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Time-averaged properties of unstable periodic orbits and chaotic orbits in ordinary differential equation systems

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It has recently been found in some dynamical systems in fluid dynamics that only a few unstable periodic orbits (UPOs) with low periods can give good approximations to the mean properties of turbulent (chaotic) solutions. By employing three chaotic systems described by ordinary differential equations, we compare time-averaged properties of a set of UPOs and those of a set of segments of chaotic orbits. For every chaotic system we study, the distributions of a time average of a dynamical variable along UPOs with lower and higher periods are similar to each other and the variance of the distribution is small, in contrast with that along chaotic segments. The distribution seems to converge to some limiting distribution with nonzero variance as the period of the UPO increases, although that along chaotic orbits inclines to converge to a δ-like distribution. These properties seem to lie in the background of why only a few UPOs with low periods can give good mean statistical properties in dynamical systems in fluid dynamics.

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Chaos in dynamical systems has been discussed in relation to unstable periodic orbits (UPOs) embedded in a chaotic attractor, as a chaotic orbit is considered to be approximate by an ensemble of UPOs which are densely distributed in the chaotic attractor.[1] Recently, in some turbulence systems in fluid dynamics, it has been shown that even only a few UPOs with relatively low periods can capture the mean properties of chaotic motions [1,2]. For the turbulent Couette flow of rather low Reynolds number in the full Navier-Stokes system, Kawahara and Kida obtained a remarkable agreement of an averaged velocity profile along a single UPO with that along a chaotic orbit in the phase space of a turbulent Couette flow. Later, van Veen *et al.* [3] performed a numerical study of an isotropic Navier-Stokes turbulence with high symmetry and found that among several UPOs there is a UPO with relatively low period where the energy dissipation rate appears to converge to a nonzero value as assumed in the Kolmogorov similarity theory in the limit of large Reynolds number. This suggests that the UPO corresponds to the isotropic turbulence of fluid motion, although the Reynolds number is not large enough to discuss the detailed properties of the fully developed turbulence because of computational difficulties. As for the universal statistical properties of fluid turbulence at high Reynolds numbers, employing the Gledzer-Ohkitani-Yamada (GOY) shell model, Kato and Yamada [4] found a single UPO which gives a fairly good approximation to the scaling exponents of structure functions of velocity, which suggests that the intermittency in the model turbulence can be interpreted as a property of a single UPO, rather than a statistical contribution of complex orbits.

In the above studies, it seems that only a few UPOs with relatively low periods are enough to capture some mean properties of a chaotic solution. However, on the other hand, the chaotic attractor is considered to include an infinite number of UPOs, and it appears that a UPO with longer period gives a better approximation to the statistical properties of chaotic solutions, as a set of long UPOs and a set of chaotic orbits are intuitively taken to have similar statistical properties. So we may have a question why in the above systems even a small number of UPOs with rather low periods can give a remarkably good approximation to the chaotic mean values. Some studies have been concerned with this problem [5–7].

Kawasaki and Sasa [6] studied a simple model of chaotic dynamical systems with a large degree of freedom and found that there is an ensemble of UPOs with the special property that the expectation values of macroscopic quantities can be calculated using one UPO sampled from the ensemble. Hunt and Ott [5] studied an optimal periodic orbit which yields the optimal (extreme) value of a time average of a given smooth performance function of dynamical variables. They obtained an implication that the optimal periodic orbit is typically a periodic orbit of low period, although they do not consider the relation of averaged statistical properties along UPOs and chaotic orbits. On the other hand, Yang *et al.* [5] reported that the optimal UPO can be a periodic orbit of high period when the system is near crisis. In a recent study on UPOs of low-dimensional map systems by Saiki and Yamada [7], it is reported that a few UPOs with low periods are not enough to approximate the time-averaged properties of chaotic orbits.

