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Integrable nonautonomous nonlinear Schrödinger equations are equivalent to the standard autonomous equation

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A class of nonautonomous nonlinear Schrödinger equations, claiming to be novel integrable systems with rich properties, continues to appear in the literature. All such equations are shown to be not new, but equivalent to the standard autonomous equation, which trivially explains their integrability features.

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Time and again, various forms of nonautonomous [with explicit time- (t-) dependent coefficients] and inhomogeneous [with explicit space- (x-) dependent coefficients] more general nonlinear Schrödinger (GNLS) equations along with their discrete variants have appeared as a central result in the literature [1–12] or are recent results under consideration for publication [13–18]. These equations, which arise in the applicable contexts of Bose-Einstein condensates, fiber optics communication, superconductivity, etc., are either suspected to be integrable due to the finding of a particular analytic ansatz or stable computer solutions, assumed to be only Painlevé integrable [16–18], or else claimed to be totally new completely integrable systems.

Complete integrability of a nonlinear system is more rich and powerful than Painlevé integrability, since while Painlevé integrability is a criterion for checking whether a given equation allows unique analytic solutions, complete integrability ensures a spectrum of important properties like the existence of infinite conserved quantities in involution, a hierarchy of integrable higher-order equations, the existence of a Lax pair related to the linear spectral problem, exact N-soliton solutions, solvability through the inverse scattering method (ISM), etc.

Apparently the solution of such GNLS equations needs a generalization of the ISM, in which the usual isospectral approach involving only the constant spectral parameter λ_0 has to be extended to *nonisospectral* flow with time-dependent $\lambda(t)$ [2,7,10,12]. Moreover, certain features of the soliton solutions (named as novel *nonautonomous solitons* [8,10]) of such nonautonomous GNLS equations, like the changing of the solitonic amplitude, shape, and velocity with time, were thought to be new and surprising discoveries.

We show here that all these GNLS equations are completely integrable systems. However, they are *not new* or independent integrable equations and in fact are *equivalent* to the standard autonomous NLS system, linked through simple gauge, scaling, and coordinate transformations. The standard NLS equation is a well-known completely integrable system with known Lax pair, soliton solutions, and usual isospectral ISM [19,20]. As we see below, a simple time-dependent gauge transformation of the standard isospectral system with constant λ_0 can create the illusion of having complicated nonisospectrality. Similarly, a time-dependent scaling of the standard NLS field $Q \to q = \rho(t)Q$ would naturally lead the constant soliton amplitude to a time-dependent one. In the same way a trivial coordinate transformation $x \to X = \rho(t)x$ would change the usual constant velocity v_0 of the NLS soliton to a time-variable quantity $v(t) = v_0/\rho(t)$ and the invariant shape of the standard soliton with constant extension $\Gamma_0 = 1/\kappa$ to a time-dependent one with variable extension $\Gamma(t) = \Gamma_0/\rho(t)$ (see Fig. 1). Therefore all rich integrability properties of the nonautonomous GNLS equation, observed in earlier papers, including more exotic and seemingly surprising features like nonisospectral flow, appearance of shape changing and accelerating soliton, etc., can be trivially explained by time-dependent transformations of the standard NLS equation. The corresponding explicit result for the nonautonomous GNLS models—namely, their Lax pair, N-soliton solutions, infinite conserved quantities, etc.—can be derived easily from their well-known counterparts in the standard NLS equation [19] through the same transformations.

We start from a recent version of the GNLS equation [10], which is generic in some sense:

$$iQ_t + \frac{1}{2}DQ_{xx} + R|Q|^2Q - (2\alpha x + \Omega^2 x^2)Q = 0, \qquad (1)$$

where the nonautonomous coefficients $D(t)$ and $R(t)$ of the dispersive and nonlinear terms are arbitrary functions of t, while the other time-dependent functions are linked as

$$\alpha(t) = s_t\rho, \quad \Omega^2(t) = \frac{1}{4}\left(\theta_t - \frac{1}{2}D\theta^2\right),$$

where

$$\rho = \frac{R}{D}, \quad \theta = \frac{\rho_t}{\rho D}, \qquad (2)$$

$s(t)$ being another arbitrary function. It is easy to see that a time-dependent scaling of the field can change the coefficient of the nonlinear term in Eq. (1) and at the same time generate an additional term from iQ_t, while a change in the phase of the field involving x^2 would yield extra terms from Q_{xx}. As a result, transforming $Q \to q = \sqrt{\rho}e^{i(\theta/4)x^2}Q$, we can rewrite the GNLS equation (1) into another form

