time derivative of the energy density, thus which is (14).

The last term of (14) can be expressed in terms of  $\mathbf{E}$ . One must be careful about the sign: Since oscillatory terms are to be discarded,  $J^2 = |\mathbf{J}|^2$  and  $E^2 = |\mathbf{E}|^2$ . Using again (8), at frequency  $\omega$ , one can replace the last term in (14) by  $\omega_p^2 E^2 / (8\pi\omega^2)$ . Taking into account (10), we obtain

$$U_1 = \frac{1}{8\pi} \left( \frac{d(\omega\epsilon)}{d\omega} E^2 + B^2 \right), \quad (A4)$$

as written in [6].

- [1] L. Šamaj and B. Jancovici, Phys. Rev. E **78**, 051119 (2008).  
 [2] S. Rytov, Dokl. Akad. Nauk SSSR **87**, 535 (1952).  
 [3] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics* (Pergamon, Oxford, 1980), Part 2, Chap. VIII.  
 [4] B. Jancovici, J. Stat. Phys. **39**, 427 (1985).

- [5] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).  
 [6] L. Landau and E. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960).  
 [7] J. M. Pitarke, V. M. Silkin, E. V. Chulkov, and P. M. Echenique, Rep. Prog. Phys. **70**, 1 (2007).