In this Rapid Communication, we employ chaotic systems described by ordinary differential equations (ODEs) and investigate the relation between the average of a dynamical quantity along a UPO and that along a chaotic orbit, especially with attention focused on the dependence of the variance of averaged values on the periods of the UPOs. At first glance, it may appear that if we take all the UPOs with period around T, for example, and take the averages of a dynamical quantity along these UPOs, the variance of the averages would decrease as T increases, because an extremely long orbit would cover most part of the chaotic attractor, capturing possible dynamical states on the attractor.

Our aim in this Rapid Communication is to see whether this intuitive discussion holds for chaotic systems simple

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[1]This is true of, for example, Axiom A systems.

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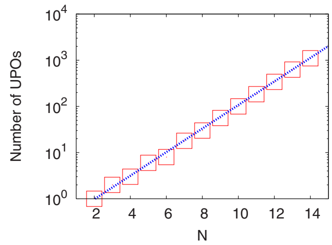

FIG. 1. (Color online) Number of detected UPOs with cycle N of the Lorenz system in comparison with 0.3×1.8^N.

enough to obtain a large number of UPOs by available numerical computation with double accuracy. For this purpose we take three chaotic systems: the Lorenz system, Rössler model, and a business cycle model. A set of UPOs in each model are obtained numerically, and time averages of variables along the UPOs are discussed especially in relation to the periods of the UPOs.

UPOs in the Lorenz system $[dx/dt=\sigma(y-x),\ dy/dt=rx-y-xz,\ dz/dt=xy-bz]$ with classical parameter values $(\sigma=10,\ b=8/3,\ r=28)$ [8] have been extensively studied. Although the Lorenz system is not uniformly hyperbolic, it was recently proved by the aid of numerical calculation with guaranteed accuracy that the Lorenz attractor is chaotic and includes an infinite number of UPOs densely [9]. Also there are several studies about the UPOs from the viewpoint of dynamical system theory on, for example, the ζ function, Hausdorff dimension, and $f(\alpha)$ spectrum [10]. Franceschini *et al.* [11] and Viswanath [12] detected UPOs in an systematic way and suggested that all UPOs are labeled by a sequence of symbols, while Zoldi showed that an ensemble of UPOs weighted by their periods and stability indices gives an approximation to a histogram of a dynamical variable in the chaotic state [13].

Here we focus our attention on the distribution of time-averaged values of a dynamical variable along UPOs of the Lorenz system. In order to detect UPOs we employ in this Rapid Communication the Newton-Raphson-Mees method in which the period of the UPO is regarded as a variable to be found in the numerical calculation [14]. We found more than 1000 UPOs of periods from 1.558 to 16.445, corresponding, respectively, from 2 to 23 rotations around a wing of the Lorenz attractor. The number of rotations (cycle number) corresponds to the period of the Poincaré map defined by the "standard" Poincaré section at $z=r-1$, $dz/dt>0$. In our numerical calculation, we identified more than 90% of the UPOs corresponding to those encoded by the symbol sequences up to 14 rotations.[2] One of the most important indices representing the complexity of a dynamical system is the topological entropy [15], which is estimated by the exponential growth rate of the number of periodic orbits, $h_{top}=\lim\sup_{N\to\infty}\log(\text{number of UPOs with cycle } N)/N$, and the topological entropy h_{top} of the Poincaré map in this case is estimated to be log (1.8) from Fig. 1. We should remark that there is a clear linear dependence of log(number of UPOs with cycle N) on N, which suggests that the number of detected UPOs in our computation is sufficient to study the statistical properties of UPOs. We now calculate the time average of z $(\langle z\rangle\equiv\int_{t=0}^{T}z/T\,dt)$ along each UPO with period T. $\langle z\rangle$'s along UPOs take similar but different values around the average value of $\langle z\rangle$'s along chaotic segments (23.55) [Fig. 2 (a)]. Figure 2 (b) shows the density distribution of $\langle z\rangle$'s along UPOs for N $(=9,11,13)$. We can see that the distribution stays similar in shape though N varies, indicating that an even longer UPO is not necessarily suitable for evaluation of z averaged along a long chaotic orbit. Although the range of N is limited, this observation may be contrary to our expectation that a UPO with longer period would give better approximations to statistical properties of chaotic orbits. Actually in Fig. 3 (a) the standard deviations of the density distribution of $\langle z\rangle$'s along UPOs with cycle N are seen to be

[2]We expect that the remaining 10% of UPOs have no significant statistical difference from the obtained UPOs.