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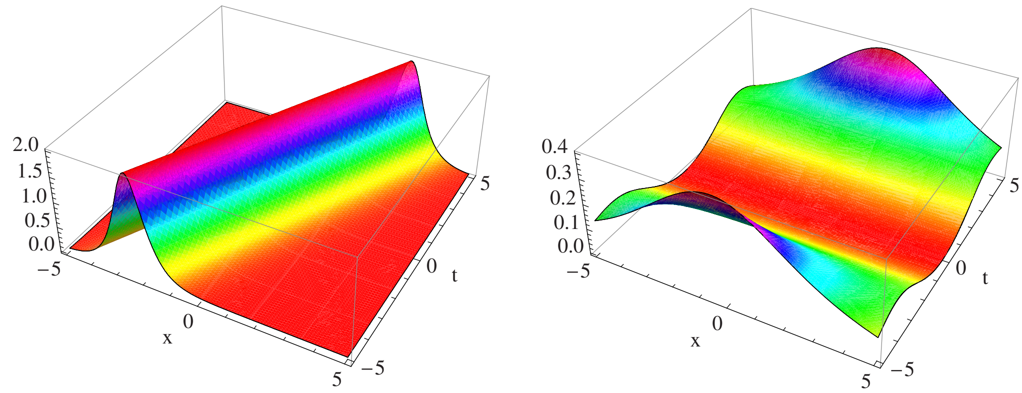

FIG. 1. (Color online) Exact soliton solutions (module) for the integrable (a) standard NLS equation (14) and (b) nonautonomous GNLS equation (3) with $A_0=2.0$, $v_0=0.5$, and the particular choice $R(t)=0.008t^2$, $D(t)=1$, $s(t)=0$. In spite of the significant differences between the appearance and dynamics of these two solutions, they are related by simple transformations (7)–(12) and belong to equivalent integrable systems.

$$iq_t+\frac{D}{2}q_{xx}+D|q|^2q-(2\alpha x+iD\theta)q-iD\theta xq_x=0. \quad (3)$$

In [10], Eq. (1) was declared to be a new discovery, and as a proof of its integrability, a Lax pair associated with Eq. (3) was presented, which we rewrite here in a compact and convenient form by introducing a matrix $U^{(0)}(q)=\sqrt{\sigma}(q\sigma^+-q^*\sigma^-)$ with $\sigma^{\pm}=\frac{1}{2}(\sigma^1\pm i\sigma^2)$ as

$$U(\lambda(t))=-i\lambda(t)\sigma^3+U^{(0)}(q),$$

$$V(\lambda(t))=DV_0(\lambda(t))-i\alpha x\sigma^3+D\theta xU(\lambda(t)), \quad (4)$$

where

$$V_0(\lambda(t))=-i\lambda^2(t)\sigma^3+\lambda(t)U^{(0)}+\frac{i}{2}\sigma^3[U_x^{(0)}-(U^{(0)})^2]. \quad (5)$$

Here σ^a, $a=1,2,3$, are standard 2×2 Pauli matrices. We can check from the above Lax pair that the flatness condition $U_t-V_x+[U,V]=0$ yields the GNLS equation (3) under the constraint $\lambda_t(t)=\alpha+D\theta\lambda(t)$. Using relations (2), one can resolve this constraint to get $\lambda(t)=\rho(t)[\lambda_0+s(t)]$ with $\lambda_0=\text{const}$, as given in [10].

We now establish the equivalence between the Lax pair $U(\lambda(t))$ and $V(\lambda(t))$ [Eqs. (4) and (5)] for the GNLS equation and the well-known Lax pair $U_{nls}(\lambda_0)$ and $V_{nls}(\lambda_0)$, Eq. (13), of the standard NLS equation [19], showing explicitly that the nonisospectral $\lambda(t)$ is convertible to the constant spectral parameter λ_0 through simple transformations.