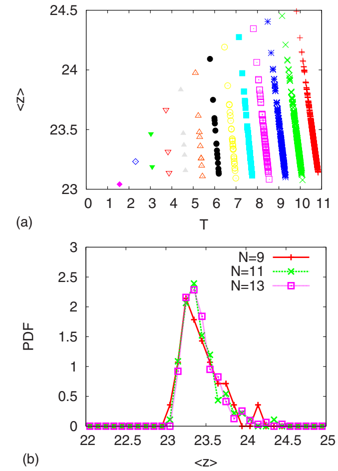

FIG. 2. (Color online) Time averages $\langle z\rangle$'s $(\langle z\rangle\equiv\int_{t=0}^{T}z/Tdt)$ along UPOs with period T (a). Density distribution of $\langle z\rangle$'s along UPOs with cycle N [=9 (+), 11 (×), 13 (□)] (b).

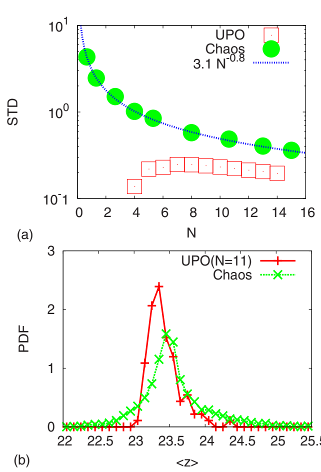

FIG. 3. (Color online) Standard deviation of density distribution of $\langle z\rangle$'s along UPOs with cycle N (□) and that along 10^5 chaotic segments with the corresponding time lengths T $(=0.753N)$ (•) and $3.1N^{-0.8}$ (line) (a). Density distribution of $\langle z\rangle$'s along UPOs with cycle 11 (average=23.43) (+) in comparison with that along 10^5 chaotic segments with the corresponding time length T $(=0.753\times11)$ (average=23.55) (×) (b).

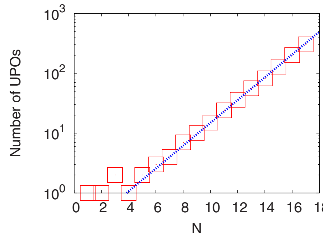

FIG. 4. (Color online) Number of detected UPOs with cycle N of the Rössler system in comparison with 0.19×1.55^N.

nearly constant as N increases. The figure also shows that the standard deviations of $\langle z\rangle$'s along segments of chaotic orbits with time length $T=0.753N$, where 0.753 stands for the corresponding recurrent time to the Poincaré section. We can see that as N increases, the latter standard deviation decreases nearly as $N^{-0.75}$. The difference between the density distribution of time averages along UPOs is clearly observed in Fig. 3 (b) in the case of the distribution of $\langle z\rangle$'s along a set of UPOs with cycle 11 and chaotic segments with the corresponding lengths.

Next, UPOs of the Rössler system $(dx/dt=-y-z,\ dy/dt=x+0.2y,\ dz/dt=0.2+xz-5.7z)$ are studied in the same way as the Lorenz system. Numerical detection of UPOs and its validity are studied by Galias [16]. Here we study time-averaged properties of more than 1000 UPOs, whose periods are between 5.881 and 140.619. More than 90% of UPOs with cycle N ($\leqslant 17$) are covered (Fig. 4) and are used for statistical analysis, where the cycle number means the period of a UPO of the Poincaré map of the Rössler system with the Poincaré section ($x=0$, $dx/dt>0$). The topological entropy of the Poincaré map is estimated to be log(1.55) from the figure.

Time averages $\langle x\rangle$'s $(\equiv\int_{t=0}^{T}x/T\,dt)$ for each UPO is shown in Fig. 5 (a). The distribution of $\langle x\rangle$'s is not very concentrated. In addition, the density distribution of $\langle x\rangle$'s along UPOs with cycle N $(=10,13,16)$ does not depend so much on N [Fig. 5 (b)]. Figure 6 (a) also indicates that the variance of the density distribution of $\langle x\rangle$'s of UPOs with

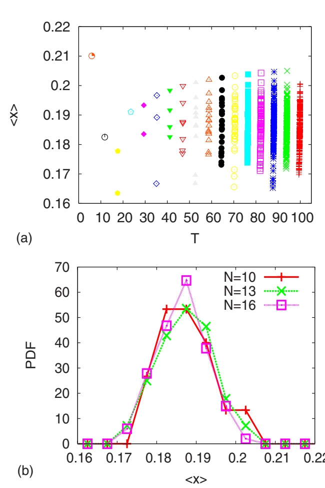

FIG. 5. (Color online) Time averages $\langle x\rangle$'s $(\langle x\rangle\equiv\int_{t=0}^{T}x/T\,dt)$ along UPOs with period T (a). Density distribution of $\langle x\rangle$'s along UPOs with cycle N [=10 (+), 13 (×), 16 (□)] (b).

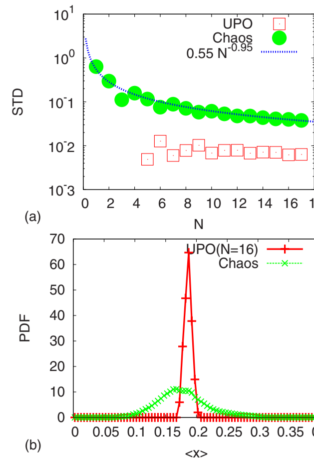

FIG. 6. (Color online) Standard deviation of the density distribution of $\langle x\rangle$'s along UPOs with cycle N (□) and that along 10^5 chaotic segments with the corresponding time length T $(=5.856N)$ (●) and $0.55N^{-0.95}$ (line) (a). Density distribution of $\langle x\rangle$'s along UPOs with cycle 16 (average=0.186) (+) in comparison with that along 10^5 chaotic segments with the corresponding time length $T=5.856\times16$ (average=0.177) (×) (b).

cycle N ($5\leqslant N\leqslant17$) are small in comparison with that of 10^5 chaotic segments with the corresponding lengths. Figure 6 (b) shows a clear difference of the density distribution of $\langle x\rangle$'s along UPOs with cycle 16 and chaotic segments with the corresponding time lengths.

As the third example, we study UPOs embedded in a chaotic business cycle model of Goodwin type, which is described by six-dimensional ODEs $(du_i/dt=0.5[0.1/(1-v_i)+\pi_i^e-0.5]u_i,\quad dv_i/dt=0.1[1.5(1-u_i)^5+3.5(u_j-u_i)^3+0.5u_i-a_iv_i+b_i]v_i,\ d\pi_i^e/dt=[v_i(0.4\pi_i^e+0.2)-0.4\pi_i^e-0.16]/(1-v_i)$, where $(i,j)=(1,2),(2,1))$ [17]. The constants are set as $a_1=0.875$, $b_1=-0.1$, $a_2=-4.2$, and $b_2=2.56$. More than 1000 UPOs, whose periods are between 25.22 and 600.05, are numerically detected. We classify UPOs according to the cycle number which corresponds to the period of the Poincaré map $(\pi_1^e=0,\ d\pi_0^e/dt>0)$ of the system (Fig. 7). Figure 7 suggests that almost all UPOs with cycle N ($\leqslant15$) are thought to be covered and the topological entropy of the map is estimated to be log(1.55). Time averages $\langle u_1\rangle$'s $[\langle u_1\rangle\equiv\int_{t=0}^{T}u_1(t)/t dt]$ along UPOs with period T (<385) are discussed, hereafter. $\langle u_1\rangle$'s along UPOs take various values [Fig. 8 (a)], and we find in Fig. 8 (b) that the density distributions of $\langle u_1\rangle$'s along UPOs with cycle N are similar for $N=9,12,15$. This figure suggests that like the former two systems the time-averaged

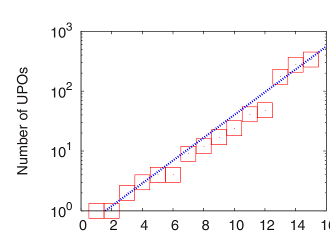

FIG. 7. (Color online) Number of detected UPOs with cycle N of the economic system in comparison with 0.5×1.55^N.