For this it is interesting to notice, first, that the structure of the NLS Lax pair (13) is hidden already in the expression of the GNLS Lax pair as $U(\lambda(t))|_{\lambda(t)=\lambda_0}=U_{nls}(\lambda_0)$ and $V_0(\lambda(t))|_{\lambda(t)=\lambda_0}=V_{nls}(\lambda_0)$. Therefore the aim should be to remove the t dependence from $\lambda(t)=\rho(t)[\lambda_0+s(t)]$ by absorbing the arbitrary functions $\rho(t)$ and $s(t)$ in a step-by-step manner. Note that the Lax pair (U,V), as evident from the associated linear problem $\partial_x\Phi=U\Phi$, $\partial_t\Phi=V\Phi$, corresponds to infinitesimal generators in the x and t directions, respectively, and therefore a simple coordinate change $(x,t)\to(\tilde{x}=\rho(t)x,\tilde{t}=t)$ resulting in $\partial_{\tilde{x}}=(1/\rho)\partial_x$, $\partial_t=\partial_{\tilde{t}}+\rho_t x\partial_{\tilde{x}}=\partial_{\tilde{t}}+D\theta x\partial_x$, would yield $U(x,t)=\rho U(\tilde{x},\tilde{t})$ and $V(x,t)=V(\tilde{x},\tilde{t})+D\theta xU(x,t)$. Therefore using such a transformation and comparing with Eq. (4), we can easily remove the $\rho(t)$ factor from $\lambda(t)$ in $U(x,t)$, which, however, would scale the field as $q\to q/\rho$ and at the same time eliminate from the transformed $V(\tilde{x},\tilde{t})$ the nonstandard term $D\theta xU(x,t)$ appearing in $V(x,t)$, Eq. (4).

For the removal of additive term $\rho(t)s(t)$ from $\lambda(t)$, present in $U(x,t)$, one can perform a gauge transformation $\Phi\to\tilde{\Phi}=g\Phi$ with $g=e^{i\rho s x\sigma^3}$, taking the Lax pair to a gauge-equivalent pair

$$\tilde{U}=g_xg^{-1}+gUg^{-1},\quad \tilde{V}=g_tg^{-1}+gVg^{-1}. \quad (6)$$

One notices that although the above transformations are enough to remove the explicit t dependence from U due to its linear dependence on $\lambda(t)$, the removal of t from $V(\lambda(t))$ becomes a bit involved due to the nonlinear entry of $\lambda^2(t)$ and $\lambda(t)U^{(0)}$ in it, which brings in more time-dependent terms like $2\lambda_0D\rho^2s$ and $D\rho^2s^2$. These extra terms, however, can be exactly compensated for by extending slightly the above coordinate and gauge transformations, by introducing additional functions $f(t)$ and $\tilde{f}(t)$ and choosing them as $f_t=2D\rho^2s$ and $\tilde{f}_t=D\rho^2s^2$.

The multiplicative factor $D\rho^2$ appearing in all terms in $V(\lambda(t))$ can be absorbed easily by a further coordinate change $t\to T=D\rho^2t$. Therefore taking the above arguments into account one finally solves the problem completely through the following three steps of simple transformations.

(i) *Coordinate transformation* $(x,t)\to(X,T)$ with

$$X=\rho(t)x+f(t),\quad T=D\rho^2t, \quad (7)$$

where $f_t=2D\rho^2s$, inducing

$$\partial_X=\frac{1}{\rho}\partial_x,\quad \partial_T=\frac{1}{D\rho^2}\partial_t+(\rho_tx+f_t)\partial_X, \quad (8)$$

with partial reduction of the Lax pair $(U,V)\to(U_1,V_1)$ as

$$U_1(\lambda_0) = U_{nls}\left(\lambda_0, U^{(0)}\left(\frac{q}{\rho}\right)\right) - is\sigma^3,$$

$$V_1(\lambda_0) = V_{nls}\left(\lambda_0, U^{(0)}\left(\frac{q}{\rho}\right)\right) + i\left(s^2 - \frac{\alpha}{D\rho^2}x\right)\sigma^3 - sU^{(0)}\left(\frac{q}{\rho}\right), \tag{9}$$

due to active cancellation of some terms by $-(\rho_t x+f_t)U_1(\lambda_0)$ appearing from coordinate change (8)

(ii) *Gauge transformation* (6), where

$$g = e^{i[s(t)X-\tilde{f}(t)]\sigma^3}, \tag{10}$$

with $\tilde{f}_t = D\rho^2 s^2$, with almost the needed reduction of the Lax pair $(U_1,V_1)\to(U_2,V_2)$ as

$$U_2(\lambda_0) = U_{nls}\left(\lambda_0, U^{(0)}\left(\frac{q}{\rho}e^{2i(sX-\tilde{f})}\right)\right) \tag{11}$$

$$V_2(\lambda_0) = V_{nls}\left(\lambda_0, U^{(0)}\left(\frac{q}{\rho}e^{2i(sX-\tilde{f})}\right)\right).$$

This dramatic reduction of Eq. (9) is caused by the above gauge transformation producing compensating terms like $i\{[1/(D\rho)]s_t x - \tilde{f}_T\}\sigma^3$ and a crucial one coming due to $gU_X^{(0)}g^{-1}\to\tilde{U}_X^{(0)}$ where $\tilde{U}^{(0)}=gU^{(0)}(Q)g^{-1}=U^{(0)}(Qe^{2i(sX-\tilde{f})})$

(iii) *Field transformation* $q\to\psi$, where

$$\psi = \frac{1}{\rho}qe^{2i(sX-\tilde{f})}, \tag{12}$$

which brings Eq. (11) to the final reduction $(U_2,V_2)\to(U_3=U_{nls}(\lambda_0,U^{(0)}(\psi)),\ V_3=V_{nls}(\lambda_0,U^{(0)}(\psi))$. Thus the above three simple transformations straightforwardly take the nonisospectral Lax pair (4) to the standard NLS Lax pair

$$U_{nls}(\lambda_0) = -i\lambda_0\sigma^3 + U^{(0)}(\psi),$$

where $U^{(0)}(\psi)=\psi\sigma^+ - \psi^*\sigma^-$,

$$V_{nls}(\lambda_0) = -i\lambda_0^2\sigma^3 + \lambda_0 U^{(0)} + \frac{i}{2}\sigma^3[U_X^{(0)} - (U^{(0)})^2]. \tag{13}$$

which proves the equivalence of the Lax pair (4) for the GNLS equation (3) and the Lax pair (13) associated with the standard NLS equation

$$i\psi_T + \frac{1}{2}\psi_{XX} + |\psi|^2\psi = 0, \tag{14}$$

obtained as the flatness condition of Eq. (13). Therefore since the Lax pair assures complete integrability and can alone derive all its associated properties, our finding trivially explains the integrability features of the GNLS equation. One can also check that under the change of independent and dependent variables, (7) and (12), the nonautonomous equation (3) is transformed directly into the autonomous equation (14).

Therefore we remark that the nonautonomous GNLS equations (1) and (3) are equivalent to the autonomous NLS equation (14), a well-known integrable system. The corresponding Lax pairs (4) and (13) are also gauge equivalent to each other, which therefore confirms and shows the triviality of the complete integrability of the GNLS equation.

All signatures of the complete integrability like the Lax pair, N-soliton solutions, infinite conserved quantities, etc., for this GNLS system can be obtained easily from the corresponding well-known expressions for the NLS system, Eqs. (13) and (14), by inverting the set of transformations (7), (10), and (12) as $(X,T)\to(x,t)$, $g\to g^{-1}$, $\psi\to q$. As a result explicit t dependence obviously enters into the Lax operators as well as in the amplitude, phase, and x dependence of the field $q(x)$ of the GNLS system, resulting in the spectral parameter $\lambda_0\to\lambda(t)$ and making the constant amplitude A_0, extension Γ_0, and velocity v_0 of the soliton to become t dependent.

Figure 1 demonstrates this situation, showing that the NLS soliton (module) (a) $|\psi|=A_0\,\text{sech}\,\xi$, $\xi=(1/\Gamma_0)(X-v_0T)$ goes to GNLS soliton (b): $|q|=A(t)\text{sech}\,\tilde{\xi}$, $\tilde{\xi}=[1/\Gamma(t)][x-v(t)]$, where $A(t)=A_0\rho(t)$, $v(t)=Dv_0\rho(t)t-f(t)/\rho(t)$, and $\Gamma(t)=\Gamma_0/\rho(t)$ under the transformations inverse of Eqs. (7), (10), and (12). Therefore, even though GNLS soliton [Fig. 1(b)] looks rather exotic and quite different from the standard NLS soliton [Fig. 1(a)], causing enthusiasm in the earlier work mentioned above, these solutions are related simply by coordinate and scale transformations and belong to equivalent integrable systems.

It is worth mentioning that, although in earlier papers one-soliton and in some cases two-soliton solutions were constructed for the GNLS equation through an ansatz or Darboux transformation, one can in fact derive the exact N-soliton solution and study the multisoliton scattering easily for the GNLS equation, bypassing all sophisticated methods and exploiting only its equivalence with the integrable NLS equation—i.e., by simply mapping the known N-soliton solution of the standard NLS equation through the same transformations (7)–(12).