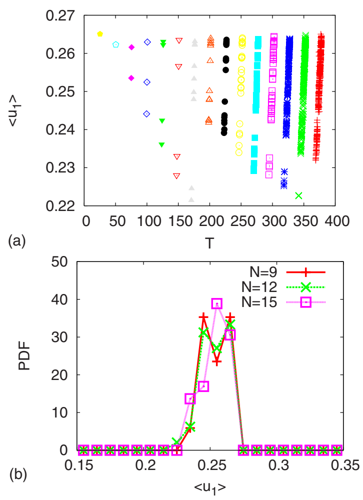

FIG. 8. (Color online) Time average $\langle u_1 \rangle$'s ($\langle u_1 \rangle \equiv \int_{t=0}^{T} u_1 dt$) of UPOs with period T (a). Density distribution of $\langle u_1 \rangle$'s along UPOs with cycle N [=9 (+), 12 (×), 15 (□)] (b).

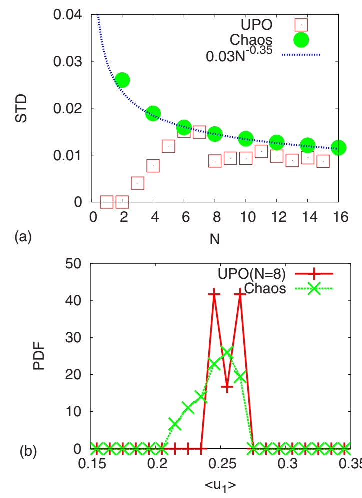

FIG. 9. (Color online) Standard deviation of the density distribution of $\langle u_1 \rangle$'s along UPOs with cycle N (□) and 10^5 chaotic segments with the corresponding time length T ($=24.541N$) (●) and $0.03N^{-0.35}$ (line) (a). Density distribution of $\langle u_1 \rangle$'s along UPOs with cycle 8 (average=0.253) (+) in comparison with that along 10^5 chaotic segments with the corresponding time length T ($=24.541 \times 8$) (average=0.246) (×) (b).

value u_1 along a UPO with a long period does not necessarily approximate that along a chaotic orbit (0.246). In fact, the standard deviation of the density distribution of $\langle u_1 \rangle$'s along UPOs with cycle N is small if N is small and does not decrease as N increases in comparison with that along chaotic segments with the corresponding time lengths [Fig. 9 (a)]. The difference between the density distribution of UPOs with cycle 8 and chaotic segments with the corresponding time lengths are clearly confirmed in Fig. 9 (b).

We have discussed time averages of dynamical variables along UPOs in three chaotic dynamical systems described by ODEs (the Lorenz system, the Rössler system, and a six-dimensional economic model). We have calculated more than 1000 UPOs for each system and found that time-averaged properties along a set of UPOs and a set of chaotic orbits with finite lengths are totally different from each other. From our numerical result with double accuracy a longer UPO is not necessarily advantageous than a shorter UPO to estimate the mean properties of the chaotic state in these models. In other words, we can employ a short UPO for the estimation of the mean properties of the chaotic state without a significant reduction of plausibility. In addition, it is conjectured that the time averages of the dynamical quantities along UPOs with the same period of the Poincaré map have a limiting distribution with nonzero variance. Recently, in some fluid dynamical systems, it has been found that only a few UPOs with low periods give fairly good approximations to some statistical properties. Our result suggests that the estimation by using a short UPO is as reliable (or unreliable) as that by using a long UPO if the limiting distribution is applicable to those variables. Thus the good approximation in the fluid dynamical models may not be improved substantially even if a UPO with a longer period is employed.

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