By redefining the field further, $Q\to b(t)Qe^{ia(t)}$ with arbitrary functions $a(t)$ and $b(t)$, we can generate more nonautonomous terms in Eq. (1) resulting an extended form of the GNLS equation:

$$iQ_t + \frac{D}{2}Q_{xx} + R|Q|^2Q - (2\alpha x + \Omega^2x^2 + a + i\gamma)Q = 0, \tag{15}$$

$$\gamma(t) = \frac{b_t}{b},$$

which is naturally equivalent again to the integrable NLS equation. The GNLS equation (15) was found to be the maximum nonautonomous and inhomogeneous NLS system, which can pass the Painlevé integrability criteria [21].

A recently proposed GNLS equation [16–18], which is simply a particular case of Eq. (15) at $a=0$ and $\alpha=0$, is therefore also equivalent to the standard NLS equation, proving wrong the assumption that the system is only Painlevé integrable and *not* completely integrable [16,17]. Equivalence with the NLS equation assures for this GNLS equation

[16–18] complete integrability along with infinite conserved quantities, N-soliton solutions, etc.

We now look into other forms of integrable GNLS equations, appearing earlier, and show their equivalence to the standard NLS equation, similar to that found above. The simplest form of inhomogeneity to the NLS equation, $2xq$, was proposed in [1] and repeated in [14], which is clearly a particular case of Eq. (3) with $\alpha=1$, $\Omega=0$, proving its equivalence with the NLS equation.

A more extended integrable GNLS equation with $F(x)Q$, $F(x)=a+\alpha x+\mu x^2$ was considered in [2], which is consistent with the general integrable GNLS equation (15), equivalent to the standard NLS equation (14). However, for constructing such integrable GNLS equation, the x-dependent spectral parameter as well as the restriction on the functions entering into $\lambda(t)$, as found in [2], actually does not arise, as we have shown here.

In [5] a variant of the GNLS equation was considered, which was suspected to be integrable through computer simulation. It is easy to see, however, that this GNLS equation, which appears again in [6,7,11], is a particular case of (15) at $\alpha=0$, $a=0$, and $\Omega=0$, but with nontrivial $R(t)$, $D(t)$, and $\gamma(t)$. Similarly the GNLS equation proposed in [8,13,15] is derivable from Eq. (15) as a particular case with nontrivial $R(t)$ and $\Omega(t)$, but $D=1$, $a=0$, $\gamma=0$, and $\alpha=0$, while the GNLS equation of [9] considers additionally a nontrivial $\gamma(t)$. Therefore all the above GNLS equations are equivalent to the standard NLS equation and hence trivially integrable.

Some integrable discrete versions of the GNLS equation—namely, generalized Ablowitz-Ladik models (GALMs)—were proposed in [3,4] and contain in addition to the standard ALM [20] an explicit n-dependent term $n\omega\psi_n$, with $\omega=1$ [3] or $\omega(t)$ as an arbitrary function [4]. We find that in spite of the discrete case, a similar reasoning found here holds true and the proposed nonautonomous ALM can be shown to be gauge equivalent to the standard ALM [20], under the discrete gauge transformation $\tilde{U}_n=g_{n+1}U_n g_n^{-1}$, $\tilde{V}_n=g_n V_n g_n^{-1}+\dot{g}_n g_n^{-1}$, with $g_n=e^{-in\Gamma(t)\sigma^3}$ and a redefinition of the field as $q_n\rightarrow\psi_n=q_n e^{i(2n+1)}\Gamma(t)$, where $\Gamma_t(t)=\omega(t)$ is an arbitrary function as found in [4].

Based on the above result, we therefore conclude that the general inhomogeneous and nonautonomous GNLS equations, if integrable, should be of the form (15). Other forms of integrable GNLS equations are only its particular cases. However, all these GNLS equations, appearing in recent years in an increasing rate, are neither new nor independent integrable systems, but are equivalent to the well-known standard NLS equation, from which all their integrable structures like the Lax pair, N-soliton solutions, infinite number of commuting conserved quantities, etc., can be obtained easily through simple mapping. The time-dependent soliton amplitude, shape, and velocity as well as the nonisospectral flow in these GNLS systems are only an artifact of the simple time-dependent coordinate, gauge, and field transformations, which generate these systems from the standard NLS equation. Therefore before proposing any *new* integrable nonautonomous GNLS equations the authors should check whether it can be linked in any way to the general integrable GNLS equation (15), the equivalence of which with the well-known NLS equation we have proved here.

Another word of caution is that, such GNLS equations have started now appearing in their vector form [22], which we believe again to be not new as an integrable system, but equivalent to the well-known vector NLS equation proposed by Manakov [23].

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